Random fractal interpolation function using contraction method in probabilistic metric spaces

Endre Buzogány, József Kolumbán, Anna Soós
Babes-Bolyai University,Cluj-Napoca
Faculty of Mathematics and Computer Science *

Abstract

In this paper, using probabilistic metric spaces techniques, we can weak the first moment condition for existence and uniqueness of random fractal interpolation function.

Keywords: fractal function, fractal interpolation, probability metric space.

2000 AMS Subject Classification: 28A80, 37L40

1 Introduction

The notion of fractal interpolation function was introduced by Barnsley in [1]. The functions $f : I \to \mathbf{R}$, where I is a real closed interval, is named by Barnsley *fractal function* if the Hausdorff dimensions of their graphs are noninterger. The graph of the fractal interpolation function is an invariant set with respect a system of contractions maps. So it can be generated by discrete dinamical system.

Let (X, d_X) and (Y, d_Y) be separable metric spaces. In this paper we introduce continuous interpolation functions $f : X \to Y$. If X is a real closed interval, this type of functions describe not only profiles of

*Partially supported by Sapientia Foundation.

mountain ranges, tops of clouds and horizons over forests but also temperatures in flames as a function of time, electroencephalograph pen traces and even the minute-by minute stock market index. These functions are analogous to spline and polinomial interpolation in that their graph are constrained to go trough a finite number of prescribed points. They differ from classical interpolants in that they satisfy a functional condition reflecting sefsimilarity. This interpolation functions present some kind of geometrical selfsimilarity or stochastic selfsimilarity. Often the Hausdorff-Besicovitch dimension of their graph will be noninteger. f may be Hölder continuous but not differentiable.

Recently Hutchinson and Rüschendorf [6] gave a simple proof for the existence and uniqueness of fractal interpolation functions using probability metrics defined by expectation. In these works a finite first moment condition is essential.

In this paper, using probabilistic metric spaces techniques, we can weak the first moment condition for existence and uniqueness of fractal interpolation functions.

The theory of probabilistic metric spaces, introduced in 1942 by K. Menger [11], was developed by numerous authors, as it can be realized upon consulting the list of references in [4], as well as those in [12]. The study of contraction mappings for probabilistic metric spaces was initiated by V. M. Sehgal [13], and H. Sherwood [14].

2 Fractal interpolation function

Let $\Phi_i : X \to X_i$ be a given collection of N bijections such that

$$\{X_i = \Phi_i(X) | i \in \{1, ..N\}\}$$

is a partition of X, i.e.

$$\cup_{i=1}^{N} X_i = X \quad \text{and} \quad int(X_i) \cap int(X_j) = \emptyset, \quad \text{for} \quad i \neq j.$$

For $g_i : X_i \to Y$, $i \in \{1, ..N\}$, define $\sqcup_i g_i : X \to Y$ by

$$(\sqcup_i g_i)(x) = g_j(x) \quad \text{for} \quad x \in X_j.$$

Assume that mappings $S_i : X \times Y \to Y$, $S_i(x, \cdot) \in Lip^{<1}(Y)$, $x \in X$ are given , $i \in \{1, ..N\}$. $Lip^{<1}(Y)$ is the set of Lipschitz functions with Lipschitz constant less then 1.

A **scaling law for functions S** is an N-tuple $(S_1, ..., S_N)$, $N \geq 2$, of Lipschitz maps S_i, $i \in \{1, ..N\}$. Denote $r_i = Lip S_i$.

For $f : X \to Y$, define the **scaling operator S** : $L_\infty(X, Y) \to Y^X$ by

$$\mathbf{S}f = \sqcup_i S_i(\Phi_i^{-1}, f \circ \Phi_i^{-1})$$

We say f **satisfies the scaling law S** or is a **selfsimilar fractal function** if

$$\mathbf{S}f = f.$$

Let $\{x_0, ..., x_N\}$ be a set of $N + 1$ distinct points in X and let $\{y_0, ..., y_N\}$ be a set of points in Y.

