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An existence result for gradient-type systems with a nondifferentiable term on unbounded strips

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Abstract

In this paper we study the existence of nontrivial solutions for a class of gradient-type systems on strip-like domains where the nonlinear term is not necessarily continuously differentiable. The proof of the main result is based on a nonsmooth version of the Mountain Pass Theorem which involves the Cerami compactness condition and on the Principle of Symmetric Criticality for locally Lipschitz functions.

Keywords: Quasilinear elliptic systems; Strip-like domain; Locally Lipschitz functions; Principle of symmetric criticality; Nonsmooth Cerami condition

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a domain with smooth boundary $\partial\Omega$, $F \in C^1(\Omega \times \mathbb{R}^2, \mathbb{R})$ and $1 < p, q < N$. Several studies have appeared dealing with the existence of nonzero solutions of the gradient-type system

$$(\mathrm{S}_{p,q,\Omega}) \quad \begin{aligned} -\Delta_p u &= F_u(x,u,v) \quad \text{in } \Omega, \\ -\Delta_q v &= F_v(x,u,v) \quad \text{in } \Omega, \\ u = v &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

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where F_u is the partial derivative of F with respect to u (similarly for F_v), and $\Delta_\alpha u = \operatorname{div}(|\nabla u|^{\alpha-2}\nabla u)$, $\alpha \in \{p, q\}$. For the case of *bounded domain*s we refer the reader to the papers of Boccardo and de Figueiredo [2], Felmer et al. [11], de Figueiredo [12], Vélin and de Thélin [22].

In this paper we consider *strip-like domains* of the form $\Omega = \omega \times \mathbb{R}^{N-m}$, where $\omega \subset \mathbb{R}^m$ $(m \geqslant 1)$ is an open bounded set, and $N - m \geqslant 2$.

The motivation to consider such domains arises from certain mechanical problems, as the nonlinear Klein–Gordon or Schrödinger equations (see, for instance, Amick [1], Esteban [9], Lions [17]). In a recent paper, Carrião and Miyagaki [3] investigated a problem related to $(\mathrm{S}_{p,p,\Omega})$ where Ω is a strip-like domain (or, in other words, an unbounded cylinder) or a domain between two infinite cylinders. In their case, the right-hand side of $(\mathrm{S}_{p,p,\Omega})$ is perturbed by the gradient-type derivative of a p^*-homogeneous term (p^* is the critical exponent), while the nonlinearity F is supposed to be autonomous and p-homogeneous. Clearly, the homogeneity assumptions play a key role in their investigations. Although we do not treat the critical case in the present paper, we allow $p \neq q$ and we do not assume any homogeneity property on the nonlinearity F.

In the above-mentioned papers [2,11,12], the regularity of the nonlinear term (i.e., F is continuously differentiable) is an indispensable condition in order to guarantee weak solutions for $(\mathrm{S}_{p,q,\Omega})$. On the other hand, reading the works of Clarke [5,6], Panagiotopoulos [20,21], Motreanu and Panagiotopoulos [18], and the very recent monograph of Motreanu and Rădulescu [19], one often encounters concrete problems in mechanics, engineering and economics as well, where the nonlinear potential is *not* differentiable. So, the following natural question arises: How can we handle (the corresponding form of) the problem $(\mathrm{S}_{p,q,\Omega})$ if we abandon the differentiability of the nonlinear term F?

In this paper we restrict our attention to such nonlinearities which are *locally Lipschitz* functions and *regular in the sense of Clarke* [5]. In this setting, $(\mathrm{S}_{p,q,\Omega})$ requires a suitable reformulation which is inspired by the theory of *hemivariational inequalities*, developed by Panagiotopoulos [20]. For simplicity, we consider only the autonomous case, i.e., F will be supposed to be x-independent. In order to do this reformulation, we assume the following growth conditions on the partial generalized gradients of the locally Lipschitz function $F:\mathbb{R}^2 \to \mathbb{R}$:

(F1) There exist $c_1 > 0$ and $r \in\,]p, p^*[$, $s \in\,]q, q^*[$ such that

$$|w_u| \leqslant c_1\big(|u|^{p-1} + |v|^{(p-1)q/p} + |u|^{r-1}\big), \tag{1}$$

$$|w_v| \leqslant c_1\big(|v|^{q-1} + |u|^{(q-1)p/q} + |v|^{s-1}\big) \tag{2}$$

for all $(u, v) \in \mathbb{R}^2$, $w_u \in \partial_1 F(u, v)$ and $w_v \in \partial_2 F(u, v)$.

