Fixed point theorem in ΛE-spaces

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The theory of metric spaces is a very useful tool in applied mathematics. However, by some practical problems this theory can not be applied. For this reason the concept of probabilistic metric space was introduced in 1942 by Menger [7]. It was developed by numerous authors, as it can be realized upon consulting the list of references in [2], as well as those in [10]. Menger proposed to replace the distance $d(x, y)$ by a distribution function $F_{x,y}$ whose value $F_{x,y}(t)$, for any real number t, is interpreted as the probability that the distance between x and y is less than t. The study of contraction mappings for probabilistic metric spaces was initiated by Sehgal [12],[13], Sherwood [16] and Bharucha-Reid [14]. Radu in [8] and [9] introduced other types of contractions in probabilistic metric spaces. The notion of E-space was introduced by Sherwood [16] in 1969 as a generalization of Menger space for random variables. For new results and applications of probabilistic analysis one can consult Constantin and Istrăţescu's book [2]. New results in fixed point theory in probabilistic metric spaces can be find in [4] and in Hadzic's book [3].

Hutchinson and Rüschendorf [5] showed that the Brownian motion can be characterized as a fixed point of a special stochastic process. They proved a fixed point theorem using a first moment condition. Our goal is to generalize this idea and to replace the first moment condition by a more less restrictive hypothesis. Using a generalization of the notion of E-space to the so called ΛE-space we will prove a new fixed point theorem. As application Brownian bridge-type stochastic fractal interpolation functions will be constructed.

In the first section we recall the notions of probabilistic metric space and E-space. The next section contains the definition and some properties of ΛE-space. The main result of this paper is the fixed point theorem in section 3. The last section contain an application of our main theorem to the stochastic fractal interpolation.

1 Probabilistic metric spaces

Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$.

A mapping $F : \mathbb{R} \to [0,1]$ is called a **distribution function** if it is non-decreasing and left continuous.

By Δ we shall denote the set of all distribution functions F. Let Δ be ordered by the relation "$\leq$", i.e. $F \leq G$ if and only if $F(t) \leq G(t)$ for all real t. Also $F < G$ if and only if $F \leq G$ but $F \neq G$.
We set $\Delta^+ := \{F \in \Delta : F(0) = 0\}$.

Let H denote the Heviside distribution function defined by

$$H(x) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0. \end{cases} \tag{1.1}$$

Let X be a nonempty set. For a mapping $\mathcal{F} : X \times X \to \Delta^+$ and $x, y \in X$ we shall denote $\mathcal{F}(x,y)$ by $F_{x,y}$, and the value of $F_{x,y}$ at $t \in \mathbb{R}$ by $F_{x,y}(t)$, respectively.

The pair $(X, \mathcal{F})$ is a **probabilistic metric space** (briefly **PM space**) if X is a nonempty set and $\mathcal{F} : X \times X \to \Delta^+$ is a mapping satisfying the following conditions:

1^0 $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X$ and $t \in \mathbb{R}$;

2^0 $F_{x,y}(t) = 1$, for every $t > 0$, if and only if $x = y$;

3^0 if $F_{x,y}(s) = 1$ and $F_{y,z}(t) = 1$ then $F_{x,z}(s+t) = 1$.

A mapping $T : [0,1] \times [0,1] \to [0,1]$ is called a **t-norm** if the following conditions are satisfied:

4^0 $T(a,1) = a$ for every $a \in [0,1]$;

5^0 $T(a,b) = T(b,a)$ for every $a, b \in [0,1]$;

6^0 if $a \geq c$ and $b \geq d$ then $T(a,b) \geq T(c,d)$;

7^0 $T(a, T(b,c)) = T(T(a,b), c)$ for every $a, b, c \in [0,1]$.

We list here the simplest:

$T_1(a,b) = max\{a+b-1, 0\}$,

$T_2(a,b) = ab$,

$T_3(a,b) = Min(a,b) = min\{a,b\}$,

A **Menger space** is a triplet $(X, \mathcal{F}, T)$, where $(X, \mathcal{F})$ is a probabilistic metric space, T is a t-norm, and instead of 3^0 we have the stronger condition:

8^0 $F_{x,y}(s+t) \geq T(F_{x,z}(s), F_{z,y}(t))$ for all $x, y, z \in X$ and $s, t \in \mathbb{R}_+$.

If the t-norm T satisfies the condition

$$sup\{T(a,a) : a \in [0,1[\} = 1,$$

then the (t, ϵ) -topology is metrizable (see [11]).

In 1966, V.M. Sehgal [13] introduced the notion of a contraction mapping in probabilistic metric spaces.