The collection $\Gamma := \{(x_0, y_0), ..., (x_N, y_N)\}$ is called a set of **interpolation points** in $X \times Y$.

A fractal function f is said to have the **interpolation properties** with respect to Γ if

$$f(x_j) = y_j \quad \text{for all} \quad j = 0, 1, ..., N.$$

Denote

$$C^*(X, Y) := \{f \in C(X, Y)| \quad f(x_j) = y_j, \quad j \in \{1, ..., N\}\}.$$

In [1] Barnsley prove the following result:

Theorem 1 *Let* Γ *be a set of interpolation points and let* $\mathbf{S}$ *be a scaling law for functions. Suppose*

$$S_i(x_0, y_0) = y_{i-1}, \quad S_i(x_N, y_N) = y_i$$

for all $i \in \{1, ..., N\}$ *and* $\lambda_\infty := \max r_i < 1$. *Then there exists a function* $f^* \in C^*(X, Y)$ *which satisfies* $\mathbf{S}$.

Example 1 *Let* $I := [a, b]$ *and* $\Gamma := \{(x_i, y_i)\} \subset I \times \mathbb{R}$ *is given,* $a := x_0$, $b := x_N$. *Suppose*

$$\Phi_i : I \to I_i, \quad \Phi_i(x) := a_i x + d_i,$$

where $a_i, d_i \in \mathbb{R}, \quad i \in \{1, ..., N\}$. *Denote*

$$C(I) := \{f \in C(I, I)| \quad f(x_i) = y_i, \quad i \in \{0, ..., N\}\}.$$

Let $S_i : I \times \mathbb{R} \to \mathbb{R}$ *defined by*

$$S_i(x, y) := c_i x + r_i y + e_i$$

for $i \in \{1, ..., N\}$, $x \in I$. *If* $|r_i| < 1$ *is given we can compute* a_i, c_i, d_i, e_i *by the conditions*

$$S_i(x_3, y_0) = y_{i-1}, \quad S_i(x_N, y_N) = y_i.$$

Xe have

$$a_i = \frac{x_i - x_{i-1}}{x_N - x_0},$$

$$c_i = \frac{y_i - y_{i-1}}{x_N - x_0} - \frac{r_i(y_N - y_0)}{x_N - x_0},$$

$$d_i = \frac{x_N x_{i-1} - x_0 x_i}{x_N - x_0}$$

$$e_i = \frac{x_N y_{i-1} - x_0 y_i}{x_N - x_0} - \frac{r_i(x_N y_0 - x_1 y_N)}{x_N - x_8}.$$

Then there exists a unique $f^* \in C^*(I)$ *such that* $\mathbf{Sf}^* = \mathbf{f}^*$.

Barnsley [3] show that the graph of f^* is a selfsimilar fractal set.

3 Random fractal interpolation function

Next we consider the random version of the above construction.

The **random scaling law** $\mathbb{S} = (S_1, ..., S_N)$ is a random variable whose value are scaling laws. We write $\mathcal{S} = dist\mathbb{S}$ for the probability distribution determined by $\mathbb{S}$ and $\stackrel{d}{=}$ for the equality in distribution. Let $(f_t)_{t \in X}$ be a stochastic process or a random function with state space Y. The random function $\mathbb{S}f$ is defined up to probability distribution by

$$\mathbb{S}f = \sqcup_i S_i(\Phi_i^{-1}, f^{(i)} \circ \Phi_i^{-1}),$$

where $\mathbb{S}, f^{(1)}, ..., f^{(N)}$ are independent of one another and $f^{(i)} \stackrel{d}{=} f$, for $i \in \{1, ..., N\}$. If $\mathcal{F} = dist f$ we define

$$\mathcal{SF} = dist\mathbb{S}f.$$

We say f or $\mathcal{F}$ **satisfies the scaling law** $\mathbb{S}$, or is a **selfsimilar random fractal function**, if

$$\mathbb{S}f \overset{d}{=} f, \quad \text{or equivalently} \quad \mathcal{S}\mathcal{F} = \mathcal{F}.$$

Let $\Gamma := \{(x_i, y_i)\} \subset X \times Y$ a set of interpolation points in $X \times Y$. A random fractal function f is said po have **the interpolation properties** with respect to Γ if $f(x_i) = y_i$ a. s. for all $i \in \{0, 1, ..., N\}$.