We denoted by $\partial_1 F(u, v)$ the (partial) generalized gradient of $F(\cdot, v)$ at the point u, and by $\partial_2 F(u, v)$ that of $F(u, \cdot)$ at v (see Clarke [5]); $\alpha^* = N\alpha/(N - \alpha)$ $(\alpha \in \{p, q\})$ is the Sobolev critical exponent.

Now, we are in the position to formulate our problem, denoted further by $(\mathrm{S}'_{p,q,\Omega})$:

Find $(u,v)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ *such that*

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla w+\int_\Omega F_1^0\big(u(x),v(x);-w(x)\big)\,dx\geqslant 0\quad\text{for all } w\in W_0^{1,p}(\Omega),$$

$$\int_\Omega |\nabla v|^{q-2}\nabla v\nabla y+\int_\Omega F_2^0\big(u(x),v(x);-y(x)\big)\,dx\geqslant 0\quad\text{for all } y\in W_0^{1,q}(\Omega).$$

Here, $F_1^0(u,v;w)$ is the (partial) generalized directional derivative of $F(\cdot,v)$ at the point $u\in\mathbb{R}$ in the direction $w\in\mathbb{R}$ (see Section 2). $F_2^0(u,v;w)$ is defined in a similar manner.

Remark 1. When $F\in C^1(\mathbb{R}^2,\mathbb{R})$ then $(u,v)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ solves $(\mathrm{S}'_{p,q,\Omega})$ if and only if (u,v) is a weak solution of $(\mathrm{S}_{p,q,\Omega})$ in the usual sense. Therefore, the formulation of $(\mathrm{S}'_{p,q,\Omega})$ recovers the classical problem $(\mathrm{S}_{p,q,\Omega})$.

We require the following further set of assumptions on F:

(F2) F is regular on $\mathbb{R}^2$ in the sense of Clarke [5], and $F(0,0)=0$.
(F3) There exist $c_2>0$ and $\mu,\nu\geqslant 1$ such that

$$-c_2\big(|u|^\mu+|v|^\nu\big)\geqslant F(u,v)+\frac{1}{p}F_1^0(u,v;-u)+\frac{1}{q}F_2^0(u,v;-v)\tag{3}$$

for all $(u,v)\in\mathbb{R}^2$.

(F4) $$\lim_{u,v\to 0}\frac{\max\{|w_u|:\ w_u\in\partial_1F(u,v)\}}{|u|^{p-1}}=\lim_{u,v\to 0}\frac{\max\{|w_v|:\ w_v\in\partial_2F(u,v)\}}{|v|^{q-1}}=0.$$

Our main result can be formulated as follows:

Theorem 1. *Let* $F:\mathbb{R}^2\to\mathbb{R}$ *be a locally Lipschitz function satisfying* (F1)–(F4) *with* $ps=qr$ *and*

$$\mu>\max\big\{p,N(r-p)/p\big\}\quad\textit{and}\quad\nu>\max\big\{q,N(s-q)/q\big\}.\tag{4}$$

Then $(\mathrm{S}'_{p,q,\Omega})$ *possesses at least a nonzero solution whose components are axially symmetric.*

An element $u\in W_0^{1,\alpha}(\Omega)$ $(\alpha\in\{p,q\})$ is *axially symmetric* if $u(x,gy)=u(x,y)$ for all $x\in\omega$, $y\in\mathbb{R}^{N-m}$ and $g\in O(N-m)$. ($O(N-m)$ is the orthogonal group in $\mathbb{R}^{N-m}$.)

Remark 2. Theorem 1 extends or complements some of the above mentioned papers, even in the differentiable case. For instance, with respect to [2,12], we allow the unboundedness of the domain; the paper [9] deals only with the scalar case involving the Laplacian operator ($p=2$); and no homogeneity property is required on F, see [3].

Remark 3. The hypothesis $ps=qr$ is imposed by a technical reason. It will be used in several times and it seems to be indispensable (see Propositions 5 and 6), taking into account the unboundedness of Ω.

Remark 4. Relation (3) is a nonsmooth type of one introduced by Costa and Magalhães [8] (see also [2,7,12]). We should mention that when $p = q$, condition (3) is implied in many cases by the following condition (of Ambrosetti–Rabinowitz type):

$$\gamma F(u,v) + F_1^0(u,v;-u) + F_2^0(u,v;-v) \leqslant 0 \quad \text{for all } (u,v) \in \mathbb{R}^2, \tag{5}$$

where $\gamma > p$. For the smooth form of (5), see for instance [2,7,12]. Indeed, from (5) and Lebourg's mean value theorem (see Proposition 1(iv) below), applied to the locally Lipschitz function $g :]0,\infty[\to \mathbb{R}$, $g(t) = t^{-\gamma} F(tu,tv)$ (with arbitrary fixed $(u,v) \in \mathbb{R}^2$) we obtain that

$$t^{-\gamma} F(tu,tv) \geqslant s^{-\gamma} F(su,sv) \quad \text{for all } t \geqslant s > 0. \tag{5$'$}$$

