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Set-valued versions of Ky Fan's inequality with application to variational inclusion theory

Alexandru Kristály * and Csaba Varga [1]

Faculty of Mathematics and Informatics, Babes-Bolyai University, 3400 Cluj-Napoca, Romania

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Abstract

In this paper, we prove two set-valued versions of Ky Fan's minimax inequality. From these results, versions of Schauder's and Kakutani's fixed point theorems can be deduced. We formulate a variational inclusion problem for set-valued maps and a differential inclusion problem, concerning the contingent derivative. Sufficient conditions for the existence of solution for these inclusion problems are obtained, generalizing classical variational inequality problems.

1. Introduction

Let X be a real normed space, $K_1, K_2 \subset X$ two nonempty sets and $\varphi : K_1 \times K_2 \to \mathbf{R}$ a given function. An important problem in the nonlinear analysis is the so-called *equilibrium problem,* i.e., find an element $\bar{x} \in K_1$ such that

$$\varphi(\bar{x}, y) \geqslant 0, \quad \forall y \in K_2. \tag{EP}$$

The most familiar existence result in this direction is the Ky Fan minimax inequality, see [5].

Theorem 1.1. *Let K be a nonempty convex, compact subset of X and $\varphi : K \times K \to \mathbf{R}$ a function satisfying*

* Corresponding author.
E-mail addresses: akristal@math.ubbcluj.ro (A. Kristály), csvarga@cs.ubbcluj.ro (C. Varga).

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(i) $\forall y \in K$, $x \to \varphi(x, y)$ *is usc on* K;
(ii) $\forall x \in K$, $y \to \varphi(x, y)$ *is quasiconvex on* K;
(iii) $\forall y \in K$, $\varphi(y, y) \geqslant 0$.

Then, there exists an element $\bar{x} \in K$ *such that*

$$\varphi(\bar{x}, y) \geqslant 0, \quad \forall y \in K.$$

This result has many applications in various branches of mathematics: mathematical economy, game theory, fixed point theorems, variational inequalities, etc.

In [3], Browder studies a particular case of the above general equilibrium problem (EP). More precisely, he poses the following *variational inequality* problem: find an element $\bar{x} \in K$ such that

$$T(\bar{x})(y - \bar{x}) \geqslant 0, \quad \forall y \in K,$$

where K is a convex subset of a topological vector space E, $T : K \to E^*$ is a continuous operator, E^* the dual of E.

The above variational inequality can be written in the following form:

$$T(\bar{x})(\bar{x} - y) \cap \mathbf{R}_- \neq \emptyset, \quad \forall y \in K. \tag{VI}$$

The aim of our paper is two-fold. First, we state a similar result in set-valued context as (VI), i.e., we formulate a *set-valued variational inclusion* problem, guaranteeing a solution for this. More precisely, let $\mathcal{T} : K \rightsquigarrow X^*$ be a set-valued map. The problem is: find an element $\bar{x} \in K$ such that

$$\mathcal{T}(\bar{x})(\bar{x} - y) \cap \mathbf{R}_- \neq \emptyset, \quad \forall y \in K. \tag{SVVI}$$

Secondly, if $T = \nabla f$, f being continuously differentiable on X, we have a particular variational inequality problem: find an element $\bar{x} \in K$ such that

$$\nabla f(\bar{x})(y - \bar{x}) \geqslant 0, \quad \forall y \in K. \tag{PVI}$$

But in several problems the function f is not differentiable and perhaps is not single-valued. Let us consider the following set-valued map $F_0 : \mathbf{R} \rightsquigarrow \mathbf{R}$ defined by

$$F_0(x) = \begin{cases} \{1\}, & x < 0, \\ \{-1, 1\}, & x = 0, \\ \{-1\}, & x > 0, \end{cases}$$

and $K_0 = [-1, 1]$. Clearly, for F_0 and K_0 we haven't a classical variational inequality problem, like (PVI).

Therefore, it's natural to pose the following *differential inclusion* problem: let $F : X \rightsquigarrow \mathbf{R}$ be a set-valued map with compact, nonempty values, X being a normed space, $K \subset X$ nonempty convex subset of X. Find $\bar{x} \in K$ such that

$$DF\big(\bar{x}, \min F(\bar{x})\big)(u - \bar{x}) \subseteq \mathbf{R}_+, \quad \forall u \in K, \tag{DI}$$

where $DF(x, y)$ is the contingent derivative at $(x, y) \in \mathit{Graph}(F)$, see Section 4. In particular, if $F(x) = \{f(x)\}$ is a single-valued continuously differentiable function on X, then the above differential inclusion problem reduces to (PVI), since $DF(x, f(x)) =$

$\nabla f(x)$. In the above inclusion, the left-hand side may be empty for some element u. In this case (as convention), the inclusion will be considered trivial. For the above set-valued map F_0, an easy calculation shows that every element from the interval $K_0 = [-1, 1]$ is solution for the corresponding (DI).