The mapping $f : X \to X$ is said to be a **contraction** if there exists $r \in]0, 1[$ such that

$$F_{f(x),f(y)}(rt) \geq F_{x,y}(t)$$

for every $x, y \in X$ and $t \in \mathbb{R}_+$.

A sequence $(x_n)_{n\in\mathbb{N}}$ from X is said to be **fundamental** if

$$\lim_{n,m\to\infty} F_{x_m,x_n}(t) = 1$$

for all $t > 0$.

The element $x \in X$ is called **limit** of the sequence $(x_n)_{n\in\mathbb{N}}$, and we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$, if $\lim_{n\to\infty} F_{x,x_n}(t) = 1$ for all $t > 0$.

A probabilistic metric (Menger) space is said to be **complete** if every fundamental sequence in that space is convergent.

For example, if (X, d) is a metric space, then the metric d induces a mapping $\mathcal{F} : X \times X \to \Delta^+$, where $\mathcal{F}(x, y) = F_{x,y}$ is defined by

$$F_{x,y}(t) = H(t - d(x, y)),\ t \in \mathbb{R}.$$

Moreover $(X, \mathcal{F}, Min)$ is a Menger space. Bharucha-Reid and Sehgal show that $(X, \mathcal{F}, Min)$ is complete if the metric d is complete (see [14]). The space $(X, \mathcal{F}, Min)$ thus obtained is called the **induced Menger space**.

The notion of E-space was introduced by Sherwood [16] in 1969. Next we recall this definition and we show that if (X, d) is a complete metric space then the E-space is also complete.

Let $(\Omega, \mathcal{K}, P)$ be a probability space and let (Y, ρ) be a metric space.

The ordered pair $(\mathcal{E}, \mathcal{F})$ is an **E-space over the metric space** (Y, ρ) (briefly, an E-space) if the elements of $\mathcal{E}$ are random variables from Ω into Y and $\mathcal{F}$ is the mapping from $\mathcal{E} \times \mathcal{E}$ into Δ^+ defined via $\mathcal{F}(x, y) = F_{x,y}$, where

$$F_{x,y}(t) = P(\{\omega \in \Omega |\ \rho(x(\omega), y(\omega)) < t\})$$

for every $t \in \mathbb{R}$.

If $\mathcal{F}$ satisfies the condition

$$\mathcal{F}(x, y) \neq H, \ \text{ if } \ x \neq y,$$

then $(\mathcal{E}, \mathcal{F})$ is said to be a **canonical E-space.** Sherwood [16] proved that every canonical E-space is a Menger space under $T = T_m$, where $T_m(a, b) = \max\{a+b-1, 0\}$. In the following we suppose that $\mathcal{E}$ is a canonical E-space.

The convergence in an E-space is exactly the probability convergence.

The E-space $(\mathcal{E}, \mathcal{F})$ is said to be **complete** if the Menger space $(\mathcal{E}, \mathcal{F}, T_m)$ is complete.

If we start with a complete metric space (X, d) then we obtain a complete E-space.

Proposition 1.1 *([6]) If (X, d) is a complete metric space then the E-space $(\mathcal{E}, \mathcal{F})$ is also complete.*

2 ΛE-spaces

Let Λ be a nonempty set and, for $\lambda \in \Lambda$, let (Y^λ, d^λ) be metric space. Denote $\mathcal{E}^\lambda$ the set of random variables from Ω into Y^λ and let

$$\mathcal{F}^\lambda : \mathcal{E}^\lambda \times \mathcal{E}^\lambda \to \Delta^+$$

be defined via $\mathcal{F}^\lambda(x, y) := F^\lambda_{x,y}$, where

$$F^\lambda_{x,y}(t) := P(\{\omega \in \Omega | d^\lambda(x^\lambda(\omega), y^\lambda(\omega)) < t\})$$

for all $t \in \mathbb{R}$. Denote

$$F_{x,y}(t) := \inf_{\lambda \in \Lambda} F^\lambda_{x,y}(t)$$

and

$$\mathcal{F}(x, y) := F_{x,y}.$$

The ordered pair $(\mathcal{E}^\lambda, \mathcal{F}^\lambda)$ is an E-space over the metric space Y^λ.

Let $Y := \prod_{\lambda \in \Lambda} Y^\lambda$, $e \in Y$ and define

$$\mathcal{E} := \{x \in \prod_{\lambda \in \Lambda} \mathcal{E}^\lambda | \quad \lim_{t \to \infty} \inf_{\lambda \in \Lambda} P(\{\omega \in \Omega | d^\lambda(x^\lambda(\omega), e^\lambda(\omega)) < t\}) = 1\}.$$

Remark. $\mathcal{E}$ is the set of bounded random functions. The convergence in $\mathcal{E}$ is similar to the uniform convergence in metric space.