If we assume in addition that

$$\liminf_{u,v\to 0} \frac{F(u,v)}{|u|^\gamma + |v|^\gamma} \geqslant a_0 > 0,$$

by (5′) we have for $(u,v) \neq (0,0)$

$$t^{-\gamma} F(tu,tv) \geqslant \liminf_{s\to 0^+} \frac{F(su,sv)}{|su|^\gamma + |sv|^\gamma} \big(|u|^\gamma + |v|^\gamma\big) \geqslant a_0\big(|u|^\gamma + |v|^\gamma\big).$$

Now, substituting $t = 1$ in the above inequality, this forces

$$pF(u,v) + F_1^0(u,v;-u) + F_2^0(u,v;-v) \leqslant (p-\gamma)F(u,v) \leqslant -c\big(|u|^\gamma + |v|^\gamma\big),$$

where $c = a_0(\gamma - p) > 0$. For $(u,v) = (0,0)$, relation (3) follows directly from (5).

Example 1. Let $p = 3/2$, $q = 9/4$, $\Omega =]a,b[\times \mathbb{R}^2$ $(a < b)$ and

$$F(u,v) = u^2 + |v|^{7/2} + 1/4 \max\big\{|u|^{5/2}, |v|^{5/2}\big\}.$$

Since F has neither homogeneity nor differentiability properties and the domain is not bounded, the earlier results (see [2,3]) cannot be applied. F being convex and locally Lipschitz function, (F2) holds (see Clarke [5, Proposition 2.3.6]), while (F4) can be verified easily. Choosing $r = 5/2$, $s = 15/4$ and $\mu = 5/2$, $\nu = 7/2$, the assumptions (F1), (F3) and (4) hold too. Therefore we can apply Theorem 1, obtaining at least a nonzero solution for $(\mathrm{S}'_{3/2,9/4,]a,b[\times\mathbb{R}^2})$.

To prove our main theorem, we define the function $\mathcal{H} : W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) \to \mathbb{R}$ by

$$\mathcal{H}(u,v) = \frac{1}{p}\int_\Omega |\nabla u|^p + \frac{1}{q}\int_\Omega |\nabla v|^q - \int_\Omega F(u,v)\,dx \tag{6}$$

for all $u \in W_0^{1,p}(\Omega)$, $v \in W_0^{1,q}(\Omega)$. We will prove that $\mathcal{H}$ is a locally Lipschitz function and $\mathcal{H}$ restricted to the subspace of axially symmetric functions of $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ satisfies the nonsmooth Cerami condition. Moreover, by means of the Mountain Pass Theorem, proved by Kourogenis and Papageorgiou [13], we obtain a critical point (in the sense of Chang [4]) of the restricted function, the components of this element being axially symmetric. Using the Principle of Symmetric Criticality for locally Lipschitz functions, proved by Krawcewicz and Marzantowicz [14], the above-mentioned point will be a critical point of $\mathcal{H}$ on the whole space $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, and consequently, a solution for our problem.

The paper is organized as follows: In Section 2, some facts about locally Lipschitz and regular functions are given; in Section 3 a key inequality is proved; in Section 4 the nonsmooth Cerami condition is verified for the function $\mathcal{H}$ restricted to the subspace of axially symmetric functions; in Section 5 we discuss the mountain pass geometry of the above-mentioned function while in the last section we will prove our theorem.

2. Basic notions

Let $(X, \|\cdot\|)$ be a real Banach space and X^* its topological dual. A function $h : X \to \mathbb{R}$ is called locally Lipschitz if each point $u \in X$ possesses a neighborhood $\mathcal{N}_u$ such that $|h(u_1) - h(u_2)| \leqslant L\|u_1 - u_2\|$ for all $u_1, u_2 \in \mathcal{N}_u$, for a constant $L > 0$ depending on $\mathcal{N}_u$.

The generalized directional derivative of h at the point $u \in X$ in the direction $z \in X$ is

$$h^0(u; z) = \limsup_{w \to u,\, t \to 0^+} \frac{h(w + tz) - h(w)}{t}$$

(see [5]). The generalized gradient of h at $u \in X$ is defined by

$$\partial h(u) = \left\{x^* \in X^*:\ \langle x^*, z\rangle_X \leqslant h^0(u; z) \text{ for all } z \in X\right\},$$

which is a nonempty, convex and w^*-compact subset of X^*, where $\langle \cdot, \cdot\rangle_X$ is the duality pairing between X^* and X.

