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28 September 1998

Physics Letters A 246 (1998) 505–510

PHYSICS LETTERS A

Response in kinetic Ising model to oscillating magnetic fields

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Received 5 May 1998; revised manuscript received 1 July 1998; accepted for publication 1 July 1998
Communicated by C.R. Doering

Abstract

Ising models obeying Glauber dynamics in a temporally oscillating magnetic field are analyzed. In the context of stochastic resonance, the response in the magnetization is calculated by means of both a mean-field theory with linear-response approximation, and the time-dependent Ginzburg–Landau equation. Analytic results for the temperature and frequency dependent response, including the resonance temperature, compare favorably with simulation data. © 1998 Elsevier Science B.V.

PACS: 64.60.Ht; 05.40.+j; 05.50.+q

1. Introduction

The Ising model with Glauber dynamics in an oscillating magnetic field was recently considered with Monte Carlo (MC) simulations in Refs. [1,2]. The phenomenon of stochastic resonance (see, e.g., Ref. [3]) was explored by viewing the Ising model as a system of coupled two-state oscillators, driven by the oscillating field and "noises" which are taken to be thermal fluctuations. The phenomenon was revealed by a characteristic peak in the correlation function $C(T)$ between the magnetic field and the magnetization $M(t)$ versus the temperature T of the system. The resonance temperature T_r (the temperature at which $C(T)$ has a maximum) was systematically computed as a function of the driving period, lattice size and driving amplitude, both for two-dimensional (2D) [1] and three-dimensional (3D) [2] systems. The one-dimensional (1D) case was analyzed by Brey and Prados [4] within linear response theory.

The present work is a natural continuation of those studies, considering analytically the 2D and 3D cases. We will present two approaches. The mean-field theory with linear response approximation will be discussed first. Then in 2D where the mean-field theory is not as good as in other dimensions, a more refined time-dependent Ginzburg–Landau (TDGL) approach will be presented, with significant improvements.

Recently, kinetic Ising systems in oscillating external fields have also been examined both experimentally and theoretically in Ref. [5]. The focus was on properties below the zero-field critical point, such as the frequency dependence of the probability distributions for the hysteresis-loop area and the residence time. The latter quantity for small systems in moderately weak fields suggests further evidences of stochastic resonance. Very recently, finite-size effects versus driving frequency have been analyzed as a dy-

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PII S0375-9601(98)00525-8

namical critical phenomena [6]. In contrast to these works, ours is focused on the temperature dependence above the zero-field critical point.

Stochastic resonance is conventionally studied by means of the signal-to-noise ratio (see, e.g., Ref. [3]). For small magnetic field, this quantity has been obtained for the Ising model from the power spectrum of the magnetization, exactly in 1D [7] and in higher dimensions by simulations and mean-field approaches [8]. The general result is that this ratio exhibits a peak at a definite temperature above T_c, weakly dependent on the driving frequency.

2. Mean-field theory and linear-response approximation

Our starting point is the master equation for the kinetic Ising model obeying Glauber dynamics [9],

$$P(\sigma;t+1)-P(\sigma;t)=\sum_{\sigma'}[w(\sigma'\to\sigma)\,P(\sigma';t) - w(\sigma\to\sigma')\,P(\sigma;t)], \quad (1)$$

where $P(\sigma;t)$ is the joint probability of finding the spin configuration σ at time t, and the w are the transition rates between two configurations which differ by one spin flip. For the heat-bath algorithm, the rate function is chosen as

$$w(\sigma\to\sigma')=\frac{1}{1+e^{-\beta[E(\sigma)-E(\sigma')]}},$$

with $\beta=1/T$ (hereafter the Boltzmann constant $k\equiv 1$), and $E(\sigma)$ is the energy of σ in a magnetic field h,

$$E(\sigma)=-J\sum_{\mathrm{nn}}S_iS_j-h(t)\sum_i S_i, \quad (2)$$

where $h(t)=A\sin(\omega t)$ and $\sum_{\mathrm{nn}}$ denotes a summation over nearest neighbors in a square or cubic lattice.