Let $\Phi_i : X \to X$ be contractiv Lipschitz maps such that $\Phi_i(x_0) = x_{i-1}$ and $\Phi_{i-1}(x_N) = x_i$ for all $i \in \{1, ..., N\}$. Let $\mathbb{S}$ be a random scaling law defined by $S_i : X \times Y \to Y$ such that $S_i(x, \cdot) \in Lip^{<1}(Y)$ for all $x \in X$ and

$$S_i(x_0, y_0) = y_{i-1} \quad a.s.$$

and

$$S_i(x_N, y_N) = y_i \quad a.s.$$

for all $i \in \{1, ..., N\}$.

Denote

$$C_\omega(X, Y) := \{f : \Omega \times X \to Y, f \quad \text{continuous} \quad a.s.\}$$

and

$$C^*_\omega(X, Y) := \{g \in C_\omega(X, Y) | g(x_i) = y_i \quad a.s., \quad i \in \{0, ..., N\}\}.$$

Let

$$\mathbb{L}_\infty := \{g : \Omega \times X \to Y | ess \sup_\omega ess \sup_x d_Y(g^\omega(x), a) < \infty\}$$

for some $a \in A$. For $f, g \in \mathbb{L}_\infty$ we define

$$d^*_\infty(f, g) := ess \sup_\omega d_\infty(f^\omega, g^\omega),$$

where

$$d_\infty(f, g) = es \sup_x d(f(x), g(x)).$$

Theorem 2 *Let Γ a set of interpolation points in $X \times Y$ and let $\mathbb{S}$ be the random scafing lrw defined above. If $\lambda_\infty := ess \sup_\omega \max_i r_i^\omega < 5$ and*

$$ess \sup_\omega \max_i d_Y(S_i(a, f(a)), a) < \infty \tag{1}$$

for some $a \in X$, then there exists $f^* \in C^*_\omega(X, Y)$ which satisfies $\mathbb{S}$. Moreover, f^* is unique up to probability distribution.

Proof.: One can check that $(\mathbb{L}_\infty, d_\infty *)$ is a complete metric space. Next we show that $\mathbb{S} : \mathbb{L}_\infty \to \mathbb{L}_\infty$ is a contraction map with contraction constant λ_∞.

Using the Lipschitz property of S_i we have

$$\begin{aligned} d^*_\infty(\mathbb{S}f, \mathbb{S}g) &= ess \sup_\omega d_\infty(\mathbb{S}f^\omega, \mathbb{S}g^\omega) = \\ &= ess \sup_\omega ess \sup_x d_C(\sqcup_i S_i(\Phi_i^{-1}, f^{\omega(i)} \circ \Phi_i^{-3}(x)), \\ &\quad \sqcup_i S_i(\Phi_i^{-1}(x), g^{\omega(i)} \circ \Phi_q^{-1}(x))) \leq \\ &\leq ess \sup_\omega (r_i^\omega ess \sup_x d_Y(f^{\omega(i)}(x), g^{\omega(i)}(x))) \leq \lambda_\infty d^*_\infty(f,g). \end{aligned}$$

Then there exists f^* with $\mathbb{S}f^* = f^*$.