A point $u \in X$ is a *critical point of* h if $0 \in \partial h(u)$, that is $h^0(u; w) \geqslant 0$ for all $w \in X$. In this case, $h(u)$ is a *critical value of* h. We define $\lambda_h(u) = \inf\{\|x^*\|_X:\ x^* \in \partial h(u)\}$ (we will use the notation $\|x^*\|_X$ instead of $\|x^*\|_{X^*}$).

The function h *satisfies the nonsmooth Cerami condition at level* $c \in \mathbb{R}$ (shortly $(C)_c$), if every sequence $\{x_n\} \subset X$ such that $h(x_n) \to c$ and $(1 + \|x_n\|)\lambda_h(x_n) \to 0$ contains a convergent subsequence in the norm of X (see [13]).

Now, we list some fundamental properties of the directional derivative and generalized gradient which will be used throughout the paper.

Proposition 1 [5].

(i) $(-h)^0(u; z) = h^0(u; -z)$ *for all* $u, z \in X$.
(ii) $h^0(u; z) = \max\{\langle x^*, z\rangle_X:\ x^* \in \partial h(u)\}$ *for all* $u, z \in X$.
(iii) *Let* $j : X \to \mathbb{R}$ *be a continuously differentiable function. Then* $\partial j(u) = \{j'(u)\}$, $j^0(u; z)$ *coincides with* $\langle j'(u), z\rangle_X$ *and* $(h + j)^0(u; z) = h^0(u; z) + \langle j'(u), z\rangle_X$ *for all* $u, z \in X$. *Moreover,* $\partial(hj)(u) \subseteq j(u)\partial h(u) + h(u)j'(u)$ *for all* $u \in X$.
(iv) (*Lebourg's mean value theorem*) *Let* u *and* v *two points in* X. *Then there exists a point* w *in the open segment between* u *and* v, *and* $x_w^* \in \partial h(w)$ *such that*

$$h(u) - h(v) = \left\langle x_w^*, u - v\right\rangle_X.$$

(v) (*Second Chain Rule*) *Let Y be a Banach space and $j:Y\to X$ a continuously differentiable function. Then $h\circ j$ is locally Lipschitz and*

$$\partial(h\circ j)(y)\subseteq\partial h\bigl(j(y)\bigr)\circ j'(y)\quad\text{for all } y\in Y.$$

We say that h is *regular at $u\in X$ in the sense of Clarke* [5] (shortly, *regular at $u\in X$*), if for all $z\in X$ the usual one-sided directional derivative

$$h'(u;z)=\lim_{t\to 0^+}\frac{h(u+tz)-h(u)}{t}$$

exists and $h'(u;z)=h^0(u;z)$. h is *regular on X in the sense of Clarke* (shortly, *regular on X*) if it is regular at every point $u\in X$.

Proposition 2. *Let $h:X\times X\to\mathbb{R}$ be a locally Lipschitz function which is regular at $(u,v)\in X\times X$. Then*

(i) $\partial h(u,v)\subseteq\partial_1 h(u,v)\times\partial_2 h(u,v)$, *where $\partial_1 h(u,v)$ denotes the* (*partial*) *generalized gradient of $h(\cdot,v)$ at the point u, and $\partial_2 h(u,v)$ that of $h(u,\cdot)$ at v.*
(ii) $h^0(u,v;w,z)\leqslant h_1^0(u,v;w)+h_2^0(u,v;z)$ *for all $w,z\in X$, where $h_1^0(u,v;w)$* (*resp. $h_2^0(u,v;z)$*) *is the* (*partial*) *generalized directional derivative of $h(\cdot,v)$* (*resp. $h(u,\cdot)$*) *at the point $u\in\mathbb{R}$* (*resp. $v\in\mathbb{R}$*) *in the direction $w\in\mathbb{R}$* (*resp. $z\in\mathbb{R}$*).

Proof. For (i), see [5, Proposition 2.3.15]. Now, let us fix $w,z\in X$. From Proposition 1(ii) it follows that there exists $x^*\in\partial h(u,v)$ such that

$$h^0(u,v;w,z)=\bigl\langle x^*,(w,z)\bigr\rangle_{X\times X}.$$

By (i) we have $x^*=(x_1^*,x_2^*)$, where $x_i^*\in\partial_i h(u,v)$ $(i\in\{1,2\})$, and using the definition of the generalized gradient, we obtain $h^0(u,v;w,z)=\langle x_1^*,w\rangle_X+\langle x_2^*,z\rangle_X\leqslant h_1^0(u,v;w)+h_2^0(u,v;z)$. □