Let us denote the configuration σ by the values of the spins $S_1,S_2,\ldots,S_V$, with system volume given by $V=N^d$. d is the spatial dimension of the system and N is its linear size. Since $S_i=\pm1$, it is easy to rewrite (1) as

$$\frac{d}{dt}P(S_1,S_2,\ldots,S_V;t) = -\sum_{j=1}^{V} w_j(S_j)\,P(S_1,S_2,\ldots,S_V;t) + \sum_{j=1}^{V} w_j(-S_j)\,P(S_1,S_2,\ldots,-S_j,\ldots,S_V;t) \quad (3)$$

with

$$w_j(S_j)=\tfrac{1}{2}[1-S_j\tanh(E_j/T)], \qquad E_j=J\sum_{k=1}^{z}S_k+h, \quad (4)$$

where the last sum runs over the z nearest neighbors of the spin S_j, with $z=2d$. Multiplying both sides of (3) by S_l and performing an ensemble average (denoted by $\langle\ \rangle$), after some simple mathematical tricks, we get the basic equation for the Glauber dynamics,

$$\frac{d}{dt}\langle S_l\rangle=-\langle S_l\rangle+\langle\tanh(E_l/T)\rangle. \quad (5)$$

Invoking the mean-field approximation, we replace E_l by $Jz\langle S\rangle+h$ to get

$$\frac{d}{dt}\langle S\rangle=-\langle S\rangle+\tanh[(h+T_c^{\mathrm{MF}}\langle S\rangle)/T], \quad (6)$$

where $T_c^{\mathrm{MF}}=Jz$ is the mean-field critical temperature. In the absence of h, the magnetization is given by the stationary solution of the well-known equation,

$$\langle S\rangle_0=\tanh[T_c^{\mathrm{MF}}\langle S\rangle_0/T]. \quad (7)$$

For small $h(t)$, we may use the linear-response theory in (6) by first writing $\langle S\rangle(t)=\langle S\rangle_0+\Delta S(t)$ and considering the $h/T\ll1$ and $\Delta S/T\ll1$ limits. Performing the Taylor expansion and keeping only the first-order terms, Eq. (6) becomes

$$\frac{d}{dt}\Delta S=-\frac{\Delta S}{\tau_{\mathrm{MF}}}+\frac{A}{T}(1-\langle S\rangle_0^2)\sin(\omega t), \quad (8)$$

where

$$\tau_{\mathrm{MF}}=\frac{1}{1-(T_c^{\mathrm{MF}}/T)(1-\langle S\rangle_0^2)} \quad (9)$$

is the relaxation time. The solution can be found easily,

$$\Delta S(t)=\Delta S_0\sin(\omega t-\theta_{\mathrm{MF}}), \quad (10)$$

with the phase shift and amplitude given by

$$\theta_{MF} = \arctan(\omega\tau_{MF}), \tag{11}$$

$$\Delta S_0 = \frac{A}{T}(1 - \langle S\rangle_0^2)\frac{1}{\sqrt{1/\tau_{MF}^2 + \omega^2}}. \tag{12}$$

The correlation function between the total magnetization $M = V\langle S\rangle$ and the external field $h(t)$ can be computed,

$$C = \overline{M(t)\,h(t)} \equiv \frac{V\omega}{2\pi}\int_0^{2\pi/\omega} \Delta S(t)\,h(t)\,\mathrm{d}t$$
$$= \frac{VA^2}{2T}(1 - \langle S\rangle_0^2)\frac{\tau_{MF}}{1 + \omega^2\tau_{MF}^2}. \tag{13}$$

Here the overline denotes a temporal average over a period $P = 2\pi/\omega$. In the $T > T_c^{MF}$ domain, $\langle S\rangle_0 = 0$, thus C becomes

$$C_{T>T_c^{MF}} = \tfrac{1}{2}VA^2\frac{T - T_c^{MF}}{(T - T_c^{MF})^2 + \omega^2T^2}. \tag{14}$$

3. Time-dependent Ginzburg–Landau approach

Before comparing (13) to simulations, we present an alternative, continuum approach to compute C. For an Ising system with non-conservative order parameter (model A [10]), the time-dependent Ginzburg–Landau (TDGL) equation for the local magnetization density $\phi(\boldsymbol{r},t)$ takes the following form,

$$\frac{\partial\phi}{\partial t} = -\Gamma\frac{\delta\mathcal{H}}{\delta\phi} + \zeta, \tag{15}$$

$$\mathcal{H} = \int \mathrm{d}\boldsymbol{r}\,\left(\tfrac{1}{2}(\nabla\phi)^2 + \tfrac{1}{2}u\phi^2 + \frac{g}{4!}\phi^4\right), \tag{16}$$

where $\mathcal{H}$ is the coarse grained Hamiltonian. For our present purpose, the white noise $\zeta(\boldsymbol{r},t)$ which accounts for the effect of thermal fluctuations is irrelevant. Conventionally, parameters Γ, u and g in (16) are understood to be obtained by coarse graining the microscopic dynamics (1). For critical properties, the sole important temperature dependence in these parameters lies in $u \propto T - T_c^{GL}$, giving rise to the spontaneous symmetry breaking below the critical temperature T_c^{GL}. For our purposes of comparing with simulations, more precise dependences on T are required. To this end, we outline here a refined mean-field approach in the continuum limit. The same approach has been successfully applied to the two-species driven diffusive systems [11]. This approximation is expected to be good outside the critical region. However, this turns out to be not a serious handicap because the presence of an oscillating field prevents the system from building up critical correlations.