The triplet $(\mathcal{E}, \mathcal{F}, T)$ is called ΛE-space.

In the following let $T := T_m$.

Proposition 2.1 *$(\mathcal{E}, \mathcal{F}, T)$ is a Menger space.*

Proof.: Conditions 1^o and 2^o are satisfied by definition. Since $F^\lambda_{x,y}$ satisfies 8^o for all $\lambda \in \Lambda$, we can write

$$\begin{aligned} F^\lambda_{x,y}(t+s) &\geq T(F^\lambda_{x,z}(t), F^\lambda_{z,y}(s)) \geq \\ &\geq \inf_\lambda \max(F^\lambda_{x,z}(t) + F^\lambda_{z,y}(s) - 1, 0) \geq \\ &\geq \max(\inf_\lambda F^\lambda_{x,z}(t) + \inf_\lambda F^\lambda_{z,y}(s) - 1, 0) = \\ &= T(F_{x,z}(t), F_{z,y}(s)) \end{aligned}$$

for all $t, s \in \mathbb{R}_+$. Taking the infimum over λ we obtain the triangle inequality:

$$F_{x,y}(t+s) = \inf_{\lambda \in \Lambda} F^{\lambda}_{x,y}(t+s) \geq T(F_{x,z}(t), F_{z,y}(s))$$

for all $t, s \in \mathbb{R}_+$. ■

Proposition 2.2 *If $(Y^{\lambda}, d^{\lambda})$ are complete metric spaces for all $\lambda \in \Lambda$, then $(\mathcal{E}, \mathcal{F}, T)$ is a complete Menger space.*

Proof.: Let $(x_n)_{n \in \mathbb{N}}$ be a Cauchy sequence of elements of $\mathcal{E}$, i.e.

$$\lim_{n,m\to\infty} F_{x_n,x_m}(t) = \lim_{n,m\to\infty} \inf_{\lambda\in\Lambda} P(\{\omega \in \Omega | d^{\lambda}(x^{\lambda}_n(\omega), x^{\lambda}_m(\omega)) < t\}) = 1 \tag{2.2}$$

for all $t > 0$ and

$$\lim_{t\to\infty} \inf_{\lambda\in\Lambda} P(\{\omega \in \Omega | d^{\lambda}(x^{\lambda}_n(\omega), e^{\lambda}(\omega)) < t\}) = 1. \tag{2.3}$$

Since, for $\lambda \in \Lambda$

$$P(\{\omega \in \Omega | d^{\lambda}(x^{\lambda}_n(\omega), x^{\lambda}_m(\omega)) < t\}) \geq F_{x_n,x_m}(t),$$

it follows that for $\epsilon > 0$, exists $n_\epsilon \in \mathbb{N}$ such that, if $n > n_\epsilon$ and $m > n_\epsilon$ then

$$P(\{\omega \in \Omega | d^{\lambda}(x^{\lambda}_n(\omega), x^{\lambda}_m(\omega)) < t\}) > 1 - \epsilon.$$

So, $(x^{\lambda}_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the E-space $(\mathcal{E}^{\lambda}, \mathcal{F}^{\lambda})$. According to Proposition 1.1 $(\mathcal{E}^{\lambda}, \mathcal{F}^{\lambda})$ is complete for all $\lambda \in \Lambda$. Denote $x^{\lambda} := \lim_{n\to\infty} x^{\lambda}_n$, and $x := (x^{\lambda} | \lambda \in \Lambda)$.

Now we have to show that

(i) $\lim_{n\to\infty} F_{x_n,x}(t) = 1$ for all $t > 0$,

and

(ii) $x \in \mathcal{E}$.

By the relation (2.2) for all $t > 0$ and $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that for $n, m > n_\epsilon$ and $\lambda \in \Lambda$

$$P(\{\omega \in \Omega | d^{\lambda}(x^{\lambda}_n(\omega), x^{\lambda}_m(\omega)) < \frac{t}{2}\}) > 1 - \frac{\epsilon}{2}.$$