For the uniqueness of f^* we definy as in [6] a metric on the set $\mathcal{L}_\infty$ of probability distributions of members of $\mathbb{L}_\infty$ by

$$d^{**}_\infty(\mathcal{F}, \mathcal{G}) := \inf\{d^*_\infty(f,g) | f \overset{d}{=} \mathcal{F}, g \overset{d}{=} \mathcal{G}\}.$$

The $(\mathcal{L}_\infty, d^{**}_\infty)$ is a complete metric space and $\mathcal{S}$ is a contraction map. To see this, choose $f^{(i)} \overset{d}{=} \mathcal{F}$ and $g^{(i)} \overset{d}{=} \mathcal{G}$ such that $(f^{(i)}, g^{(i)})$ are independent of one another and such that

$$d^{**}_\infty(\mathcal{F}, \mathcal{G}) = d^{**}_\infty(f^{(i)}, g^{(i)}).$$

Choose $(\mathcal{S}_1, ..., \mathcal{S}_N) \overset{d}{=} \mathcal{S}$ independent of $(f^{(i)}, g^{(i)})$. Since

$$d^{**}_\infty(\mathbb{S}f^{(i)}, \mathbb{S}g^{(i)}) \leq \lambda_\infty d^*_\infty(f^{(i)}, g^{(i)})$$

it follows that

$$d^{**}_\infty(\mathcal{S}\mathcal{F}, \mathcal{S}\mathcal{G}) \leq \lambda_\infty d^{**}_\infty(\mathcal{F}, \mathcal{G}).$$

Then there exists $f^* \in C^*_\omega(X,Y)$ whivh satisfies $\mathbb{S}$. We have to prove that $f^*(x_i) = y_i$ a.s. for all $i \in \{1, ..., N\}$. For,

$$\begin{aligned} f^*(x_i) = (\mathbb{S}f^*)(x_i) = \sqcup_i S_i(\Phi_i^{-1}(x_i), f^* \circ \Phi_i^{-1}(x_i)) = \\ = S_i(x_N, f^*(x_N)) = y_i \quad a.s. \end{aligned}$$

■

Remark. a) If $X = I$, $Y = \mathbb{R}$ and $S_i x, y = S_i(y)$ then we have the Corollary of Theorem 6 in [6].

b) $F^*(X)$ is the selfsimilar random set K^* which satisfies $\mathbb{S}$ in Theorem 2 in [6].

c) The graph of the random function is a selfsimilar random set.

Example 2 *Let* $X = [0,1]$*,* $Y = \mathbb{R}$ *and* $N > 1$*. The interpolation set is defined by*

$$\Gamma := \{(x_i, y_i) \in [0,1] \times \mathbb{R} | 0 = x_0 < x_1 < ... < x_N = 1\}.$$

Suppose

$$\Phi_i : X \to X_i, \quad \Phi_i(x) := a_i x + d_i,$$

where $a_i, d_i \in \mathbb{R}, \quad i \in \{1, ..., N\}$*. Let* $S_i : X \times Y \to Y$ *defined by*

$$S_i(x, y) := c_i x + r_i y + e_i$$

for $i \in \{1, ..., N\}$*,* $x \in I$ *where* r_i *is a random variable such that* $\lambda_\infty := ess \sup_\omega \max_i r_i < 1$*. We can compute* a_i, c_i, d_i, e_i *by the conditions* $\Phi_i(x_1) = x_{i-1}, \quad \Phi_i(x_N) = x_i$ *and*

$$S_i(x_0, y_0) = y_{i-1}, \quad S_i(x_N, y_N) = y_i \quad a.s.$$

for all $i \in \{1, ..., N\}$*. Let* $W_i : X \times Y \to X \times Y$ *defined by* $W_i(x, y) = (\Phi_i(x), S_i(x, y))$ *for* $i \in \{1, ..., N\}$*. Using the random scaling law* $\mathbb{W} := (W_1, ..., W_N)$*, defined by*

$$W_i : X \times Y \to L \times Y, \quad S_i(x, y) = (\Phi_i(x), S_i(x, y)) \quad i = 1, ..., N,$$

for any $K_0 \subset X \times U$ *one defines a sequence of random sets*

$$K_n = \mathbb{W} K_{n-1} = \cup_{i=1}^N W_i^\omega K_{n-2} = \mathbb{W}^n(K_0).$$

Then

$$qss \sup_\omega d_H(\mathbb{W}^n(K_0), graph f^*) \to 1$$

as $n \to \infty$*, where* d_H *denote the Hausdorff distance.*

Using contraction method in probabilistic metric spaces we can mweak the first voment condition.