3. A key inequality

Throughout the paper, the usual norm of $L^\beta(\Omega)$ will be denoted by $\|\cdot\|_\beta$ $(\beta>1)$. Since Ω has the cone property, we have the Sobolev embedding $W_0^{1,\alpha}(\Omega)\hookrightarrow L^\beta(\Omega)$ $(\beta\in[\alpha,\alpha^*],\ \alpha\in\{p,q\})$, and $W_0^{1,\alpha}(\Omega)$ can be endowed with the norm

$$\|u\|_{1,\alpha}=\left(\int_\Omega|\nabla u|^\alpha\right)^{1/\alpha}\quad\bigl(\alpha\in\{p,q\}\bigr).$$

Let $c_{\beta,\alpha} > 0$ be the embedding constant, i.e., $\|u\|_\beta \leqslant c_{\beta,\alpha}\|u\|_{1,\alpha}$ for all $u \in W_0^{1,\alpha}(\Omega)$. The product space $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ will be endowed with the norm $\|(u,v)\|_{1,p,q} = \|u\|_{1,p} + \|v\|_{1,q}$ and we define the function $\mathcal{F}: W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) \to \mathbb{R}$ by

$$\mathcal{F}(u,v) = \int_\Omega F(u,v)\,dx$$

for $u \in W_0^{1,p}(\Omega),\ v \in W_0^{1,q}(\Omega)$.

Proposition 3. *If $F:\mathbb{R}^2 \to \mathbb{R}$ is a locally Lipschitz function which verifies* (F1) *and* (F2), *then the function $\mathcal{F}$ is well-defined and locally Lipschitz. Suppose in addition that E_p and E_q are closed subspaces of $W_0^{1,p}(\Omega)$ and $W_0^{1,q}(\Omega)$, respectively. If $\mathcal{F}_E$ denotes the restriction of $\mathcal{F}$ to $E = E_p \times E_q$ then*

$$\mathcal{F}_E^0(u,v;w,y) \leqslant \int_\Omega F^0\bigl(u(x),v(x);w(x),y(x)\bigr)\,dx$$

for all $u,w \in E_p$ and $v,y \in E_q$.

Proof. Let us fix $u,v,w,y \in \mathbb{R}$. By Lebourg's mean value theorem we have an element $z \in \partial F(\theta u + (1-\theta)w, \theta v + (1-\theta)y)$ with $\theta \in \,]0,1[$ such that

$$F(u,v) - F(w,y) = \bigl\langle z, (u-w, v-y)\bigr\rangle_{\mathbb{R}^2}.$$

Since F is regular on $\mathbb{R}^2$, using Proposition 2(i), we have $z_i = z_i(\theta,u,v,w,y) \in \partial_i F(\theta u + (1-\theta)w, \theta v + (1-\theta)y)$ $(i \in \{1,2\})$ such that

$$F(u,v) - F(w,y) = z_1(u-w) + z_2(v-y).$$

From relations (1), (2) and from the fact that for all $\beta \in \,]0,\infty[$ there is a constant $c(\beta) > 0$ such that

$$(x+y)^\beta \leqslant c(\beta)(x^\beta + y^\beta) \quad \text{for all } x,y \in [0,\infty[,$$

we have

$$\begin{aligned}
&\bigl|F(u,v) - F(w,y)\bigr| \\
&\quad\leqslant c_3\Bigl[|u-w|\bigl(|u|^{p-1} + |w|^{p-1} + |v|^{(p-1)q/p} + |y|^{(p-1)q/p} + |u|^{r-1} + |w|^{r-1}\bigr) \\
&\qquad + |v-y|\bigl(|v|^{q-1} + |y|^{q-1} + |u|^{(q-1)p/q} + |w|^{(q-1)p/q} + |v|^{s-1} + |y|^{s-1}\bigr)\Bigr],
\end{aligned} \tag{7}$$

where $c_3 = c_3(c_1,p,q,r,s) > 0$. Now, we fix $u,w \in W_0^{1,p}(\Omega)$ and $v,y \in W_0^{1,q}(\Omega)$ arbitrary. Using Holder's inequality, from (7) we have

$$\begin{aligned}
\bigl|\mathcal{F}(u,v) - \mathcal{F}(w,y)\bigr| \leqslant c_3\bigl\|(u,v)-(w,y)\bigr\|_{1,p,q}\Bigl[&c_{p,p}^p\bigl(\|u\|_{1,p}^{p-1} + \|w\|_{1,p}^{p-1}\bigr) \\
&+ c_{p,p}c_{q,q}^{(p-1)q/p}\bigl(\|v\|_{1,q}^{(p-1)q/p} + \|y\|_{1,q}^{(p-1)q/p}\bigr)
\end{aligned}$$

$$+ c_{r,p}^r\big(\|u\|_{1,p}^{r-1} + \|w\|_{1,p}^{r-1}\big) + c_{q,q}^q\big(\|v\|_{1,q}^{q-1} + \|y\|_{1,q}^{q-1}\big)$$
$$+ c_{q,q}c_{p,p}^{(q-1)p/q}\big(\|u\|_{1,p}^{(q-1)p/q} + \|w\|_{1,p}^{(q-1)p/q}\big)$$
$$+ c_{s,q}^s\big(\|v\|_{1,q}^{s-1} + \|y\|_{1,q}^{s-1}\big)\Big].$$