To solve the above problems, we need set-valued versions of Ky Fan-type result. For this, we formulate two *set-valued equilibrium problems*. Let $F : K \times K \rightsquigarrow \mathbf{R}$ be a set-valued map. Find $\bar{x}_1 \in K$ resp. $\bar{x}_2 \in K$ such that

$$F(\bar{x}_1, y) \subseteq \mathbf{R}_+, \quad \forall y \in K, \tag{SVEP1}$$

resp.

$$F(\bar{x}_2, y) \cap \mathbf{R}_- \neq \emptyset, \quad \forall y \in K. \tag{SVEP2}$$

The main purpose of Section 2 is to present existence results for (SVEP1) and (SVEP2). It will be pointed out that from our main result of this section (Theorem 2.1) we can deduce a special form of the Ky Fan's minimax inequality (Corollary 2.1), remarking that this result does not cover the complete generality of the single-valued case. In the third section, we give simple proofs for versions of Schauder's and Kakutani's fixed point theorems. In the last section, we give sufficient conditions to guarantee the existence of solutions for (SVVI) and (DI); in particular, containing a result of Browder's type, see [3].

2. Set-valued versions of Ky Fan's inequality

Let Z and Y be metric spaces, $F : Z \rightsquigarrow Y$ be a set-valued map with nonempty values. We define the graph of the function F by

$$\mathit{Graph}(F) = \{(z, y) \in Z \times Y \mid y \in F(z)\}.$$

We say that the set-valued map $F : Z \rightsquigarrow Y$ is *upper semicontinuous at* $z \in Z$ (usc at z) if and only if for any neighborhood U of $F(z)$, $\exists \eta > 0$ such that for every $z' \in B_Z(z, \eta)$ we have $F(z') \subset U$. The set-valued function $F : Z \rightsquigarrow Y$ is *lower semicontinuous at* $z \in Z$ (lsc at z) if and only if for any $y \in F(z)$ and for any sequence of elements (z_n) in Z converging to z, there exists a sequence of elements $y_n \in F(z_n)$ converging to y.

The set-valued function F is *upper* (resp. *lower*) *semicontinuous on* Z if F is upper (resp. lower) semicontinuous at every point $z \in Z$.

We shall say that the set-valued map F is *continuous* at z if it is both usc and lsc at z, and that it is continuous on Z if and only if it is continuous at every point of Z.

Let M be a subset of Y. We denote

$$F^{-1}(M) = \{z \in Z \mid F(z) \cap M \neq \emptyset\},$$
$$F^{+1}(M) = \{z \in Z \mid F(z) \subseteq M\}.$$

The subset $F^{-1}(M)$ is called the *inverse image* of M by F and $F^{+1}(M)$ is called the *core* of M by F.

We need the following characterization of the upper (resp. lower) semicontinuity of F.

Proposition 2.1 [1, Proposition 1.4.4]. *A set-valued map $F : Z \rightsquigarrow Y$ with nonempty values is upper semicontinuous on Z if and only if the inverse image of any closed subset is closed, and is lower semicontinuous on Z if and only if the core of any closed subset is closed.*

Let K be a convex subset of a vector space X and $F : X \rightsquigarrow \mathbf{R}$ a set-valued code.

We say that F is *convex on K* (resp. *concave on K*), see [1, p. 57], if and only if $\forall x_1, x_2 \in K$ and $\lambda \in [0, 1]$,

$$\lambda F(x_1) + (1-\lambda)F(x_2) \subseteq (\text{resp. } \supseteq) F\big(\lambda x_1 + (1-\lambda)x_2\big).$$

Remark 2.1. F is convex on K (resp. concave on K) if and only if for all $n \geqslant 2$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \geqslant 0$ such that $\sum_{i=1}^n \lambda_i = 1$ and for all $x_1, x_2, \ldots, x_n \in K$,

$$\sum_{i=1}^n \lambda_i F(x_i) \subseteq (\text{resp. } \supseteq) F\left(\sum_{i=1}^n \lambda_i x_i\right).$$

We use the intersection theorem due to Ky Fan, known in the literature as Ky Fan's lemma.

Lemma 2.1 [4]. *Let X be a Hausdorff topological vector space, K a subset of X and for each $x \in K$, let $S(x)$ be a closed subset of X such that*

(i) *there exists $x_0 \in K$ such that the set $S(x_0)$ is compact*;
(ii) *for each $x_1, x_2, \ldots, x_n \in K$*, $\operatorname{co}\{x_1, x_2, \ldots, x_n\} \subseteq \bigcup_{i=1}^n S(x_i)$.

Then

$$\bigcap_{x \in K} S(x) \neq \emptyset.$$

The main result of this section can be formulated as follows.

Theorem 2.1. *Let X be a real normed space, K a nonempty convex, compact subset of X and $F : K \times K \rightsquigarrow \mathbf{R}$ a set-valued map satisfying*

(i) *$\forall y \in K$, $x \rightsquigarrow F(x, y)$ is lsc on K*;
(ii) *$\forall x \in K$, $y \rightsquigarrow F(x, y)$ is convex on K*;
(iii) *$\forall y \in K$, $F(y, y) \subseteq \mathbf{R}_+$.*

Then, there exists an element $\bar{x} \in K$ such that

$$F(\bar{x}, y) \subseteq \mathbf{R}_+, \quad \forall y \in K, \tag{2.1}$$

i.e., $\bar{x}$ is a solution for (SVEP1).