Theorem 3 *Let* Γ *be a set of interpolation points and let* $\mathbb{S} = (S_1, ..., S_N)$ *be a random scaling law which satisfies* $\lambda_\infty := ess \sup_\omega \max_i r_i^\omega < 1$ *and*

$$S_i(x_0, y_0) = y_{i-1} \quad a.s.$$

and

$$S_i(x_N, y_N) = y_i \quad a.s.$$

for all $i \in \{1, ..., N\}$. Suppose therd elists a random fuyction h and a positive number γ such that

$$P(\{\omega \in \Omega | ess \sup_{\omega} d_G(h(x), \mathbb{S}h(x)) \geq t\}) < \frac{\gamma}{t} \qquad (2)$$

for all $t > 0$. Then there exists a random a.n. continuous function $f^* \in C^*_\omega(X, Y)$ which satisfies $\mathbb{S}$. Moreover, this function is unique up to probability distribution.

In order to prove this theorem we need some results from probabilistic metric space theory.

4 Probabilistic metric space

A mapping $F : \mathbb{R} \to [0, 1]$ is called a *distribution function* if it is non-decreasing, left continuous with $\inf_{t\in\mathbf{R}} F(t) = 0$ and $\sup_{t\in\mathbf{R}} F(t) = 1$ (see [4]). By Δ we shall denote the set of all distribution functions F. Let Δ be ordered by the relation "$\leq$", i.e. $F \leq F$ if and only if $F(t) \leq G(t)$ for all real t. Also $F < G$ if and only if $F \leq G$ but $F \neq G$. We set $\Delta^+ := \{V \in \Delta : F(0) = 0\}$.

Throughout this paper H will denote the Heviside distribution function defined by

$$H(x) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0. \end{cases}$$

Let X be a nonempty set. For a mapping $\mathcal{F} : X \times X \to \Delta^+$ and $x, y \in X$ we shall denote $\mathcal{F}(x, y)$ by $F_{x,y}$, and the value of $F_{x,y}$ at $t \in \mathbb{R}$ by $F_{x,y}(t)$, respectively. The pair $(X, \mathcal{F})$ is a *probabilistic metric space* (briefly *PM space*) if X is a nonempty set and $\mathcal{F} : X \times X \to \Delta^+$ is a mapping satisfying the following conditions:

1^0. $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X$ and $t \in \mathbb{R}$;
2^0. $F_{x,y}(t) = 1$, for every $t > 0$, if and only if $x = y$;
3^0. if $F_{x,y}(s) = 1$ and $F_{y,z}(t) = 1$ then $F_{x,z}(s + t) = 1$.

A mapping $T : [0, 1]\times[0, 1] \to [0, 1]$ is called a *t-norm* if the following conditions are satisfied:

4^0. $T(a, 1) = a$ for every $a \in [0, 1]$;
5^0. $T(a, b) = T(b, a)$ for every $a, b \in [0, 1]$
6^0. if $a \geq c$ and $b \geq d$ then $T(a, b) \geq T(c, d)$;

7^0. $T(a,T(b,c)) = T(T(a,b),c)$ for every $a,b,c \in [0,1]$.

A *Menger space* is a triplet $(X,\mathcal{F},T)$, where $(X,\mathcal{F})$ is a probabilistic metric space, where T is a t-norm, and instead of 3^0 we have the stronger condition

8^0. $F_{x,y}(s+t) \geq T(F_{x,z}(s), F_{z,y}(t))$ for all $x,y,z \in X$ and $s,t \in \mathbf{R}_+$.