Since

$$P(\{\omega \in \Omega | d^{\lambda}(x^{\lambda}_n(\omega), x^{\lambda}(\omega)) < t\}) \geq$$

$$\geq P(\{\omega \in \Omega | d^{\lambda}(x^{\lambda}_n(\omega), x^{\lambda}_m(\omega)) < \frac{t}{2}\}) + P(\{\omega \in \Omega | d^{\lambda}(x^{\lambda}_m(\omega), x^{\lambda}(\omega)) < \frac{t}{2}\}) - 1 > 1 - \epsilon$$

we have

$$\inf_{\lambda\in\Lambda} F^{\lambda}_{x_n,x}(t) > 1-\epsilon$$

for all $t>0$ and $\epsilon>0$. So

$$\lim_{n\to\infty} F_{x_n,x}(t) = 1, \quad \text{for all} \quad t>0.$$

In order to show (ii) we use relation (2.3). For $\epsilon>0$ there exists $t_\epsilon>0$ such that for all $t\geq t_\epsilon$ the following inequalities hold

$$F_{x,e}(2t) \geq T(F_{x_n,x}(t), F_{x_n,e}(t)) \geq T(F_{x_n,x}(1), F_{x_n,e}(t)) >$$

$$> 1-\frac{\epsilon}{2} + F_{x_n,e}(t) - 1 > 1-\epsilon.$$

■

3 The main result

The main result of this paper is the following fixed point theorem:

Theorem 3.1 *Let $(\mathcal{E},\mathcal{F},T)$ be a complete ΛE- space, and let $f:\mathcal{E}\to\mathcal{E}$ be a contraction with ratio r. Suppose there exists $z\in\mathcal{E}$ and a real number γ such that*

$$\sup_{\lambda\in\Lambda} P(\{\omega\in\Omega|\ d^\lambda(z^\lambda(\omega), f(z^\lambda)(\omega)) \geq t\}) \leq \frac{\gamma}{t} \textit{ for all } t>0.$$

Then there exists a unique $x_0\in\mathcal{E}$ such that $f(x_0)=x_0$.

Proof.: Let $a_0=z$ and $a_n=f(a_{n-1})$ for $n\geq 1$.
First we show that $(a_n)_{n\in\mathbb{N}}$ is a fundamental sequence in $(\mathcal{E},\mathcal{F},T)$.
Let $f_n = f\circ\cdots\circ f$ n-times.
Since $a_{n+k}=f_n(a_k)$ and $a_n=f_n(a_0)$, we have

$$F_{a_n,a_{n+k}}(s) = F_{f_n(z),f_n(a_k)}(s) \geq \inf_{\lambda\in\Lambda} P(\{\omega\in\Omega|\ r^n d^\lambda(z^\lambda(\omega), a^\lambda_k(\omega)) < s\}) =$$

$$\geq \inf_{\lambda\in\Lambda} P(\{\omega\in\Omega|\ r^n d^\lambda(z^\lambda(\omega), a^\lambda_k(\omega)) < s\cdot(1+\sqrt{r}+\cdots+\sqrt{r}^{k-1})(1-\sqrt{r})\}) \geq$$

$$\geq P(\{\omega\in\Omega|\ r^n[d^\lambda(z^\lambda(\omega), f(z^\lambda(\omega))) + d^\lambda(f(z^\lambda(\omega)), f_2(z^\lambda(\omega))) + \cdots +$$

$$+ d^\lambda(f_{k-1}(z^\lambda(\omega)), f_k(z^\lambda(\omega)))] < s\cdot(1+\sqrt{r}+\cdots+\sqrt{r}^{k-1})(1-\sqrt{r})\}) \geq$$

$$\geq \inf_{\lambda\in\Lambda}[P(\{\omega\in\Omega|\ d^\lambda(z^\lambda(\omega), f(z^\lambda(\omega))) < \frac{s(1-\sqrt{r})}{r^n}\}) +$$

$$+P(\{\omega\in\Omega|\ d^\lambda(f(z^\lambda(\omega)),f_2(z^\lambda(\omega)))<\frac{s(1-\sqrt{r})\sqrt{r}}{r^n}\})+\cdots+$$

$$+P(\{\omega\in\Omega|\ d^\lambda(f_{k-1}(z^\lambda(\omega)),f_k(z^\lambda(\omega)))<\frac{s(1-\sqrt{r})\sqrt{r}^{k-1}}{r^n}\})]-(k-1)\geq$$

$$\geq\inf_{\lambda\in\Lambda}[P(\{\omega\in\Omega|\ d^\lambda(z^\lambda(\omega),f(z^\lambda(\omega)))<\frac{s(1-\sqrt{r})}{r^n}\})+$$

$$+P(\{\omega\in\Omega|\ rd^\lambda(z^\lambda(\omega)),f(z^\lambda(\omega)))<\frac{s(1-\sqrt{r})\sqrt{r}}{r^n}\})+\cdots+$$