Proof. For all $y \in K$, let $S_y = \{x \in K \mid F(x, y) \subseteq \mathbf{R}_+\}$. In order to prove relation (2.1) it's enough to prove that $\bigcap_{y \in K} S_y \neq \emptyset$. From (iii) it follows that $S_y \neq \emptyset$, $(y \in S_y)$. From (i) and Proposition 2.1, the sets S_y are closed for all $y \in K$, and since K is supposed to be compact, they are compact, too. Therefore (i) from Lemma 2.1 is satisfied. We shall show that for all $y_1, y_2, \dots, y_n \in K$, $\operatorname{co}\{y_1, y_2, \dots, y_n\} \subseteq \bigcup_{i=1}^n S_{y_i}$. Indeed, supposing the contrary, there exist $y_1, y_2, \dots, y_n \in K$ and $\lambda_1, \lambda_2, \dots, \lambda_n \geqslant 0$, $\sum_{i=1}^n \lambda_i = 1$ such that

$$\sum_{i=1}^n \lambda_i y_i \notin S_{y_i}, \quad \forall i = \overline{1,n}. \tag{2.2}$$

Let $I = \{i \in \{1, 2, \dots, n\} \mid \lambda_i > 0\}$. Of course, $I \neq \emptyset$. From (2.2) we have that $F(\sum_{i=1}^n \lambda_i y_i, y_i) \nsubseteq \mathbf{R}_+$, $\forall i = \overline{1,n}$. From this, we get

$$\lambda_i F\left(\sum_{i\in I} \lambda_i y_i, y_i\right) \cap \mathbf{R}_-^* \neq \emptyset, \quad \forall i \in I. \tag{2.3}$$

Using (ii), (iii) and (2.3) we obtain

$$\begin{aligned}\emptyset \neq \left\{\sum_{i\in I} \lambda_i F\left(\sum_{i\in I} \lambda_i y_i, y_i\right)\right\} \cap \mathbf{R}_-^* &\subseteq F\left(\sum_{i\in I} \lambda_i y_i, \sum_{i\in I} \lambda_i y_i\right) \\ &\cap \mathbf{R}_-^* \subseteq \mathbf{R}_+ \cap \mathbf{R}_-^* = \emptyset.\end{aligned}$$

This contradiction shows that (ii) from Lemma 2.1 holds. Therefore $\bigcap_{y\in K} S_y \neq \emptyset$, i.e., there exists an element $\bar{x} \in K$ such that $F(\bar{x}, y) \subseteq \mathbf{R}_+$, $\forall y \in K$. □

Remark 2.2. The above theorem remains true for any Hausdorff topological vector space.

In an analogous way, we can obtain the "dual" of the above result. For the sake of completeness, we give the proof.

Theorem 2.2. *Let X, K and F as above, satisfying*

(i) $\forall y \in K$, $x \rightsquigarrow F(x, y)$ *is usc on* K;
(ii) $\forall x \in K$, $y \rightsquigarrow F(x, y)$ *is concave on* K;
(iii) $\forall y \in K$, $F(y, y) \cap \mathbf{R}_- \neq \emptyset$.

Then, there exists an element $\bar{x} \in K$ such that

$$F(\bar{x}, y) \cap \mathbf{R}_- \neq \emptyset, \quad \forall y \in K, \tag{2.4}$$

i.e., $\bar{x}$ is a solution for (SVEP2).

Proof. For all $y \in K$, let $S_y = \{x \in K \mid F(x, y) \cap \mathbf{R}_- \neq \emptyset\}$. We apply again Lemma 2.1. From (iii) it follows that $S_y \neq \emptyset$ ($y \in S_y$). From (i) and Proposition 2.1, the sets S_y are closed for all $y \in K$, and since K is supposed to be compact, they are compact, too. Therefore (i) from Lemma 2.1 is satisfied. We shall show that for all $y_1, y_2, \dots, y_n \in K$, $\operatorname{co}\{y_1, y_2, \dots, y_n\} \subseteq \bigcup_{i=1}^n S_{y_i}$. Supposing the contrary, there exist $y_1, y_2, \dots, y_n \in K$ and $\lambda_1, \lambda_2, \dots, \lambda_n \geqslant 0$, $\sum_{i=1}^n \lambda_i = 1$ such that

$$\sum_{i=1}^n \lambda_i y_i \notin S_{y_i}, \quad \forall i = \overline{1,n}. \tag{2.5}$$

Let $I = \{i \in \{1, 2, \dots, n\} \mid \lambda_i > 0\}$. Of course, $I \neq \emptyset$. From (2.5) we have that $F(\sum_{i=1}^n \lambda_i y_i, y_i) \subseteq \mathbf{R}_+^*$, $\forall i = \overline{1, n}$. From this, we have

$$\lambda_i F\left(\sum_{i\in I} \lambda_i y_i, y_i\right) \subseteq \mathbf{R}_+^*, \quad \forall i \in I. \tag{2.6}$$

Using (ii) and (2.6) we obtain

$$F\left(\sum_{i\in I} \lambda_i y_i, \sum_{i\in I} \lambda_i y_i\right) \subseteq \sum_{i\in I} \lambda_i F\left(\sum_{i\in I} \lambda_i y_i, y_i\right) \subseteq \mathbf{R}_+^*,$$

which contradicts (iii). This completes the proof. □

Lemma 2.2. *Let X be a normed space, $K \subset X$ and $f : K \to \mathbf{R}$. We define the set-valued map $F : K \rightsquigarrow \mathbf{R}$ by $F(x) = [f(x), \infty)$. Then*

(i) *if f is continuous on K then F is continuous on K;*
(ii) *if K is convex and f is convex on K, so is F on K;*
(iii) *if K is convex and f is concave on K then F is concave on K.*

As a first application, from Theorem 2.1 we obtain a special case of the Ky Fan's minimax inequality.

