Fractal stochastic processes

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Abstract

In this paper we construct fractal stochastic processes as fixed point for a scaling law. Using probabilistic metric spaces techniques, we can weak the first moment condition for existence and uniqueness of fractal processes.

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In 1986 Barnsley [1] introduced a class of fractal functions defined on a compact interval of $\mathbb{R}$ which satisfies certain interpolation properties. These functions are analogous to spline and polynomial interpolation in that their graph are constrained to go trough a finite number of prescribed points. They differ from classical interpolants in that they satisfy a functional condition reflecting selfsimilarity. These interpolation functions present some kind of geometrical selfsimilarity or stochastic selfsimilarity. Often the Hausdorff-Besicovitch dimension of their graph will be noninteger. The box-dimension of such interpolation function is greater than 1 if the sum of the absolute values of scaling factors is greater than 1. These functions are Hölder continuous but not differentiable. This type of functions describe not only profiles of mountain ranges, tops of clouds and horizons over forests but also temperatures in flames as a function of time, electroencephalograph pen traces and even the minute-by minute stock market index.

In a series of papers Dubuc [3], [4] introduced an iterative interpolation process for interpolating dates defined on a closed discrete subset of $\mathbb{R}^n$. In [10] Massopust gave a general construction based on a Read-Bajraktarevič operator for the fractal interpolation function. Recently, using probability metrics, Hutchinson and Rüschendorf [7] proved the existence and uniqueness conditions for fractal interpolation function. .

Herburt and Malysz [6] generalize Barnslay's fractal interpolation function to fractal interpolation processes. He prove that for α-selfsimilar processes which include α-fractional Brownian motion's trajectories of such fractal interpolations converges to a trajectory of the interpolated process. Convergence of the trajectory in fractal interpolation of stochastic processes is equivalent to the convergence of trajectories in linear interpolation. Moreover, he show that the box-dimension of trajectories of such interpolations for self similar process with stationary increments converge to $2-\alpha$.

Herburt and Malysz worked with Gaussian processes where the first moment condition is essential. Using contraction methods in Λ E-spaces, we can weak existence and uniqueness condition for fractal interpolation stochastic processes.

1 λE-space

Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$.

A mapping $F : \mathbb{R} \to [0,1]$ is called a **distribution function** if it is non-decreasing and left continuous.

By Δ we shall denote the set of all distribution functions F and $\Delta^+ := \{F \in \Delta : F(0) = 0\}$.

H will denote the Heviside distribution function defined by

$$H(x) = \begin{cases} 0, & x \le 0, \\ 1, & x > 0. \end{cases} \tag{1.1}$$

Let X be a nonempty set. For a mapping $\mathcal{F} : X \times X \to \Delta^+$ and $x, y \in X$ we shall denote $\mathcal{F}(x, y)$ by $F_{x,y}$, and the value of $F_{x,y}$ at $t \in \mathbb{R}$ by $F_{x,y}(t)$, respectively.

The pair $(X, \mathcal{F})$ is a **probabilistic metric space** if X is a nonempty set and $\mathcal{F} : X \times X \to \Delta^+$ is a mapping satisfying the following conditions:

1^0 $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X$ and $t \in \mathbb{R}$;

2^0 $F_{x,y}(t) = 1$, for every $t > 0$, if and only if $x = y$;

3^0 if $F_{x,y}(s) = 1$ and $F_{y,z}(t) = 1$ then $F_{x,z}(s + t) = 1$.

A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a **t-norm** if the following conditions are satisfied:

4^0 $T(a, 1) = a$ for every $a \in [0, 1]$;

5^0 $T(a, b) = T(b, a)$ for every $a, b \in [0, 1]$;

6^0 if $a \ge c$ and $b \ge d$ then $T(a, b) \ge T(c, d)$;

7^0 $T(a, T(b, c)) = T(T(a, b), c)$ for every $a, b, c \in [0, 1]$.