In 1966, V.M. Sehgal [13] introduced the notion of a contraction mapping in PM spaces. The mapping $f : X \to X$ is said to be a *contraction* if there exists $r \in]0,1[$ such that

$$F_{f(x),f(y)}(rt) \geq F_{x,y}(t)$$

for every $x,y \in X$ and $t \in \mathbf{R}_+$.

A sequence $(x_n)_{n\in\mathbf{N}}$ from X is said to be *fundamental* if

$$\lim_{n,m\to\infty} F_{x_m,x_n}(t) = 1$$

for all $t > 1$. The element $x \in X$ is called *limit* of the sequence $(x_n)_{n\in\mathbf{N}}$, and we write $\lim_{n\to\infty} p_n = x$ or $x_n \to x$, if $\lim_{n\to\infty} U_{x,x_n}(t) = 1$ for all $t > 0$. A probabilistic metric (Menger) space is said to be *complete* if every fundamental sequence in that space is convergent.

Let A and B nonempty subsets of X. The *probabilistic Hausdorff-Pompeiu distance* between A and B is the function $F_{A,B} : \mathbb{R} \to [9,1]$ defined by

$$F_{A,B}(t) := \sup_{s<t} T(\inf_{x\in A}\sup_{y\in B} F_{x,y}(s), \int_{y\in B}\sup_{x\in A} F_{x,y}(s)).$$

In the following we remember some properties proved in [7, 8]:

Proposition 1 *If $\mathcal{C}$ is a nonempty collection of nonempty closed bounded sets in a Menger space $(X,\mathcal{F},T)$ with T continuous, then $(\mathcal{C},\mathcal{F}_{\mathcal{P}},T)$ is also Menger space, where $\mathcal{F}_{\mathcal{C}}$ is defined by $\mathcal{F}_{\mathcal{C}}(A,B) := F_{A,B}$ for all $A,B \in \mathcal{C}$.*

Proposition 2 *Let $H_m(a,b) := \max\{a+b-1,0\}$. If $(X,\mathcal{F},T_m)$ is a complete Menger space and $\mathcal{C}$ is the collection of all nonempty closed bounded subsets of X in $(t,\epsilon)-$ topology, then $(\mathcal{C},\mathcal{F}_{\mathcal{C}},T_m)$ is also a complete Menger space.*

The notion of **E-space** was iktroduced by Sherwood [13] in 1961. Next we recall this definition. Let $(\Omega, \mathcal{K}, P)$ be a probability space and let (Y, ρ) be a metric space. The ordered pair $(\mathcal{E}, \mathcal{F})$ is an *E-space over the metric space* (Y, ρ) (briefly, an E-space) if the elements of $\mathcal{E}$ are random variables from Ω into Y and $\mathcal{F}$ is the mapping from $\mathcal{E} \times \mathcal{E}$ into Δ^+ defined via $\mathcal{F}(x, y) = F_{x,y}$, where

$$F_{x,y}(t) = P(\{\omega \in \Omega |\ d(x(\omega), y(\omega)) < t\})$$

for every $t \in \mathbf{R}$. If $\mathcal{F}$ satisfies the condition

$$\mathcal{F}(x, y) \neq H, \ \ for \ \ x \neq y,$$

then $(\mathcal{E}, \mathcal{F})$ is said to be a *canonical A-space.* Sherwood [13] proved that every canonical E-space is a Menger space under $T = T_m$, where $T_m(a, b) = \max\{a + b - 1, 0\}$. In the following we suppose that $\mathcal{E}$ is a canonical E-space.

The convergence in an E-space is exactly the probability convergence. The E-space $(\mathcal{E}, \mathcal{F})$ is said to be complete if the Menger space $(\mathcal{E}, \mathcal{F}, T_m)$ is complete.