Corollary 2.1. *Let X be a real normed space, K a nonempty convex, compact subset of X and $f : K \times K \to \mathbf{R}$ a function satisfying*

(i) $\forall y \in K$, $x \to f(x, y)$ *is continuous on* K;
(ii) $\forall x \in K$, $y \to f(x, y)$ *is convex on* K;
(iii) $\forall y \in K$, $f(y, y) \geqslant 0$.

Then, there exists an element $\bar{x} \in K$ such that

$$f(\bar{x}, y) \geqslant 0, \quad \forall y \in K.$$

Proof. It's easy to verify that the function $F : K \times K \rightsquigarrow \mathbf{R}$ defined by $F(x, y) = [f(x, y), \infty)$ satisfies the hypotheses from Theorem 2.1, using Lemma 2.2. Therefore, there exists $\bar{x} \in K$ such that $F(\bar{x}, y) \subseteq \mathbf{R}_+$, $\forall y \in K$. From this, we have necessarily that $f(\bar{x}, y) \geqslant 0, \forall y \in K$. □

The compactness of K in Theorem 2.1 and Theorem 2.2 is a rather restrictive condition. In the classical theory this condition can be weakened by assuming a so-called *coercivity condition*, see [2], due to Brézis, Nirenberg and Stampacchia. We give a set-valued version of their result.

Theorem 2.3. *Let X be a real normed space, K a nonempty convex, closed subset of X and $F : K \times K \rightsquigarrow \mathbf{R}$ a set-valued map satisfying*

(i) $\forall y \in K$, $x \rightsquigarrow F(x, y)$ *is lsc on* K;
(ii) $\forall x \in K$, $y \rightsquigarrow F(x, y)$ *is convex on* K;
(iii) $\forall y \in K$, $F(y, y) \subseteq \mathbf{R}_+$;
(iv) *there exist a compact set* $K_0 \subset X$ *and an element* $y_0 \in K \cap K_0$ *such that*

$$F(x, y_0) \cap \mathbf{R}_-^* \neq \emptyset, \quad \forall x \in K \setminus K_0. \tag{2.7}$$

Then, there exists an element $\bar{x} \in K \cap K_0$ *such that*

$$F(\bar{x}, y) \subseteq \mathbf{R}_+, \quad \forall y \in K$$

i.e., $\bar{x}$ *is a solution for* (SVEP1).

Proof. It's similar to the proof of Theorem 2.1. For all $y \in K$, let $S_y = \{x \in K \mid F(x, y) \subseteq \mathbf{R}_+\}$. We prove that $S_{y_0} \subseteq K_0$. Indeed, supposing the contrary, there exists an element $z \in S_{y_0}$ such that $z \notin K_0$. From the definition of S_{y_0} we have that $F(z, y_0) \subseteq \mathbf{R}_+$ which is in contradiction with the relation (2.7). Therefore, $S_{y_0} \subseteq K_0$ and since K_0 is compact, S_{y_0} is compact, too. The rest of the proof is the same as in Theorem 2.1. □

3. Application to fixed point theorems

From Theorem 2.2 we can deduce directly a version of Schauder fixed point theorem, and in particular the Brouwer fixed point theorem.

Corollary 3.1. *Let K be a convex compact subset of a real normed space and* $f : K \to K$ *a continuous function. Then* f *has a fixed point.*

Proof. Let $F : K \times K \rightsquigarrow \mathbf{R}$ be the set-valued map defined by $F(x, y) = -\|y - f(x)\| + \|x - f(x)\| + [0, \infty)$. We verify (i)–(iii) from Theorem 2.2.

(i) and (ii) follow from Lemma 2.2, using the concavity of $y \mapsto -\|y - f(x)\|$ for all $x \in K$. For (iii), we have

$$\bigl\{-\bigl\|y - f(y)\bigr\| + \bigl\|y - f(y)\bigr\| + [0, \infty)\bigr\} \cap \mathbf{R}_- = \{0\} \neq \emptyset, \quad \forall y \in K.$$

Therefore, there exists $\bar{x} \in K$, such that

$$\bigl\{-\bigl\|y - f(\bar{x})\bigr\| + \bigl\|\bar{x} - f(\bar{x})\bigr\| + [0, \infty)\bigr\} \cap \mathbf{R}_- \neq \emptyset, \quad \forall y \in K.$$

We have necessarily that $-\|y - f(\bar{x})\| + \|\bar{x} - f(\bar{x})\| \leqslant 0, \forall y \in K$. Let $y := f(\bar{x}) (\in K)$ and we get $\|\bar{x} - f(\bar{x})\| \leqslant 0$, i.e., $\bar{x} = f(\bar{x})$. □

If M is a subset of a normed space X, and $x \in X$ then we denote by $\operatorname{dist}(x, M) = \inf\{\|x - y\| \mid y \in M\}$.