We list here the simplest:

$T_1(a, b) = max\{a + b - 1, 0\}$,

$T_2(a, b) = ab$,

$T_3(a, b) = Min(a, b) = min\{a, b\}$,

A **Menger space** is a triplet $(X, \mathcal{F}, T)$, where $(X, \mathcal{F})$ is a probabilistic metric space, T is a t-norm, and instead of 3^0 we have the stronger condition:

8^0 $F_{x,y}(s + t) \ge T(F_{x,z}(s), F_{z,y}(t))$ for all $x, y, z \in X$ and $s, t \in \mathbb{R}_+$.

The notion of E-space was introduced by Sherwood [11] in 1969.

Let $(\Omega, \mathcal{K}, P)$ be a probability space and let (Y, ρ) be a metric space.

The ordered pair $(\mathcal{E}, \mathcal{F})$ is an **E-space over the metric space** (Y, ρ) (briefly, an E-space) if the elements of $\mathcal{E}$ are random variables from Ω into Y and $\mathcal{F}$ is the mapping from $\mathcal{E} \times \mathcal{E}$ into Δ^+ defined via $\mathcal{F}(x, y) = F_{x,y}$, where

$$F_{x,y}(t) = P(\{\omega \in \Omega | \ \rho(x(\omega), y(\omega)) < t\})$$

for every $t \in \mathbb{R}$. If $\mathcal{F}$ satisfies the condition

$$\mathcal{F}(x, y) \ne H, \ \text{ if } \ x \ne y,$$

then $(\mathcal{E}, \mathcal{F})$ is said to be a **canonical E-space.** Sherwood [11] proved that every canonical E-space is a Menger space under $T = T_m$, where $T_m(a, b) = \max\{a + b - 1, 0\}$. In the following we suppose that $\mathcal{E}$ is a canonical E-space.

The convergence in an E-space is exactly the probability convergence.

The E-space $(\mathcal{E}, \mathcal{F})$ is said to be **complete** if the Menger space $(\mathcal{E}, \mathcal{F}, T_m)$ is complete.

Let Λ be a nonempty set and, for $\lambda \in \Lambda$, let (Y^λ, d^λ) be metric space. Denote $\mathcal{E}^\lambda$ the set of random variables from Ω into Y^λ and let

$$\mathcal{F}^\lambda : \mathcal{E}^\lambda \times \mathcal{E}^\lambda \to \Delta^+$$

defined via $\mathcal{F}^\lambda(x, y) := F^\lambda_{x,y}$, where

$$F^\lambda_{x,y}(t) := P(\{\omega \in \Omega | d^\lambda(x^\lambda(\omega), y^\lambda(\omega)) < t\})$$

for all $t \in \mathbb{R}$. Denote

$$F_{x,y}(t) := \inf_{\lambda \in \Lambda} F^\lambda_{x,y}(t)$$

and

$$\mathcal{F}(x, y) := F_{x,y}.$$

The ordered pair $(\mathcal{E}^\lambda, \mathcal{F}^\lambda)$ is an E-space over the metric space Y^λ.

Let $e \in Y := \prod_{\lambda\in\Lambda} Y^\lambda$ and define

$$\mathcal{E} := \{x \in \prod_{\lambda\in\Lambda} \mathcal{E}^\lambda | \quad \lim_{t\to\infty} \inf_{\lambda\in\Lambda} P(\{\omega \in \Omega | d^\lambda(x^\lambda(\omega), e^\lambda(\omega)) < t\}) = 1\}.$$

A ΛE-space is the triplet $(\mathcal{E}, \mathcal{F}, T)$.

In the next section we need the following corollary:

Corollary 1.1 *[13] Let $(\mathcal{E}, \mathcal{F}, T)$ be a complete ΛE- space, and let $f : \mathcal{E} \to \mathcal{E}$ be a contraction with ratio r. Suppose there exists $z \in \mathcal{E}$ and a real number γ such that*

$$\sup_{\lambda\in\Lambda} P(\{\omega \in \Omega | \ d^\lambda(z^\lambda(\omega), f(z^\lambda)(\omega)) \geq t\}) \leq \frac{\gamma}{t} \text{ for all } t > 0.$$

Then there exists a unique $x_0 \in \mathcal{E}$ such that $f(x_0) = x_0$.