$$+P(\{\omega\in\Omega|\ r^{k-1}d^\lambda(z^\lambda(\omega),f(z^\lambda(\omega)))<\frac{s(1-\sqrt{r})\sqrt{r}^{k-1}}{r^n}\})]-(k-1)=$$

$$=1-\sup_{\lambda\in\Lambda}[P(\{\omega\in\Omega|\ d^\lambda(z^\lambda(\omega),f(z^\lambda(\omega)))\geq\frac{s(1-\sqrt{r})}{r^n}\})+$$

$$+P(\{\omega\in\Omega|\ d^\lambda(z^\lambda(\omega),f(z^\lambda(\omega)))\geq\frac{s(1-\sqrt{r})\sqrt{r}}{r^{n+1}}\})+\cdots+$$

$$+P(\{\omega\in\Omega|\ d^\lambda(z^\lambda(\omega),f(z^\lambda(\omega)))<\frac{s(1-\sqrt{r})\sqrt{r}^{k-1}}{r^{n+k-1}}\})]\geq$$

$$\geq 1-\gamma\cdot r^n\left(\frac{1}{s(1-\sqrt{r})}+\frac{r^{1/2}}{s(1-\sqrt{r})}+...+\frac{r^{(k-1)/2}}{s(1-\sqrt{r})}\right)>$$

$$>1-\gamma\frac{r^n}{s(1-\sqrt{r})^2}.$$

Since

$$\lim_{n\to\infty}\left(1-\gamma\frac{r^n}{s(1-\sqrt{r})^2}\right)=1,$$

we have, for $t>0$,

$$\lim_{n\to\infty}F_{a_n,a_{n+k}}(t)=1,$$

uniformly with respect to k. The space $(\mathcal{E},\mathcal{F},T)$ being complete, (a_n) is convergent. Let x_0 be its limit.

Next we show that x_0 is a fixed point of f.

For we have

$$F_{a_n,f(x_0)}(\frac{t}{2})\geq F_{a_{n-1},x_0}(\frac{t}{2})\ \text{for all}\ t>0.$$

Using 8^0 it follows

$$F_{x_0,f_(x_0)}(t)\geq T(F_{x_0,a_n}(\frac{t}{2}),F_{a_n,f(x_0)}(\frac{t}{2}))\geq T(F_{x_0,a_n}(\frac{t}{2}),F_{a_{n-1},x_0}(\frac{t}{2})).$$

Since $\lim_{n\to\infty} a_n = x_0$, we have

$$F_{x_0,f(x_0)}(t) = 1 \;\; for\ all \;\; t > 0,$$

therefore

$$f(x_0) = x_0.$$

For the uniqueness we suppose that there exists an other element $x' \in \mathcal{E}$ such that $f(x') = x'$. For $n \in \mathbb{N}$ and $t > 0$, we have

$$F_{x_0,x'}(t) = F_{f^n(x_0),f^n(x')}(t) \geq F_{x_0,x'}\left(\frac{t}{r^n}\right).$$

Since $\lim_{n\to\infty} r^n = 0$, we have

$$F_{x_0,x'}(t) = 1 \;\text{ for all } t > 0,$$

therefore $x_0 = x'$.

■

4 Application: stochastic fractal interpolation

In [5] Hutchinson and Rüschendorf showed that the Brownian bridge can be characterized as the fixed point of a "scaling" function. Indeed, let $(\Omega, \mathcal{K}, P)$ be a probability space and let $\Lambda = \mathbb{R}_+$, the set of positive real numbers. Define the Brownian bridge as the stochastic process $(X_t^\lambda)_{t\in\mathbb{R}_+}$ with the following properties:

$$P(\{\omega \in \Omega | \quad t \mapsto X^\lambda(t,\omega) \quad \text{is continuous}\}) = 1,$$

and, for every $t \geq 0$ and every $h > 0$,

$$X^\lambda(t+h) - X^\lambda(t) \overset{d}{=} N(0, \lambda h),$$

thus

$$P(\{\omega \in \Omega | X^\lambda(t+h,\omega) - X^\lambda(t,\omega) < x\}) = \frac{1}{\sqrt{2\pi} h\lambda} \int_{-\infty}^{x} e^{-\frac{t^2}{2\lambda^2 h^2}} dt.$$

$N(a, b)$ denote the normal distribution with mean a and variance b.

We suppose

$$X^\lambda(0,\omega) = 0 \quad a.s. \quad \text{and} \quad X^\lambda(1,\omega) = 1 \quad a.s..$$

Denote $I = [0, 1]$, and define the functions

$$\Phi_1 : I \to [0, \frac{1}{2}], \quad \Phi_1(s) = \frac{s}{2},$$

and

$$\Phi_2 : I \to [\frac{1}{2}, 1], \quad \Phi_1(s) = \frac{s+1}{2}.$$

Let $\lambda \in \Lambda$ and denote p^λ the random point with distribution $N(0, \frac{\lambda}{2})$.