Now we deduce a special form of Kakutani's fixed point theorem from Theorem 2.1. First, we need the following result, which is known in the literature as Maximum Theorem. Let us consider the set-valued map $F : Z \rightsquigarrow Y$, Z and Y being metric spaces and a function $f : \mathit{Graph}(F) \to \mathbf{R}$. We define the *marginal function* $g : Z \to \mathbf{R} \cup \{+\infty\}$ by

$$g(x) = \sup_{y \in F(x)} f(x, y).$$

We have the Maximum Theorem.

Lemma 3.1 [1, Theorem 1.4.16, p. 48]. *Let $F : Z \rightsquigarrow Y$ be a set-valued map and a function $f : Graph(F) \to \mathbf{R}$. Then*

(i) *if f and F are lsc, so is the marginal function*;
(ii) *if f and F are usc and if the values of F are compact, so is the marginal function.*

Theorem 3.1. *Let X be a normed space, K be a convex, compact subset of X and $G : K \rightsquigarrow X$ be a continuous set-valued map on K with nonempty compact convex values. If $K \subseteq G^{-1}(K)$ then G has a fixed point $\bar{x} \in K \cap G(\bar{x})$.*

Proof. We define the set-valued map $F : K \times K \rightsquigarrow \mathbf{R}$ by

$$F(x, y) = \operatorname{dist}\big(y, G(x)\big) - \operatorname{dist}\big(x, G(x)\big) + [0, \infty).$$

Applying Lemma 3.1 for $f(x, y) = -\|x - y\|$, $Z := K$, $Y := X$ and $F := G$, we obtain that $g(x) = -\operatorname{dist}(x, G(x))$ is continuous on K. The continuity of the function $x \mapsto \operatorname{dist}(y, G(x))$ for y fixed, can be proved analogously, adapting the proof of the previous Lemma 3.1, see [1]. Using Lemma 2.2(i) we deduce that $x \rightsquigarrow F(x, y)$ is lsc on K for all $y \in K$. Since $y \mapsto \operatorname{dist}(y, G(x))$ is convex function for all $x \in K$, using Lemma 2.2(ii) we obtain that $y \rightsquigarrow F(x, y)$ is convex for all $x \in K$. Moreover, $F(y, y) = \mathbf{R}_+$. Therefore, from Theorem 2.1 we have $\bar{x} \in K$ such that $\operatorname{dist}(y, G(\bar{x})) - \operatorname{dist}(\bar{x}, G(\bar{x})) + [0, \infty) \subseteq \mathbf{R}_+$, $\forall y \in K$. From this, we obtain that

$$\operatorname{dist}\big(y, G(\bar{x})\big) - \operatorname{dist}\big(\bar{x}, G(\bar{x})\big) \geqslant 0, \quad \forall y \in K. \tag{3.1}$$

Since $K \subseteq G^{-1}(K)$, i.e., for all $x \in K$, $G(x) \cap K \neq \emptyset$, we may choose an element $y \in G(\bar{x}) \cap K$. Substituting in (3.1), we obtain that $\operatorname{dist}(\bar{x}, G(\bar{x})) \leqslant 0$. Therefore $\bar{x} \in G(\bar{x})$. □

Remark 3.1. The Kakutani's fixed point theorem is a natural set-valued version of Brouwer's fixed point theorem. Of course, we can deduce the latter one from the previous theorem.

4. Application to variational inclusion theory

Let X be a real normed space, K be a subset of X and $\mathcal{T} : K \rightsquigarrow X^*$ be a set-valued map. We have $\mathcal{T}(x)(y) = \bigcup_{x^* \in \mathcal{T}(x)} x^*(y)$.

The main result of this section is

Theorem 4.1. *Let K be a convex, compact subset of X and $\mathcal{T} : K \rightsquigarrow X^*$ be an upper semicontinuous set-valued map such that* $\operatorname{card} \mathcal{T}(x) < \infty$, *$\forall x \in K$. Then there exists $\bar{x} \in K$ such that*

$$\mathcal{T}(\bar{x})(\bar{x} - y) \cap \mathbf{R}_- \neq \emptyset, \quad \forall y \in K,$$

i.e., $\bar{x} \in K$ is a solution for (SVVI).

Proof. Let $F : K \times K \rightsquigarrow \mathbf{R}$ defined by $F(x, y) = \mathcal{T}(x)(x - y)$. We verify the hypotheses from Theorem 2.2. Let $x \in K$ be fixed.

To prove (i), let U be a neighborhood of $F(x, y)$, $y \in K$ is fixed. Since card $\mathcal{T}(x) < \infty$, then there exists $\delta > 0$ such that

$$B_{\mathbf{R}}\big(x^*(x-y), \delta\big) \subset U, \quad \forall x^* \in \mathcal{T}(x). \tag{4.1}$$

Let

$$\delta_1 := \min\left\{\frac{\delta}{3(\|y\|+1)}, \frac{\delta}{3(\|x\|+1)}\right\}.$$

Since $\mathcal{T}$ is upper semicontinuous on K, for $\bigcup_{x^* \in \mathcal{T}(x)} B_{X^*}(x^*, \delta_1)$ (which is a neighborhood of $\mathcal{T}(x)$ in X^*), there exists $\eta^* > 0$ such that

$$\mathcal{T}(w) \subset \bigcup_{x^* \in \mathcal{T}(x)} B_{X^*}(x^*, \delta_1), \quad \forall w \in B_K(x, \eta^*). \tag{4.2}$$