2 Brownian motion

In [7] Hutchinson and Rüschendorf, starting from the idea of Graf [5], showed that the Brownian motion can be characterized as the fixed point of a modified scaling law.

Let $(\Omega, \mathcal{K}, P)$ be a probability space and let Λ be a nonempty set.

A **Brownian motion** is a stochastic process $B^\lambda = (B_t^\lambda)_{t\in\mathbb{R}}$ characterized by the following properties:

$$P(\{\omega \in \Omega | B^\lambda(0, \omega) = 0 \quad \text{and} \quad t \mapsto B^\lambda(t, \omega) \quad \text{is continuous}\}) = 1,$$

and for every $t > 0$ and every $h > 0$

$$B^\lambda(t + h) - B^\lambda(t) \stackrel{d}{=} N(0, \lambda h),$$

thus

$$P(\{\omega \in \Omega | B^\lambda(t + h, \omega) - B^\lambda(t, \omega) < x\}) = \frac{1}{\sqrt{2\pi h\lambda}} \int_{-\infty}^{x} e^{-\frac{t^2}{2\lambda^2 h^2}} dt.$$

$N(a, b)$ denote the normal distribution with mean a and variance b.

Remarks.

1. B_t^λ has a normal distribution with mean 0 and variance $t\lambda$.
2. For $h > 0$ the distribution $B^\lambda(t + h) - B^\lambda(t)$ is independent of t.

For each $\lambda > 0$, let $B^\lambda : [0, 1] \times \Omega \to \mathbb{R}$ denote the **constrained Brownian motion** given by

$$B^\lambda(0, \omega) = 0 \quad a.s., \quad \text{and} \quad B^\lambda(1, \omega) = 1 \quad a.s..$$

Using the properties of Brownian motions, we have $B^\lambda(\frac{1}{2}) \stackrel{d}{=} N(0, \frac{\lambda}{2})$.

Next we construct the scaling law $\mathbb{S}$ such that the constrained Brownian motion $\{B^\lambda, \lambda > 0\}$ satisfies the family of scaling laws.

Let $\lambda \in \Lambda$ and denote p^λ the random point with distribution $N(0, \frac{\lambda}{2})$. For $p^\lambda \in \mathbb{R}$ consider the constrained Brownian motion $B^\lambda \Big|_{B^\lambda(\frac{1}{2})=p^\lambda}$.

Denote $I = [0, 1]$, and define the functions

$$\Phi_1 : I \to [0, \frac{1}{2}], \quad \Phi_1(s) = \frac{s}{2},$$

and

$$\Phi_2 : I \to [\frac{1}{2}, 1], \quad \Phi_1(s) = \frac{s + 1}{2}.$$

Let $\varphi_1, \varphi_2 : \mathbb{R} \times \Lambda \to \mathbb{R}$ be the unique affine transformations characterized by $\varphi_1(0,\lambda) = 0$, $\varphi_1(1,\lambda) = \varphi_2(0,\lambda) = p^\lambda$, $\varphi_2(1,\lambda) = 1$ for all $\lambda \in \Lambda$. Denote $r_1^\lambda = Lip\varphi_1 = |p^\lambda|$, $\quad r_2^\lambda = Lip\varphi_2 = |1 - p^\lambda|$. For φ_1, φ_2 we obtain

$$\varphi_1(x,\lambda) = p^\lambda x \quad \text{and} \quad \varphi_2(x,\lambda) = (1 - p^\lambda)x + p^\lambda.$$

Denote $\mathbf{L}$ the set of functions from $\mathbb{R} \times \Lambda$ to $\mathbb{R}$

$$\mathbf{L} := \{f : \mathbb{R} \times \Lambda \to \mathbb{R}\}.$$

Let $\psi_1, \psi_2 : \mathbf{L} \to \mathbf{L}$ be mappings satisfying the following property:

$$\psi_i(f)(a,\lambda) = f(a, \frac{\lambda}{2r_i^2}) \quad i = 1, 2.$$

Let us define the transformations $S_1, S_2 : \mathbb{R} \to \mathbb{R}$ by

$$S_i(f(a,\lambda)) := \varphi_i(\psi_i(f)(a,\lambda))$$

for all $f \in \mathbf{L}$.