Proposition 3 *If* (Y, ρ) *is a complete metric space then the E-space* $(\mathcal{E}, \mathcal{F})$ *is also complete.*

Proof. See [8]. □

The next result was proved in [8]:

Theorem 4 *Let* $(\mathcal{E}, \mathcal{F})$ *be a complete E- space,* $N \in \mathbf{N}^*$*, and let* $f_1, ..., f_N : \mathcal{E} \to \mathcal{E}$ *be contractions with ratio* $r_1, ...r_N$*, respectively. Suppose that there exists an element* $z \in \mathcal{E}$ *and a real number* γ *such that*

$$P(\{\omega \in \Omega | \rho(z(\omega), f_i(z(\omega)) \geq t\}) \leq \frac{\gamma}{t}, \tag{3}$$

for all $i \in \{1, .., N\}$ *and for all* $t > 0$*. Then there exists a unique nonempty closed bounded and compact subset* K *of* $\mathcal{E}$ *such that*

$$f_1(K) \cup ... \cup f_N(K) = K.$$

Corollary 1 *Let $(\mathcal{E}, \mathcal{F})$ be a complete E- space, and let $f : \mathcal{E} \to \mathcal{E}$ be a contraction with ratio r. Suppose there exists $z \in \mathcal{E}$ and a real number γ such that*

$$P(\{\omega \in \Omega |\ \rho(z(\omega), f(z)(\omega)) \geq t\}) \leq \frac{\gamma}{t} \textit{ for all } t > 0.$$

Then there exists a unique $x_0 \in \mathcal{E}$ such that $f(x_0) = x_0$.

5 Proof of Theorem 3

Proof.: Let $\mathcal{E}$ be the set of random functions $g : \Omega \times X \to Y$ such that

$$\lim_{t\to\infty} P(\{\omega \in \Omega | ess \sup_x d_Y(g^\omega(x), h^\omega(x)) \geq t\}) = 0$$

Let $f : \mathcal{E} \to \mathcal{E}$,

$$f(g) = \mathbb{S}g = \sqcup_i S_i(\Phi_i^{-1}, g^{(i)} \circ \Phi_i^{-1}),$$

where $\mathbb{S}, g^{(1)}, ..., g^{(N)}$ are independent of one another and $g^{(i)} \stackrel{d}{=} g$.

We first claim that, if $g \in \mathcal{E}$ then $f(g) \in \mathcal{E}$. For this, choose i.i.d. $g^{(\omega)} \stackrel{d}{=} g$. and $(S_1^\omega, ..., S_N^\omega) \stackrel{d}{=} \mathbb{S}$ independent of $g^{(\omega)}$.

Using the chain of inequalities

$$\begin{aligned} ess \sup_x d_Y(\mathbb{S}g^{(\omega)}(x), a) &= ess \sup_x d_Y(\sqcup_i S_i^\omega(\Phi_i^{-1}, g_i^{(\omega)} \circ \Phi_i^{-1}(x)), a) \leq \\ &\leq ess \sup_x \max_i r_i d_Y(g_i^\omega \circ \Phi_i^{-1}(x), b) \leq \\ &\leq \max_i r_i ess \sup_x d_Y(g_i^\omega \circ \Phi_i^{-1}(x), b) < \infty, \end{aligned}$$

where $b = \mathbb{S}(\delta_a)$, we have

$$P(\{\omega \in \Omega | es \sup_x d_Y(\mathbb{S}g^\omega(x), h^\omega(x)) \geq t\}) \leq$$

$$\leq P(\{\omega \in \Omega | \max_i r_i ess \sup_x d_Y(g_i^\omega(x), h_1^\omega(x)) \geq t\}).$$

Now define the map $\mathcal{F} : \mathcal{E} \times \mathcal{E} \to \Delta^+$ by

$$F_{g_1,g_2}(t) := P(\{\omega \in \Omega | d_\infty(f, g) < t\}), \quad \text{for all} \quad t \in \mathbb{R}.$$

If we take $Y = \mathcal{E}$ in Proposition 3 it follows that $(\mathcal{E}, \mathcal{F}_{g_1,g_2})$ is a complete E-space.