Let $\varphi_1^\lambda, \varphi_2^\lambda : \mathbb{R} \times \Lambda \to \mathbb{R}$ be the affine transformations characterized by $\varphi_1^\lambda(0, \lambda) = 0$, $\varphi_1^\lambda(1, \lambda) = \varphi_2^\lambda(0, \lambda) = p^\lambda$, $\varphi_2^\lambda(1, \lambda) = 1$ for all $\lambda \in \Lambda$. Denote $r_1^\lambda = Lip\varphi_1^\lambda = |p^\lambda|, \quad r_2^\lambda = Lip\varphi_2^\lambda = |1 - p^\lambda|$. For φ_1^λ, φ_2^λ we obtain

$$\varphi_1^\lambda(a, \lambda) = p^\lambda a \quad \text{and} \quad \varphi_2^\lambda(a, \lambda) = (1 - p^\lambda)a + p^\lambda.$$

Denote $\mathbf{L}$ the set of functions from $\mathbb{R} \times \Lambda$ to $\mathbb{R}$,

$$\mathbf{L} := \{u : \mathbb{R} \times \Lambda \to \mathbb{R}\}.$$

Let $\psi_1, \psi_2 : \mathbf{L} \to \mathbf{L}$ be mappings satisfying the following property:

$$\psi_i(u)(a, \lambda) = u(a, \frac{\lambda}{2r_i^2}), \qquad i = 1, 2.$$

Let

$$S_i^\lambda = \varphi_i^\lambda \circ \psi_i.$$

Using the definition of the process, we have

$$X^\lambda|_{X^\lambda(\frac{1}{2})=p^\lambda}(t) \stackrel{d}{=} S_1^\lambda \circ X^\lambda(2t), \quad t \in [0, \frac{1}{2}].$$

Similarly

$$X^\lambda|_{X^\lambda(\frac{1}{2})=p^\lambda}(t) \stackrel{d}{=} S_2^\lambda \circ X^\lambda(2t - 1), \quad t \in [\frac{1}{2}, 1].$$

This relations can be written as follows

$$X^\lambda|_{X^\lambda(\frac{1}{2})=p^\lambda}(t) \stackrel{d}{=} \sqcup_i S_i^\lambda \circ X^\lambda \circ \Phi_i^{-1}(t), \quad t \in [0, 1].$$

For each $\lambda > 0$, we have

$$X^\lambda \stackrel{d}{=} \sqcup_i S_i^\lambda \circ X^{\lambda(i)} \circ \Phi_i^{-1},$$

where $X^{\lambda(i)} \stackrel{d}{=} X^\lambda$ are chosen independently of one another.

Let $Y^\lambda = L_1([0,1])$ and d^λ the Euclidean metric in $\mathbb{R}$, for all $\lambda \in \Lambda$. In this case $\mathcal{E}^\lambda$ is the space of real random variables and $\mathcal{E}$ is their product space. By Theorem 2.2 $(\mathcal{E}, \mathcal{F}, T)$ is a complete ΛE- space. Consider the function $f : \mathcal{E} \to \mathcal{E}$, defined by $f := (f^\lambda | \lambda \in \Lambda)$ where

$$f^\lambda(X) := \sqcup_i S_i^\lambda \circ X^{\lambda(i)} \circ \Phi_i^{-1}$$

for all $X \in \mathcal{E}$. If X_0 is a fixed point of f then, for all $\lambda \in \Lambda$,

$$X_0^\lambda \overset{d}{=} f^\lambda(X_0^\lambda).$$

Hutchinson and Rüschendorf [5] proved that, if the set of all functions $Z \in \mathcal{E}$ such that

$$\sup_{\lambda \in \Lambda} \lambda^{-\frac{1}{2}} E_\omega \int_I |Z(t,\lambda,\omega)| dt < \infty$$

there exists a fixed point of f. Motivated by this result, we consider the following problem.

Let Λ be a nonempty set and let $0 = t_0 < t_1 < \ldots < t_N = 1$, $t_i \in \mathbb{R}$, $i \in \{0, \ldots, N\}$ be $N+1$ given points. Consider N bijections

$$\Phi_i : I \to [t_{i-1}, t_i] = I_i$$

for $i \in \{1, \ldots, N\}$, with Lipschitz constant α_i.