Using the definition of constrained Brownian motion, we have

$$B^\lambda|_{B^\lambda(\frac{1}{2})=p^\lambda}(t) \stackrel{d}{=} S_1 \circ B^\lambda(2t), \quad t \in [0, \frac{1}{2}].$$

Similarly

$$B^\lambda|_{B^\lambda(\frac{1}{2})=p^\lambda}(t) \stackrel{d}{=} S_2 \circ B^\lambda(2t-1), \quad t \in [\frac{1}{2}, 1].$$

This relations can be written as follows

$$B^\lambda|_{B^\lambda(\frac{1}{2})=p^\lambda}(t) \stackrel{d}{=} \sqcup_i S_i \circ B^\lambda \circ \Phi_i^{-1}(t), \quad t \in [0,1].$$

If $\mathbb{S}$ denote the random scaling law whose values are scaling laws $S := (S_1, S_2)$, then, for each $\lambda > 0$, we have

$$B^\lambda \stackrel{d}{=} \sqcup_i S_i \circ B^{\lambda(i)} \circ \Phi_i^{-1},$$

where $B^{\lambda(i)} \stackrel{d}{=} B^\lambda$ are chosen independently of one another.

3 Random fractal interpolation process

In this section, starting from the model of Brownian motion, we give the notion of random scaling law for random fractal interpolation process.

Let Λ be a nonempty set and let $t_0 < t_1 < ... < t_N$, $t_i \in \mathbb{R}$, $i \in \{0, ..., N\}$ be $N+1$ given points. Consider N bijections

$$\Phi_i : I = [t_0, t_N] \to [t_{i-1}, t_i] = I_i$$

for $i \in \{1, ..., N\}$. Define the stochastic process $y : I \times \Lambda \times \Omega \to \mathbb{R}$.

The collection $\Gamma := \{(t_i, y(t_i, \lambda, \omega)), i \in \{0, ..., N\}\}$ is called a set of **random interpolation points**.

A random fractal function $f : I \times \Lambda \times \Omega \to \mathbb{R}$ is said to have the interpolation property with respect to Γ if

$$f(t_i, \lambda, \omega) = y(t_i, \lambda, \omega) \quad a.s., \quad i \in \{0, ..., N\}.$$

Let $Y^\lambda := L_p(I)$ for $p > 0$ and $\gamma(\lambda) > 0$. For $u, v \in Y^\lambda$, define

$$d^\lambda(u,v) := \begin{cases} \gamma(\lambda) \left(\int_I |u(a) - v(a)|^p da\right)^{\frac{1}{p}}, & if \quad p \geq 1 \\ \gamma(\lambda) \left(\int_I |u(a) - v(a)| da\right), & if \quad p < 1 \\ \gamma(\lambda) ess \sup_a |u(a) - v(a)|, & if \quad p = \infty. \end{cases}$$

Let $\mathcal{E}$ be the set of random functions and let $e \in I \times \Lambda \times \Omega$ be fix. Denote

$$\begin{aligned} Z: &= \{z : I \times \Lambda \times \Omega \to \mathcal{E} | \forall \lambda \in \Lambda, \forall \omega \in \Omega : z(\cdot, \lambda, \omega) \in Y^\lambda, \\ & \forall a \in I, \lim_{t\to\infty} \inf_{\lambda \in \Lambda} P(\{\omega \in \Omega | d^\lambda(z(a,\lambda,\omega), e(a,\lambda,\omega)) < t\}) = 1\}. \end{aligned}$$