Since $F(0,0)=0$, $\mathcal{F}$ is well-defined. Moreover, the Lipschitz property for $\mathcal{F}$ is verified on bounded sets of $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$.

Now, we fix $u, w \in E_p$ and $v, y \in E_q$. By definition, $F^0(u(x), v(x); w(x), y(x))$ can be written as the upper limit of

$$\frac{F(z^u + tw(x), z^v + ty(x)) - F(z^u, z^v)}{t},$$

where $(z^u, z^v) \to (u(x), v(x))$ take values in a countable dense subset of $\mathbb{R}^2$ and $t \to 0^+$ take rational values. Being the upper limit of measurable functions of $x \in \Omega$, the function $\Omega \ni x \mapsto F^0(u(x), v(x); w(x), y(x))$ is also measurable. Moreover, due to Proposition 1(ii) and relations (1) and (2), the above function belongs to $L^1(\Omega)$.

Since $E_p \times E_q$ is a closed subspace of a separable Banach space, there exist elements $u_n \in E_p$, $v_n \in E_q$ and numbers $t_n \to 0^+$ such that (u_n, v_n) converges (strongly) to (u, v) in $E_p \times E_q$ and

$$\mathcal{F}_E^0(u, v; w, y) = \lim_{n\to\infty} \frac{\mathcal{F}_E(u_n + t_n w, v_n + t_n y) - \mathcal{F}_E(u_n, v_n)}{t_n}.$$

Moreover, without loss of generality, we may assume that

$$u_n(x) \to u(x), \quad v_n(x) \to v(x) \quad \text{a.e. } x \in \Omega, \tag{8}$$

and there exist $h_\alpha \in L^\alpha(\Omega, \mathbb{R}_+)$ ($\alpha \in \{p, q, r, s\}$) such that

$$\big|u_n(x)\big| \leqslant \min\big\{h_p(x), h_r(x)\big\} \quad \text{and} \quad \big|v_n(x)\big| \leqslant \min\big\{h_q(x), h_s(x)\big\} \tag{9}$$

a.e. $x \in \Omega$. Let $g_n : \Omega \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\begin{aligned} g_n(x) = {} & -\frac{F(u_n(x) + t_n w(x), v_n(x) + t_n y(x)) - F(u_n(x), v_n(x))}{t_n} \\ & + c_3\Big[\big|w(x)\big|\big(\big|u_n(x)\big|^{p-1} + \big|u_n(x) + t_n w(x)\big|^{p-1} + \big|u_n(x)\big|^{r-1} \\ & + \big|u_n(x) + t_n w(x)\big|^{r-1} + \big|v_n(x) + t_n y(x)\big|^{(p-1)q/p} + \big|v_n(x)\big|^{(p-1)q/p}\big) \\ & + \big|y(x)\big|\big(\big|v_n(x)\big|^{q-1} + \big|v_n(x) + t_n y(x)\big|^{q-1} + \big|v_n(x)\big|^{s-1} \\ & + \big|v_n(x) + t_n y(x)\big|^{s-1} + \big|u_n(x) + t_n w(x)\big|^{(q-1)p/q} + \big|u_n(x)\big|^{(q-1)p/q}\big)\Big]. \end{aligned}$$

The function g_n is measurable, and due to (7), it is nonnegative. Fatou's lemma implies that

$$A = \int_\Omega \limsup_{n\to\infty}\big[-g_n(x)\big]\,dx \geqslant \limsup_{n\to\infty} \int_\Omega \big[-g_n(x)\big]\,dx = B.$$

Let $D_n = g_n + C_n$, where

$$C_n(x) = \frac{F(u_n(x) + t_n w(x), v_n(x) + t_n y(x)) - F(u_n(x), v_n(x))}{t_n}.$$

By the Lebesgue dominated convergence theorem (using (8) and (9)), we have

$$\lim_{n\to\infty} \int_\Omega D_n\,dx = 2c_3 \int_\Omega \big[|w|\big(|u|^{p-1} + |u|^{r-1} + |v|^{(p-1)q/p}\big) + |y|\big(|v|^{q-1} + |v|^{s-1} + |u|^{(q-1)p/q}\big)\big]\,dx.$$