Let $Y^\lambda := L_1(I)$ and let $\beta(\lambda) > 0$ for all $\lambda \in \Lambda$. For $u, v \in Y^\lambda$, define

$$d^\lambda(u,v) := \beta(\lambda) \left(\int_I |u(a) - v(a)| da \right).$$

Let $\mathcal{E}$ be defined as in previous section with $e = 0$.

For all $\lambda \in \Lambda$ and $i \in \{1, \ldots, N\}$ define the random function $\varphi_i^\lambda : \mathbb{R} \to \mathbb{R}$, $\varphi_i^\lambda \in Lip^{(<1)}$, and r_i^λ denote its Lipschitz constant. Let $\gamma_i : \Lambda \to \mathbb{R}$ be real functions. Consider the mappings $\psi_i : \mathbf{L} \to \mathbf{L}$ such that

$$\psi_i(u)(a,\lambda) := u(a, \gamma_i(\lambda)),$$

and S_i^λ be defined as above, i.e. $S_i^\lambda := \varphi_i^\lambda \circ \psi_i$. Suppose for $\lambda \in \Lambda$ there exists $\delta(\lambda) > 0$ such that the following Lipschitz condition will be satisfied:

$$\begin{aligned} &\inf_\lambda P(\{\omega \in \Omega | \delta(\lambda) \int_I |u(a,\gamma_i(\lambda),\omega) - v(a,\gamma_i(\lambda),\omega)| da < s\}) \geq \\ &\geq \inf_\lambda P(\{\omega \in \Omega | \beta(\lambda) \int_I |u(a,\lambda,\omega) - v(a,\lambda,\omega)| da < s\}) \end{aligned}$$

for all $u, v \in \mathcal{E}$.

Let p_i^λ be given random variable ($i \in \{0, ..., N\}$). Suppose the next interpolation properties are fulfilled:

for $u \in \mathcal{E}$, $\lambda \in \Lambda$ and $i \in \{1, ..., N-1\}$

$$\varphi_1^\lambda(u(0,\lambda,\omega)) = p_0^\lambda(\omega) \quad a.s. \tag{4.4}$$

$$\varphi_{i+1}^\lambda(u(0,\lambda,\omega)) = \varphi_i^\lambda(u(1,\lambda,\omega)) = p_i^\lambda(\omega) \quad a.s. \tag{4.5}$$

$$\varphi_N^\lambda(u(1,\lambda,\omega)) = p_N^\lambda(\omega) \quad a.s. \tag{4.6}$$

If $x \in \mathcal{E}$ then the random function $f(x)$ is defined by

$$f^\lambda(x) = \sqcup_i S_i^\lambda \circ x \circ \Phi_i^{-1}, \tag{4.7}$$

Theorem 4.1 *Suppose*

$$ess \sup_\omega \sup_{\lambda\in\Lambda} \sum_{i=1}^N \frac{r_i^\lambda(\omega)\alpha_i\beta(\lambda)}{\delta(\lambda)} < 1 \tag{4.8}$$

and there exists a real number γ such that

$$\sup_{\lambda\in\Lambda} P(\{\omega \in \Omega| \sum \alpha_i|\varphi_i^\lambda(0)| \geq t\}) \leq \frac{\gamma}{t} \, for \, all \, t > 0. \tag{4.9}$$

Then there exists a random fractal interpolation function $x^* \in \mathcal{E}$ such that

$$f(x^*) = x^*$$

and

$$x^*(t_i, \lambda, \omega) = p_i^\lambda(\omega) \quad a.s., \quad i \in \{0, ..., N\}, \, \lambda \in \Lambda. \tag{4.10}$$

Proof.: For the random functions $x, z : I \times \Lambda \times \Omega \to \mathbb{R}$, $i \in \{1, ..., n\}$ let as define

$$F_{x,z}(t) := \inf_\lambda P(\{\omega \in \Omega|\beta(\lambda) \left(\int_I |x(a,\lambda,\omega) - z(a,\lambda,\omega)|da\right) < t\}).$$

Assuming this has been done, in order to show that f is a contraction map we compute

$$F_{f(x),f(z)}(t) = \inf_{\lambda\in\Lambda} P(\{\omega \in \Omega | d^{\lambda}(f(x), f(z)) < t\}) =$$

$$= \inf_{\lambda\in\Lambda} P(\{\omega \in \Omega|\beta(\lambda)(\sum_{i=1}^{N} \int_{I_i} |\varphi_i^{\lambda}(\psi_i(x(\Phi_i^{-1}(a), \lambda, \omega)) - \varphi_i^{\lambda}(\psi_i(z(\Phi_i^{-1}(a), \lambda, \omega))|da) < t\}) \geq$$