In a mean-field approximation, the joint probabilities in (1) are factorized into singlet probabilities $p(\boldsymbol{r};t)$ for finding the spin up at site $\boldsymbol{r}$ at time t. Since a spin flip depends on a total of $z+1$ spins in (1), the factorization effectively produces a series expansion of $\mathcal{H}$ in powers of ϕ up to ϕ^{z+1}. This is followed by the continuum limit, i.e., expansions in the derivatives such as

$$p(x\pm 1,y;t) \to p(x,y;t) \pm \frac{\partial p(x,y;t)}{\partial x} + \frac{1}{2}\frac{\partial^2 p(x,y;t)}{\partial x^2} + \dots$$

For long-distance behavior, we stop at the order as shown, consistent with (16). By identifying p as $(\phi+1)/2$ and collecting terms according to powers of ϕ, we obtain from (1) a kinetic equation for ϕ after some algebra. For $h = 0$, we find precisely the deterministic part of (15) with

$$\Gamma = \tfrac{1}{8}(-2W_4 + 2W_{-4} - W_8 + W_{-8}), \tag{17}$$

$$u = \frac{1}{8\Gamma}(6W_0 + 12W_4 - 4W_{-4} + 5W_8 - 3W_{-8}), \tag{18}$$

$$g = \frac{3}{2\Gamma}(-6W_0 - 4W_4 + 4W_{-4} + 5W_8 + W_{-8}), \tag{19}$$

where $W_n \equiv 1/(1+\mathrm{e}^{n\beta J})$ contains the desired explicit T dependence. The coefficient for ϕ^5 happens to vanish for heat-bath rates. When a small uniform field h is applied, to $O(h)$ we have finally the deterministic kinetic equation

$$\frac{\partial\phi}{\partial t} = -\Gamma(-\nabla^2\phi + u\phi + \tfrac{1}{6}g\phi^3 - \mu h), \tag{20}$$

where $\mu = \beta(3W_0^2 + 4W_4W_{-4} + W_8W_{-8})/2\Gamma$. It is useful to note that Γ, g and μ in (20) are positive definite for all T, whereas u has one zero at $T_c^{GL} \approx 3.0901J \approx 1.3618T_c$, where $T_c = -2J/\ln(\sqrt{2}-1) \approx 2.2692J$ is exact. This is an improvement over $T_c^{MF} = 4J$ from the last section. Moreover, we reproduce the first few terms of the high-temperature series expansions of thermodynamic quantities such as the susceptibility and the relaxation time. In the $\beta \to 0$ limit, we recover the mean-field results of the last section: $u \approx 1/\beta J - 4$, $\Gamma \approx \beta J$, $g \approx 48(\beta J)^2$, and $\mu \approx 1/J$.

For small h and $T > T_c^{GL}$, the nonlinear term $g\phi^3$ in (20) is negligible. The total magnetization $M(t) = \int \mathrm{d}\boldsymbol{r}\, \phi(\boldsymbol{r},t) = \tilde{\phi}(\boldsymbol{q}=0,t)$ in response to an external field can then be computed easily, where $\tilde{\phi}$ denotes the spatial Fourier transform of ϕ. It satisfies $\partial M/\partial t = -\Gamma u M + \Gamma\mu\tilde{h}(\boldsymbol{q}=0,t)$. We readily find

$$M(t) = \frac{V\mu A\Gamma}{\sqrt{(\Gamma u)^2+\omega^2}} \sin(\omega t - \theta_{\mathrm{GL}}), \qquad (21)$$

where the phase shift is $\theta_{\mathrm{GL}} = \arctan(\omega/\Gamma u)$. The correlation function with h is then given by

$$C_{T>T_c^{\mathrm{GL}}} = \frac{VA^2\Gamma^2\mu u}{2[(\Gamma u)^2+\omega^2]}. \qquad (22)$$

Note that this coincides with the mean-field result (14) in the high-temperature limit.