Therefore,

$$\begin{aligned} B &= \limsup_{n\to\infty} \frac{\mathcal{F}_E(u_n + t_n w, v_n + t_n y) - \mathcal{F}_E(u_n, v_n)}{t_n} - \lim_{n\to\infty} \int_\Omega D_n\,dx \\ &= \mathcal{F}_E^0(u, v; w, y) - 2c_3 \int_\Omega \big[|w|\big(|u|^{p-1} + |u|^{r-1} + |v|^{(p-1)q/p}\big) \\ &\quad + |y|\big(|v|^{q-1} + |v|^{s-1} + |u|^{(q-1)p/q}\big)\big]\,dx. \end{aligned}$$

On the other hand, $A \leqslant A_1 - A_2$, where

$$A_1 = \int_\Omega \limsup_{n\to\infty} C_n(x)\,dx \quad \text{and} \quad A_2 = \int_\Omega \liminf_{n\to\infty} D_n(x)\,dx.$$

By (8), we have

$$A_2 = 2c_3 \int_\Omega \big[|w|\big(|u|^{p-1} + |u|^{r-1} + |v|^{(p-1)q/p}\big) + |y|\big(|v|^{q-1} + |v|^{s-1} + |u|^{(q-1)p/q}\big)\big]\,dx$$

while

$$\begin{aligned} A_1 &= \int_\Omega \limsup_{n\to\infty} \frac{F(u_n(x) + t_n w(x), v_n(x) + t_n y(x)) - F(u_n(x), v_n(x))}{t_n}\,dx \\ &\leqslant \int_\Omega \limsup_{(z^u, z^v, t)\to(u(x), v(x), 0^+)} \frac{F(z^u + t w(x), z^v + t y(x)) - F(z^u, z^v)}{t}\,dx \\ &= \int_\Omega F^0\big(u(x), v(x); w(x), y(x)\big)\,dx. \end{aligned}$$

This completes the proof. □

Remark 5. We point out that, while the inequality given in Proposition 3 is proved for (subspaces of) the Sobolev space $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, under suitable growth conditions on the nonlinear term F, a similar inequality is known in the case of integral functionals on the $L^p(\Omega) \times L^q(\Omega)$ spaces, see [5, pp. 82–85]. We emphasize that in [5], the fact that Ω has finite measure plays an indispensable role.

We have the following relation between the critical points of $\mathcal{H}$ and the solutions of $(\mathrm{S}'_{p,q,\Omega})$.

Proposition 4. *Under the conditions of Proposition* 3, *the function* $\mathcal{H}$ (*from* (6)) *is well-defined and locally Lipschitz on* $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$. *Moreover, every critical point* $(u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ *of* $\mathcal{H}$ *is a solution of* $(\mathrm{S}'_{p,q,\Omega})$.

Proof. Since the function $W_0^{1,\alpha}(\Omega) \ni u \mapsto \frac{1}{\alpha}\|u\|_{1,\alpha}^{\alpha}$ is of class C^1 ($\alpha \in \{p, q\}$), the first part follows from Proposition 3. Now, we choose $E_p = W_0^{1,p}(\Omega)$ and $E_q = W_0^{1,q}(\Omega)$ in Proposition 3. Due to Proposition 1(i), for all $(w, y) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ we have

$$\begin{aligned}
0 &\leqslant \mathcal{H}^0(u, v; w, y) \\
&= \int_\Omega |\nabla u|^{p-2}\nabla u\nabla w + \int_\Omega |\nabla v|^{q-2}\nabla v\nabla y + (-\mathcal{F})^0(u, v; w, y) \\
&= \int_\Omega |\nabla u|^{p-2}\nabla u\nabla w + \int_\Omega |\nabla v|^{q-2}\nabla v\nabla y + \mathcal{F}^0(u, v; -w, -y) \\
&\leqslant \int_\Omega |\nabla u|^{p-2}\nabla u\nabla w + \int_\Omega |\nabla v|^{q-2}\nabla v\nabla y \\
&\quad + \int_\Omega F^0\big(u(x), v(x); -w(x), -y(x)\big)\,dx.
\end{aligned}$$

By using Proposition 2(ii) and taking $y = 0$, respectively $w = 0$ in the above inequality, we obtain the corresponding inequalities from $(\mathrm{S}'_{p,q,\Omega})$. $\square$