Define the functions $\varphi_i : \mathbb{R} \to \mathbb{R}$, $\varphi_i \in Lip^{(<1)}$ for $i \in \{1, ..., N\}$, r_i denote its Lipschitz constant. Let $\gamma_i : \mathbb{R} \to \mathbb{R}$ be real functions and denote

$$\mathbf{L}(\mathbb{R} \times \Lambda, \mathbb{R}) := \{f : \mathbb{R} \times \Lambda \to \mathbb{R}\}.$$

Consider the mappings $\psi_i : \mathbf{L}(\mathbb{R} \times \Lambda, \mathbb{R}) \to \mathbf{L}(\mathbb{R} \times \Lambda, \mathbb{R})$ such that

$$\psi_i(u(a, \lambda)) := u(a, \gamma_i(\lambda)),$$

and suppose for $\lambda \in \Lambda$ there exists $q_i(\lambda)$ with the next property

$$\int_I |\psi_i(u(a,\lambda)) - \psi_i(v(a,\lambda))| da \leq \sup \gamma_i(\lambda) \int_I |u(a,\lambda) - v(a,\lambda)| da.$$

Let $S_i : \mathbb{R} \to \mathbb{R}$ defined by $S_i := \varphi_i \circ \psi_i$ such that

$$S_i(u(a,\lambda)) := \varphi_i(\psi_i(u(a,\lambda))).$$

Let $\mathbb{S} = \{S_1, ..., S_N\}$ be the scaling law defined above.

If $f : I \times \Lambda \times \Omega \to \mathbb{R}$ is a stochastic process, then the random function $(\mathbb{S}f)$ is defined up to probability distribution by

$$(\mathbb{S}f) \stackrel{d}{=} \sqcup_i S_i \circ f^{(i)} \circ \Phi_i^{-1},$$

where $f^{(i)} \stackrel{d}{=} f$ for $i \in \{1, ..., N\}$ are chosen independently of one another.

The stochastic processes or random functions f **satisfies the scaling law** $\mathbb{S}$ or is a **selfsimilar random fractal function** if

$$(\mathbb{S}f) \stackrel{d}{=} f.$$

Using the contraction method in ΛE-space, we will prove an existence and uniqueness theorem for random fractal interpolation function.

Theorem 3.1 *Let $\mathbb{S}$ be the random scaling law defined above with the next interpolation property: for $u \in Z$, $\lambda \in \Lambda$ and $i \in \{1, ..., N-1\}$*

$$S_1(u(t_0, \lambda, \omega)) = t_0 \quad a.s. \tag{3.2}$$

$$S_{i+1}(u(t_0, \lambda, \omega)) = S_i(u(t_N, \lambda, \omega)) = y(t_i, \lambda, \omega) \quad a.s. \tag{3.3}$$

$$S_n(u(t_n, \lambda, \omega)) = t_n \quad a.s. \tag{3.4}$$

Suppose

$$ess \sup_\omega \sup_{\lambda \in \Lambda} \sum_{i=1}^N r_i(\lambda, \omega) q_i(\lambda) |I_i| < 1. \tag{3.5}$$

Then there exists a random fractal interpolation function $f^* \in Z$ which satisfies $\mathbb{S}$ and

$$f^*(t_i, \lambda, \omega) = y(t_i, \lambda, \omega) \quad a.s., \quad i \in \{0, ..., N\}$$

Proof.: For the random functions $f, g : I \times \Lambda \times \Omega \to \mathbb{R}$, $p > 0$ and $\gamma > 0$ let as define

$$F_{f,g}(t) := \inf_{\lambda} P(\{\omega \in \Omega | \gamma(\lambda) \left(\int_I |f(a,\lambda,\omega) - g(a,\lambda,\omega)|^p da \right)^{\frac{1}{p}} < t\}).$$

We define the scaling operator $\mathbb{S} : Z \to Z$ by

$$\mathbb{S}(f) := \sqcup_i S_i \circ f^{(i)} \circ \Phi_i^{-1}$$

more precisely

$$\mathbb{S}(f)(a,\lambda,\omega) = \varphi_i(f(\Phi_i^{-1}(a), \gamma_i(\lambda), \omega))$$

for $a \in I_i$.