For $T < T_c^{GL}$, the term proportional to g is needed to break the symmetry, leading to the spontaneous magnetization $m = \sqrt{-6u/g}$ (recall that $g > 0$ for all T, and $u < 0$ for $T < T_c^{GL}$.) Linearizing about m, we find precisely the same form of C as $T > T_c^{GL}$ except that u is replaced by $-2u$ in (22).

Examining (20), one may ask why one should expect stochastic resonance above T_c where the potential has a single well. Besides, C is computed without ever using the noise term ζ in (15). The resolution of these apparent contradictions with conventional stochastic resonance lies in the fact that thermal effects, regarded as the "noises" here, have been separated for mathematical convenience into a deterministic and a stochastic part in (15). Essentially, the deterministic part (the entropic effect) has been incorporated with the two-state nature of the spins, resulting in a single-well free energy functional, whereas ζ accounts for the remaining stochastic part. Hence, our analysis is based on a transformed description in which part of the noises are integrated with the double-well potential. We are not aware of a similar formulation in conventional studies of stochastic resonance.

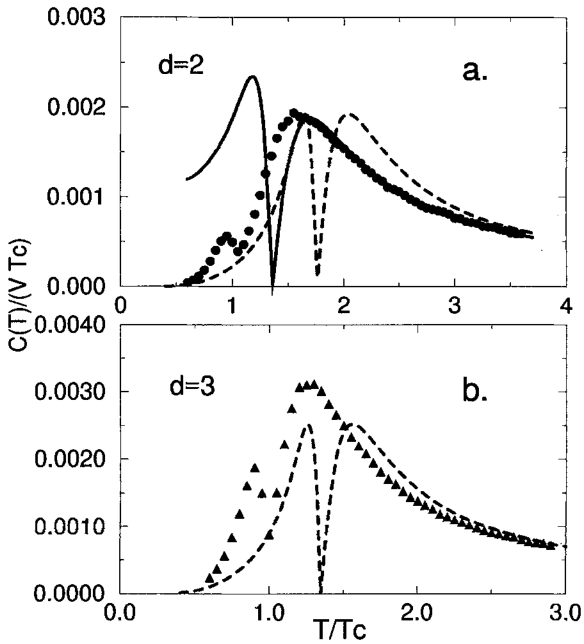

Fig. 1. $C(T)/VT_c$ versus temperature for $P = 40$ and $A = 0.05T_c$ for 2D in (a) and 3D in (b). Dots are MC simulation results in 2D ($N = 200$), triangles are MC simulations in 3D ($N = 40$), the continuous line is from the TDGL approximation and the dashed line is the mean-field result.

4. Discussion and comparison with simulations

From the simulation data in Refs. [1,2], we learn that the system has a maximum response to external driving at a definite temperature T_r which depends on the driving frequency. Hence T_r can be designated as the *resonance temperature*. From the analytically determined correlation functions in (14) and (22), we find two peaks in C above and below the respective T_c, and also $C(T_c) = 0$, as shown in Fig. 1. This double-peak structure in C is consistent with simulations for larger lattice sizes (up to $N = 200$ for 2D and $N = 40$ for 3D) and with smaller steps in T than reported in Refs. [1,2]. The reason for missing the peak below T_c in our earlier simulations may be the use of small lattice sizes. Note that the peak below T_c is much smaller than the one above and its position is less sensitive to the driving period. The reason for the overestimated theoretical values of the peaks below T_c may the frustration of the system to order in the presence of $h(t)$. Such frustration probably arises from nucleation of droplets of the stable phase inside the metastable phase [5]. Such local excitations have not been taken into account in our calculations. Instead, a uniform response of the system about one of the two local minima below T_c has been assumed.

We believe that this also explains the discrepancy at T_c, where simulations show a small but finite $C(T)$. Finite-size effects are not of great concern here because, as mentioned above, the correlation length even at T_c is truncated by h. In simulations, we have checked the convergence in $C(T)$ for $N \geqslant 50$ in 2D.

Focusing on $T > T_c$ from now on, the TDGL predictions for $C(T)$ are more accurate than those of the mean-field theory in general. They both converge to the simulations in the tails at $T \gg T_c$ (see Fig. 1). In 3D the mean-field theory is already acceptable except for the peak position, which is affected by the inaccuracy of T_c^{MF}.