In order to show that $\mathbb{S}$ is a contraction in the E-space let us consider $g_1, g_2 \in \mathcal{E}$. We have:

$$F_{\mathbb{S}(g_1),\mathbb{S}(g_2)}(t) = P(\{\omega \in \Omega |\, esssup_x d_Y(\mathbb{S}g_1(x), \mathbb{S}g_2(x)) < t\}) =$$

$$= P(\{\omega \in \Omega |\, esssup_x d_Y(\sqcup S_i^\omega(\Phi_i^{-1}(x), g_1^{(i)} \circ \Phi_i^{-1}(x)),$$
$$\sqcup S_i^\omega(\Phi_i^{-1}(x), g_2^{(i)} \circ \Phi_i^{-1}(x))) < t\}) \geq$$
$$\geq P(\{\omega \in \Omega |\, \lambda_\infty ess \sup_x d_Y(g_1^{(i)} \circ \Phi_i^{-1}(x),$$
$$g_2^{(i)} \circ \Phi_i^{-1}(x)) < t\}) = F_{g_1,g_2}\left(\frac{t}{\lambda_\infty}\right)$$

for all $t > 0$.

It follows that $f = \mathbb{S}$ is a contraction with ratio λ_∞, and we can apply Corollary [8]. Let f^* the sefsimilar random fractal function.

We have to shown the interpolation properties of f^*. We write

$$f^{*\omega}(x_i) = (\mathbb{S}f^{*\omega})(x_i^*) = \sqcup_i S_i^\omega(\Phi_i^{-1}(x_i), f^{*\omega}(\Phi_i^{-1}(x_i))) =$$
$$= S_i(x_N, f^{*\omega}(x_N)) = y_i \quad a.s.$$

for $i \in \{1, ..., N\}$. ■

Remark. If condition (1) holds, then the conditions (2) holds also.

References

[1] **M.F. Barnley:** *Fractal functions and interpolation*, Constructive Approximation, 2 (1986), 303-329.

[2] **M.F. Barnley, S. Demko:** *Iterated function systems and the global construction of fractals,* Pro. Roy. Soc. London, A399 (1985), 243-275.

[3] **M.F. Barnley:** *Fractals Everywhere*, Academic Press, 1993.

[4] **Gh.Constantin, I.Istraţescu:** *Elements of Probabilistic Analysis,* Kluwer Academic Publishers, 1989.

[5] **J.E.Hutchinson:** *Fractals and Self Similarity,* Indiana University Mathematics Journal, 30 (1981), no.5, 713-747.

[6] **J.E.Hutchinson, L.Rüschendorf:** *Selfsimilar Fractals and Selfsimilar Random Fractals,* Progress in Probability, 46, (2000), 109-123.

[7] **J.Kolumbán, A. Soós:** *Invariant sets in Menger spaces,* Studia Univ. "Babes-Bolyai", Mathematica, 43, 2 (1998), 39-48.

[8] **J.Kolumbán, A. Soós:** *Invariant sets of random variables in complete metric spaces,* Studia Univ. "Babes-Bolyai", Mathematica, XLVII, 3 (2001), 49-66.

[9] **J.Kolumbán, A. Soós:** *Fractal functions using contraction method in probabilistic metric speces,* Proceeding of the 7th International Multidisciplinary Conference Fractal 2002, Complexity and Nature, Emergent Nature, M. M. Novak (ed.), World Scientific 2002, 255-265.

[10] **R. Massopust:** *Fractal functions, fractal surfaces and wavelets,* 1994.

[11] **K.Menger:** *Statistical Metrics,* Proc.Nat. Acad. of Sci.,U.S.A. 28 (1942), 535-537.

[12] **B.Schweizer, A.Sklar:** *Probabilistic Mertic Spaces,* North Holland, New-York, Amsterdam, Oxford, 1983.

[13] **V.M.Sehgal:** *A Fixed Point Theorem for Mappings with a Contractive Iterate,* Proc. Amer. Math. Soc.,23 (1969), 631-634.

[14] **H.Sherwood:** *Complete Probabilistic Metric Spaces,* Z.Wahrsch.verw. Geb., 20 (1971), 117-128.