$$\geq \inf_{\lambda\in\Lambda} P(\{\omega \in \Omega|\beta(\lambda)(\sum_{i=1}^{N} r_i^{\lambda}(\omega)\alpha_i \int_{I} |\psi_i(x(a, \lambda, \omega)) - \psi_i(z(a, \lambda, \omega))|da) < t\}) \geq$$

$$\geq \inf_{\lambda\in\Lambda} P(\{\omega \in \Omega| \left(\sum_{i=1}^{N} \frac{\alpha_i r_i^{\lambda}(\omega)\beta(\lambda)}{\delta(\lambda)}\right) \delta(\lambda) \left(\int_{I} |\psi_i(x(a, \lambda, \omega)) - \psi_i(z(a, \lambda, \omega))|da\right) < t\}) \geq$$

$$\geq \inf_{\lambda\in\Lambda} P(\{\omega \in \Omega|r \left(\int_{I} \beta(\lambda)|x(a, \lambda, \omega) - z(a, \lambda, \omega)|da\right) < t\}).$$

So we have

$$F_{f(x),f(z)}(t) \geq F_{x,z}(\frac{t}{r}).$$

Using Theorem 3.1 for the contraction f there exists a fractal interpolation function x^*.

Next we have to show the interpolation property of x^*. For $i \in \{1, ..., N\}$ we have the following equalities

$$x^*(t_i, \lambda, \omega) = f(x^*(t_i, \lambda, \omega)) = S_i^{\lambda}(x^*(t_N, \lambda, \omega)) = p_i^{\lambda}(\omega).$$

■

This fractal interpolation function x^* can be considered a generalized Brownian motion.

Remark: If

$$\sup_{\lambda\in\Lambda} \beta(\lambda) E_{\omega} \sum \alpha_i |\varphi_i^{\lambda}(0)| < \infty$$

then, by Tchebysev inequality, (4.9) is fulfilled.

References

[1] **A.T.Bharucha-Reid:***Fixed point theorems in probabilistic analysis*, Bull. Amer. Math. Soc. 82(1976), 641-657.

[2] **Gh.Constantin, I.Istraţescu:** *Elements of Probabilistic Analysis,* Kluwer Academic Publishers, 1989.

[3] **O.Hadžić, E.Pap:** *Fixed point theory in probabilistic metric spaces*, Kluwer Acad. Publ., 2001.

[4] **O.Hadzić, R.Meciar, E.Pap**: *A Fixed Point Theorem in Probabilistic Metric Spaces and Application,* Journal of Mathematical Analysis and Application, **202** (1996), 433-449.

[5] **J.E.Hutchinson, L.Rüschendorf:** *Selfsimilar Fractals and Selfsimilar Random Fractals,* Progress in Probability, 46, (2000), 109-123.

[6] **J.Kolumbán, A.Soós:** *Invariant sets of random variables in complete metric spaces,* Studia Univ. Babes-Bolyai, Mathematica, XLVII, 3 (2001), 49-66.

[7] **K.Menger:** *Statistical Metrics,* Proc.Nat. Acad. of Sci.,U.S.A. 28 (1942), 535-537.

[8] **V.Radu:** *On some contraction type mappings in Menger spaces,* Sem. Teor. Prob. Apl. Univ. Timisoara 78, 1985.

[9] **V.Radu:** *On some fixed point theorem in PM-spaces,* Lectures Notes in Mathematics, Springer Verlag, 1233(1987), 125-133.

[10] **B.Schweizer, A.Sklar:** *Probabilistic Metric Spaces,* North Holland, New-York, Amsterdam, Oxford, 1983.

[11] **B.Schweizer, A.Sklar, E.Thorp:** *The Metrization of Statistical Metric Spaces,* Pacific Journal of Mathematics, 10 (1960), no. 2, 673-675.

[12] **V.M.Sehgal:** *Some Fixed Point Theorems in Functional Analysis and Probability*, Ph.D.Thesis, Wayne State University, 1966.

[13] **V.M.Sehgal:** *A Fixed Point Theorem for Mappings with a Contractive Iterate,* Proc. Amer. Math. Soc.,23 (1969), 631-634.

[14] **V.M.Sehgal, A.T.Bharucha-Reid:** *Fixed Points of Contraction Mappings on Probabilistic Metric Spaces,* Math. Systems Theory, 6 (1972), 92-102.

[15] **H.Sherwood:** *Complete Probabilistic Metric Spaces,* Z.Wahrsch.verw. Geb., 20 (1971), 117-128.

[16] **H.Sherwood:** *E-spaces and their relation to other classes of probabilistic metric spaces,* J.London Math. Soc., 44 (1969), 441-448.