Let

$$\eta := \min\left\{\frac{\delta}{3(M+1)}, \eta^*, 1\right\},$$

where $M := \max\{\|x^*\| \mid x^* \in \mathcal{T}(x)\}$. We prove, that for all $z \in B_K(x, \eta)$: $F(z, y) \subset U$, which means that $x \rightsquigarrow F(x, y)$ is usc on K. For this let $z \in K$ such that $\|z - x\| < \eta$ and let $z^* \in \mathcal{T}(z)$. From (4.2) and from the fact that $\eta \leqslant \eta^*$, we have that $\mathcal{T}(z) \subset \bigcup_{x^* \in \mathcal{T}(x)} B_{X^*}(x^*, \delta_1)$. Therefore, there exists $x_0^* \in \mathcal{T}(x)$ such that $z^* \in B_{X^*}(x_0^*, \delta_1)$, i.e., $\|z^* - x_0^*\| < \delta_1$. We have

$$\begin{aligned}
&\big|z^*(z-y) - x_0^*(x-y)\big| \\
&\quad = \big|\big(z^* - x_0^*\big)(z) + x_0^*(z-x) - \big(z^* - x_0^*\big)(y)\big| \\
&\quad \leqslant \big\|z^* - x_0^*\big\| \cdot \|z\| + \big\|x_0^*\big\| \cdot \|z - x\| + \big\|z^* - x_0^*\big\| \cdot \|y\| \\
&\quad \leqslant \frac{\delta}{3(\|x\|+1)} \cdot \big(\|x\| + \eta\big) + \frac{\delta}{3(M+1)} \cdot \big\|x_0^*\big\| + \frac{\delta}{3(\|y\|+1)} \cdot \|y\| < \delta.
\end{aligned}$$

Therefore, $z^*(z-y) \in \bigcup_{x^* \in \mathcal{T}(x)} B_{\mathbf{R}}(x^*(x-y), \delta)$, $\forall z^* \in \mathcal{T}(z)$, i.e., $F(z, y) \subset U$, using (4.1).

To prove (ii), let $x^* \in \mathcal{T}(x)$ and $y_1, y_2 \in K$, $\lambda \in [0, 1]$. Since x^* is linear, we have

$$\begin{aligned}
x^*\big(x - \lambda y_1 - (1-\lambda) y_2\big) &= \lambda x^*(x - y_1) + (1-\lambda) x^*(x - y_2) \\
&\in \lambda \mathcal{T}(x)(x - y_1) + (1-\lambda)\mathcal{T}(x)(x - y_2).
\end{aligned}$$

From this, we have $F(x, \lambda y_1 + (1-\lambda) y_2) \subseteq \lambda F(x, y_1) + (1-\lambda) F(x, y_2)$, i.e., $y \rightsquigarrow F(x, y)$ is concave.

Since $F(x, x) = \mathcal{T}(x)(0) = \bigcup_{x^* \in \mathcal{T}(x)} x^*(0) = \{0\}$, then $F(x, x) \cap \mathbf{R}_- = \{0\} \neq \emptyset$.

Therefore, from Theorem 2.2 there exists $\bar{x} \in K$ such that $F(\bar{x}, y) \cap \mathbf{R}_- \neq \emptyset$, $\forall y \in K$, which is exactly the desired conclusion. □

Specializing the above statement to single-valued map, we obtain the well-known result of Browder.

Corollary 4.1 [3]. *Let K be a convex, compact subset of X and let $T : K \to X^*$ be continuous. Then, there exists $\bar{x} \in K$ such that*

$$T(\bar{x})(y - \bar{x}) \geqslant 0, \quad \forall y \in K,$$

i.e., $\bar{x}$ is a solution for (VI).

Proof. Take $\mathcal{T}(x) = \{T(x)\}$, $\forall x \in K$. We remark, that the continuity of T is equivalent with the upper semicontinuity of $\mathcal{T}$ (see [1]). From Theorem 4.1, we obtain $\bar{x} \in K$ such that $\mathcal{T}(\bar{x})(\bar{x} - y) \cap \mathbf{R}_- \neq \emptyset$, $\forall y \in K$. From this, $T(\bar{x})(\bar{x} - y) \leqslant 0$, $\forall y \in K$, which completes the proof. $\square$

In the rest of the paper, we will be interested to guarantee solution for (DI). For this, we recall some notions from [1].

In the sequel, let $F : X \rightsquigarrow \mathbf{R}$ be a set-valued map with nonempty and compact values. First of all, we define the contingent cone.

Let K be a subset of a normed space X and $x \in \overline{K}$, $\overline{K}$ being the closure of K. The *contingent cone* $T_K(x)$ is defined by

$$T_K(x) = \left\{ v \,\middle|\, \liminf_{h \to 0^+} \frac{\operatorname{dist}(x + hv, K)}{h} = 0 \right\}.$$

We say that F is *Lipschitz around* $x \in X$ if there exist a positive constant L and a neighborhood U of x such that

$$\forall x_1, x_2 \in U, \quad F(x_1) \subset F(x_2) + L\|x_1 - x_2\| \cdot [-1, 1].$$

Let $K \subseteq X$. We say that F is *K-locally Lipschitz* if it is Lipschitz around all $x \in K$.