Turning our attention to the amplitude dependence, replotting the simulation data from Refs. [1,2] suggests that the height of the peak $C(T_r) \propto A^2$, in agreement with (14) and (22). For not too large frequencies and small A, the theoretical proportionality constant agrees well with simulations. For example, the slope of $C(T_r)/VT_c$ versus A^2/T_c^2 for $P = 50$ in 2D gives 0.92 from simulations [1], 0.96 from TDGL and 0.99 from mean-field approach. In 3D the same slope is 0.88 from simulations [2], and 1.29 from mean-field approach (In 3D the comparison are worse because T_r is much closer now to T_c.) This proportionality is a manifestation of the linear response of the system to h, which breaks down at large enough amplitudes. Our new simulations show that this happens for $A/T_c > 0.15$ in 2D for $P = 40$.

A quantity of significant interest is the resonance temperature $T_r(P)$. It can be determined analytically from (14)

$$T_r^{MF} = T_c^{MF}\left(1 + \sqrt{1 - \frac{1}{\omega^2 + 1}}\right), \tag{23}$$

and numerically from (22) for T_r^{GL}. These together with simulation results are presented in Fig. 2. The agreements are reasonable. As expected the mean-field approximation is quite good in 3D but in 2D the TDGL approximation is better.

The results in Fig. 2 confirm the earlier observation in Refs. [1,2] that for $P \to \infty$ we get $T_r \to T_c$. This result is also consistent with the one obtained by Brey and Prados [4] in 1D where the above limit becomes $T_r \to T_c = 0$. In the opposite limit $P \to 1$ (in unit of Monte Carlo steps $P \geqslant 1$) both the theory in 1D [4] and our approximations in 2D and 3D suggest $T_r \to$ const. Unfortunately, in Refs. [1,2] the wrong conclusion $T_r \to \infty$ was drawn in this limit. Similarly, the position of the peak below T_c also converges to T_c in the $P \to \infty$ limit.

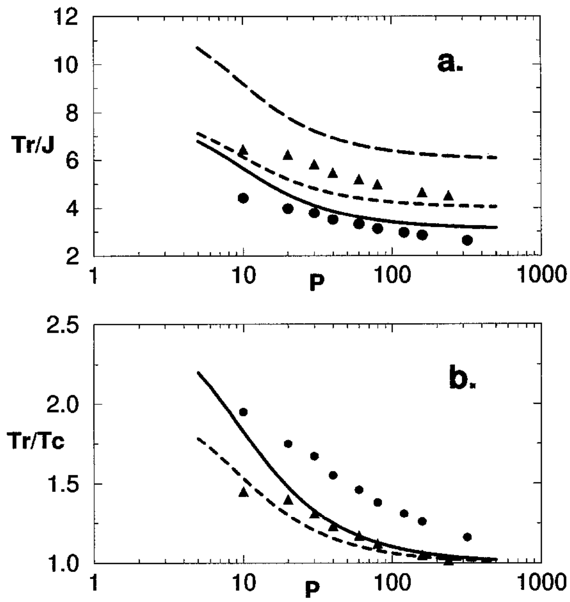

Fig. 2. Resonance temperature above T_c versus driving period P for $A = 0.05$, on absolute scale T_r/J in (a) and on relative scale T_r/T_c in (b). The long-dashed and short-dashed lines in (a) are the mean-field results for 3D and 2D respectively, in the rest the symbols mean the same as in Fig. 1.

In passing, we also derive [8] the relationship between the correlation function and the hysteresis-loop area $\mathcal{A}$,

$$\mathcal{A} = 2\pi C \tan\theta, \tag{24}$$

where θ is the phase shift between h and M. This result has also been derived recently by Acharyya [12], and relates our results of C to that of $\mathcal{A}$ as observed in Ref. [5].

5. Conclusions

Using mean-field with linear-response and TDGL approximations, the characteristics of the resonance peaks observed in kinetic Ising models in oscillating magnetic fields [1,2] are reproduced. New simulations improve earlier results by confirming the analytically predicted double peaks. Focusing mostly on the behavior above T_c (where our approaches work better), we determine the dependence of the resonance temperature as a function of driving frequency and amplitude. We confirm the already predicted result in Refs. [1,2] that $T_r \rightarrow T_c$ for the limit of practically interesting driving frequencies ($P \rightarrow \infty$), and corrected the wrong extrapolation in the opposite limit $P \rightarrow 1$. We introduce a refined TDGL approach which improves significantly the mean-field results in 2D, but in 3D the mean-field approximation is already acceptable. We have thus demonstrated that the stochastic resonance in kinetic Ising models above T_c can be understood by means of rather simple theoretical approaches for small driving amplitudes.

Acknowledgement

We are grateful to the NSC of ROC for their support through the grant NSC87-2112-M-001-006.

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