Assuming this has been done, in order to show that $\mathbb{S}$ is a contraction map, for $p \geq 1$ we compute

$$\begin{aligned}
F_{\mathbb{S}f,\mathbb{S}g}(t) &= \inf_{\lambda \in \Lambda} P(\{\omega \in \Omega | d^\lambda(\mathbb{S}f, \mathbb{S}g) < t\}) = \\
&= \inf_{\lambda \in \Lambda} P(\{\omega \in \Omega | \gamma(\lambda) (\sum_{i=1}^N \int_{I_i} |\varphi_i(\psi_i(f(\Phi_i^{-1}(a), \lambda, \omega)) - \\
&- \varphi_i(\psi_i(g(\Phi_i^{-1}(a), \lambda, \omega))|^p da)^{\frac{1}{p}} < t\}) \geq \\
&\geq \inf_{\lambda \in \Lambda} P(\{\omega \in \Omega | \gamma(\lambda) (\sum_{i=1}^N r_i^p(\lambda,\omega)|I_i| \int_I |\psi_i(f(a,\lambda,\omega)) - \\
&- \psi_i(g(a,\lambda,\omega))|^p da)^{\frac{1}{p}} < t\}) \geq \\
&\geq \inf_{\lambda \in \Lambda} P(\{\omega \in \Omega | \left(\sum_{i=1}^N \frac{|I_i| r_i^p(\lambda,\omega)\gamma(\lambda)}{q_i^p(\lambda)} \right)^{\frac{1}{p}} \gamma(\lambda) \\
&\left(\int_I |f(a,\lambda,\omega) - g(a,\lambda,\omega)|^p da \right)^{\frac{1}{p}} < t\})
\end{aligned}$$

Taking

$$r := ess \sup_{\omega} \sup_{\lambda \in \Lambda} \sum_{i=1}^N r_i(\lambda,\omega) q_i(\lambda) |I_i| < 1$$

we have

$$F_{\mathbb{S}f,\mathbb{S}g}(t) \geq F_{f,g}(\frac{t}{r}).$$

For $0 < p < 1$ and $p = \infty$ one works similarly.

Since, for $f \in Z$, the function $f(a,\lambda,\cdot)$ is a real random variable for all $a \in I$ and $\lambda \in \Lambda$ the triplet $(Z, \mathcal{F}, T)$ is a ΛE-space, where $\mathcal{F} : Z \times Z \to \Delta^+$ is defined by

$$\mathcal{F}(f,g) := F_{f,g}$$

and $T := T_m(a,b) = \max\{a+b-1, 0\}$. Using Corollary 1.1 from for the contraction $\mathbb{S}$ there exists a selfsimilar random fractal function f^*.

Next we have to show the interpolation property of f^*. For $i \in \{1,...,N\}$ we have the following equalities

$$f^*(t_i,\lambda,\omega) = \mathbb{S}f^*(t_i,\lambda,\omega) = S_i(f^*(t_N,\lambda,\omega)) = y(t_i,\lambda,\omega).$$

■

Remark 1. In the case $N = 2$ the demonstration is published in [12].

Remark 2. If, for $f : I \times \Lambda \times \Omega \to \mathbb{R}$ we suppose

$$\sup_{\lambda\in\Lambda} \lambda^{-\frac{1}{2}} E_\omega \int_I |f(t, \lambda\omega)| dt < \infty$$

then $f \in Z$. Taking $\gamma(\lambda) := \frac{1}{\sqrt{\lambda}}$, $N = 2$ and $p = 1$ the Brownian motion can be characterized as a fixed point of a scaling law. In this case $r = \frac{1}{\sqrt{2}}$.

This result was proved by Hutchinson and Rüschendorf in [7] using d^* metric defined by

$$d^*(f, g) := \sup_{\lambda\in\Lambda} \lambda^{\frac{-1}{2}} \int_I |f(\lambda, t) - g(\lambda, t)| dt.$$

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