Proposition 4.1. *If $F : X \rightsquigarrow \mathbf{R}$ is K-locally Lipschitz then the restriction $F|_K : K \rightsquigarrow \mathbf{R}$ is continuous on K.*

The *contingent derivative* $DF(x, y)$ of $F : X \rightsquigarrow \mathbf{R}$ at $(x, y) \in \mathit{Graph}(F)$, see [1, p. 181], is the set-valued map from X to $\mathbf{R}$ defined by

$$\mathit{Graph}\big(DF(x, y)\big) := T_{\mathit{Graph}(F)}(x, y),$$

where $T_{\mathit{Graph}(F)}(x, y)$ is the contingent cone at (x, y) to the $\mathit{Graph}(F)$.

We can characterize the contingent derivative by a limit of differential quotient. Let $(x, y) \in \mathit{Graph}(F)$ and suppose that F is Lipschitz around x. We have

$$v \in DF(x, y)(u) \iff \liminf_{h \to 0^+} \operatorname{dist}\left(v, \frac{F(x + hu) - y}{h}\right) = 0, \tag{4.3}$$

see [1, Proposition 5.1.4, p. 186].

Remark 4.1. Let us consider the case where F is single-valued, i.e., $F(x) = \{f(x)\}$, $\forall x \in X$. Suppose that $f : X \to \mathbf{R}$ is continuously differentiable. From [1, Proposition 5.1.3, p. 184] we have that

$$DF\big(x, f(x)\big)(h) = \nabla f(x)h, \quad \forall h \in X. \tag{4.4}$$

We say that $F : X \rightsquigarrow \mathbf{R}$ is *sleek* at $(x, y) \in Graph(F)$ if the map

$$Graph(F) \ni (x', y') \rightsquigarrow Graph\big(DF(x', y')\big)$$

is lower semicontinuous at (x, y). F is sleek if it is sleek at every point $(x, y) \in Graph(F)$.

$F : X \rightsquigarrow \mathbf{R}$ is *lower semicontinuously differentiable* (see [1, p. 188]) if the map

$$(x, y, u) \in Graph(F) \times X \rightsquigarrow DF(x, y)(u)$$

is lower semicontinuous.

Of course the lower semicontinuous differentiability of F implies that this is sleek.

Remark 4.2. If F is a closed set-valued map (i.e., $Graph(F)$ is closed) and is sleek at $(x, y) \in Graph(F)$, then the contingent derivative at (x, y) is a closed convex (process) set-valued map, due to Theorem 4.1.8 from [1, p. 130].

The first result, concerning the (DI) problem is the following

Theorem 4.2. *Let X be a real normed space, K a compact, convex, nonempty subset of X and $F : X \rightsquigarrow \mathbf{R}$ be a K-locally Lipschitz set-valued map with compact and nonempty values. Then there exists $\bar{x} \in K$ such that*

$$DF\big(\bar{x}, \min F(\bar{x})\big)(u - \bar{x}) \subseteq \mathbf{R}_+, \quad \forall u \in K,$$

i.e., $\bar{x}$ is a solution for (DI).

Proof. Since F is K-locally Lipschitz, then $F|_K$ is usc and lsc on K, see Proposition 4.1. Applying Lemma 3.1 for $F := F|_K$, $Z := K$, $Y := \mathbf{R}$ and $f(x, y) = -y$ we observe that $x \mapsto \min F|_K(x)$ is continuous on K. K being compact, there exists $\bar{x} \in K$ such that $\min F(\bar{x}) \leqslant \min F(x)$, $\forall x \in K$, i.e.,

$$F(x) - \min F(\bar{x}) \subseteq \mathbf{R}_+, \quad \forall x \in K. \tag{4.5}$$

Let $v \in DF(\bar{x}, \min F(\bar{x}))(u - \bar{x})$ be a fixed element, $u \in K$ being also fixed. From the relation (4.3) we have that

$$\liminf_{h \to 0^+} \operatorname{dist}\left(v, \frac{F(\bar{x} + h(u - \bar{x})) - \min F(\bar{x})}{h}\right) = 0, \tag{4.6}$$

since F is K-locally Lipschitz (in particular is Lipschitz around $\bar{x}$). Because $\bar{x} + h(u - \bar{x}) \in K$, using (4.5), we have that

$$\frac{F(\bar{x} + h(u - \bar{x})) - \min F(\bar{x})}{h} \subseteq \mathbf{R}_+.$$

Suppose that $v < 0$. Then

$$0 < |v| = \operatorname{dist}(v, \mathbf{R}_+) \leqslant \operatorname{dist}\left(v, \frac{F(\bar{x} + h(u - \bar{x})) - \min F(\bar{x})}{h}\right)$$

which is in contradiction with (4.6). Therefore, $v \geqslant 0$. Since $u \in K$ and v were arbitrary, v in $DF(\bar{x}, \min F(\bar{x}))(u - \bar{x})$, the proof is complete. $\square$

Theorem 4.2 reduces to a classical result concerning variational inequalities.

Corollary 4.2. *Let $K \subset X$ be compact convex and $f : X \to \mathbf{R}$ continuously differentiable. Then there exists $\bar{x} \in K$ such that*

$$\nabla f(\bar{x})(u - \bar{x}) \geqslant 0, \quad \forall u \in K,$$

i.e., $\bar{x}$ is a solution for (PVI).

Example 4.1. Let X be a real normed space and K be a compact convex subset of X. Let us consider two locally Lipschitz functions $f, g : X \to \mathbf{R}$ and we define a set-valued map $F : X \rightsquigarrow \mathbf{R}$ such that for all $x \in X$, $F(x)$ is the interval (maybe degenerate) between $f(x)$ and $g(x)$. Let us suppose that $f(x) \leqslant g(x)$, $\forall x \in K$. Naturally, F is K-locally Lipschitz and we can apply the above theorem.

Example 4.2. Let $f, g : \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = \begin{cases} -2x - 1, & x < 0, \\ -1, & x \geqslant 0, \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 1, & x < 0, \\ -2x + 1, & x \geqslant 0, \end{cases}$$

and $F : \mathbf{R} \rightsquigarrow \mathbf{R}$ defined as in the above example. Let $K := [-1, 1]$. Clearly, f and g are locally Lipschitz functions, and $f(x) \leqslant g(x)$, $\forall\ x \in K$. Applying the above example, F is K-locally Lipschitz, therefore we can apply Theorem 4.2, obtaining solution for the problem (DI). Calculating effectively the contingent derivatives, we obtain that the solutions for (DI) corresponding to F and K are the points in the interval $[0, 1]$.

When the set K is not compact, the problem is more delicate.

Theorem 4.3. *Let X be a real normed space, K a closed, convex, nonempty subset of X and $F : X \rightsquigarrow \mathbf{R}$ be a closed, lower semicontinuously differentiable, K-locally Lipschitz set-valued map with compact and nonempty values. We suppose that*:
there exist a compact subset K_0 of X and an element $y_0 \in K \cap K_0$ such that

$$\inf DF\big(x, \min F(x)\big)(y_0 - x) < 0, \quad \forall x \in K \setminus K_0.$$

Then there exists $\bar{x} \in K \cap K_0$ such that

$$DF\big(\bar{x}, \min F(\bar{x})\big)(u - \bar{x}) \subseteq \mathbf{R}_+, \quad \forall u \in K,$$

i.e., $\bar{x}$ is a solution for (DI).

Proof. Let $G : K \times K \rightsquigarrow \mathbf{R}$ defined by $G(x, u) = DF(x, \min F(x))(u - x)$. We shall show that G satisfies the hypotheses from Theorem 2.3.

To prove (i), it's enough to prove that the function $x \mapsto \min F(x)$ is continuous on K. This fact can be deduced similarly as in Theorem 4.2, using again Lemma 3.1. Let us consider $x \in K$ fixed and let $w \in G(x, u) = DF(x, \min F(x))(u - x)$, for $u \in K$ fixed and $\{x_n\} \subset K$ an arbitrary sequence which converges to x. Since $\min F(x_n) \to \min F(x)$ and using the lower semicontinuous differentiability of F, there exist $w_n \in DF(x_n, \min F(x_n))(u - x_n) = G(x_n, u)$ such that $w_n \to w$. Therefore, $K \ni x \rightsquigarrow G(x, u)$ is lsc, $\forall u \in K$.

(ii) follows from Remark 4.2, that is the contingent derivative is a convex set-valued map. Therefore, $K \ni u \rightsquigarrow G(x, u)$ is convex, $\forall x \in K$.

For (iii), let $x \in K$ and $v \in DF(x, \min F(x))(0)$. Using the characterization of the contingent derivative and the fact that F is K-locally Lipschitz (see (4.3)), we have

$$\liminf_{h\to 0^+} \operatorname{dist}\left(v, \frac{F(x) - \min F(x)}{h}\right) = 0.$$

Since $\frac{F(x)-\min F(x)}{h} \subseteq \mathbf{R}_+$, we obtain that $v \geqslant 0$. Therefore, $G(x,x) = DF(x, \min F(x))(0) \subseteq \mathbf{R}_+$, $\forall x \in K$.

From our hypothesis, we can deduce that $DF(x, \min F(x))(y_0 - x) \cap \mathbf{R}^*_- \neq \emptyset$, $\forall x \in K \setminus K_0$.

From Theorem 2.3, there exists an element $\bar{x} \in K$ such that $G(\bar{x}, u) \subseteq \mathbf{R}_+$, $\forall u \in K$, which is exactly the desired relation. □

In the finite dimensional case, we can use the following coerciveness hypothesis instead of the above one:
there exists $y_0 \in K$ *such that*

$$\limsup_{\substack{\|x\|\to\infty \\ x\in K}} \inf DF\big(x, \min F(x)\big)(y_0 - x) < 0.$$

Indeed, this hypothesis implies that there exist $\varepsilon > 0$ and $a > 0$ such that

$$\sup_{\substack{\|x\|\geqslant a \\ x\in K}} \inf DF\big(x, \min F(x)\big)(y_0 - x) \leqslant -\varepsilon < 0.$$

Let K_0 be the closed ball $\overline{B_X}(0, \max\{a, \|y_0\|\})$. Since $\dim X < \infty$ then K_0 is compact. Moreover, $y_0 \in K \cap K_0$. Using the above relation, we have $DF(x, \min F(x))(y_0 - x) \cap \mathbf{R}^*_- \neq \emptyset$, $\forall x \in K \setminus K_0$.

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