Remark 6. A natural question arises: Can the converse of the last part of Proposition 4 be proved, i.e., can one characterize the critical points of $\mathcal{H}$ by means of the solutions of $(\mathrm{S}'_{p,q,\Omega})$? Unfortunately, it seems that this cannot be done. To see why, let us put ourselves within the assumptions of Proposition 4. Using Fatou's lemma, a calculation similar to that in the proof of Proposition 3 shows that

$$\mathcal{F}^0(u, v; w, y) = \mathcal{F}'(u, v; w, y) = \int_\Omega F^0\big(u(x), v(x); w(x), y(x)\big)\,dx. \tag{10}$$

(For a similar relation on domains with finite measure and suitable growth conditions on the term F, see [5, p. 85].) From (10), one has

$$\begin{aligned}
\mathcal{H}^0(u, v; w, y) = {}& \int_\Omega |\nabla u|^{p-2}\nabla u\nabla w + \int_\Omega |\nabla v|^{q-2}\nabla v\nabla y \\
& + \int_\Omega F^0\big(u(x), v(x); -w(x), -y(x)\big)\,dx.
\end{aligned} \tag{11}$$

If we were dealing with a *scalar problem*, where the nonlinear term is *regular*, then we would be able to obtain a formula similar to (11). In such a case, an element would be a critical point for the corresponding functional *if and only if* it would be a solution of the studied problem. But, in our *nonscalar case*, the problem has a different behaviour. The main difficulty is caused by the inequality from Proposition 2(ii), which may be *strict*. Indeed, let us consider for instance $F:\mathbb{R}^2 \to \mathbb{R}$, defined by $F(u,v) = \max\{|u|^{5/2}, |v|^{5/2}\}$. It is clear that F is regular on $\mathbb{R}^2$ in the sense of Clarke and for every $\alpha, \beta > 0$, $F^0(\alpha,\alpha;\beta,\beta) = F_1^0(\alpha,\alpha;\beta) = F_2^0(\alpha,\alpha;\beta) = 5\alpha^{3/2}\beta/2$.

Now, if we suppose that (u,v) is a solution of $(\mathrm{S}'_{p,q,\Omega})$, we *cannot* assert that $\mathcal{H}^0(u,v;w,y) \geqslant 0$ for all $(w,y) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, taking into account that the inequality from Proposition 2(ii) may be strict. So, it seems we need *more* regularity on F, not only the regularity in the sense of Clarke, in order to prove this implication. In spite of the fact that formula (11) is more precise than in the proof of Proposition 4, the latter cannot be improved. This fact is another point where our approach differs from the scalar case.

4. The Cerami condition

Since the embeddings $W_0^{1,\alpha}(\Omega) \hookrightarrow L^\beta(\Omega)$ for $\beta \in [\alpha, \alpha^*]$ $(\alpha \in \{p,q\})$ are not compact, we introduce the action of $G = \mathrm{id}^m \times O(N-m)$ on $W_0^{1,\alpha}(\Omega)$ as

$$gu(x,y) = u\big(x, g_0^{-1}y\big)$$

for all $(x,y) \in \omega \times \mathbb{R}^{N-m}$, $g = \mathrm{id}^m \times g_0 \in G$ and $u \in W_0^{1,\alpha}(\Omega)$ $(\alpha \in \{p,q\})$. Moreover, the action G on $W_0^{1,\alpha}(\Omega)$ is isometric, that is $\|gu\|_{1,\alpha} = \|u\|_{1,\alpha}$ for all $g \in G$, $u \in W_0^{1,\alpha}(\Omega)$ $(\alpha \in \{p,q\})$. Let us denote by

$$W_{0,G}^{1,\alpha}(\Omega) = \big\{u \in W_0^{1,\alpha}(\Omega)\colon\ gu = u \text{ for all } g \in G\big\} \quad \big(\alpha \in \{p,q\}\big),$$

which is exactly the closed subspace of axially symmetric functions of $W_0^{1,\alpha}(\Omega)$. The embeddings $W_{0,G}^{1,\alpha}(\Omega) \hookrightarrow L^\beta(\Omega)$, $\alpha < \beta < \alpha^*$ $(\alpha \in \{p,q\})$ are compact (see [10,17], and [9] for $\alpha = 2$). In the sequel, we denote by $\mathcal{F}_G$ and $\mathcal{H}_G$ the restrictions of $\mathcal{F}$ and $\mathcal{H}$ to $W_{0,G}^{1,p}(\Omega) \times W_{0,G}^{1,q}(\Omega)$, respectively.

Proposition 5. *Under the conditions of Theorem* 1, $\mathcal{H}_G$ *satisfies* $(C)_c$ *for all* $c > 0$.

Proof. Let $\{(u_n, v_n)\}$ be a sequence from $W_{0,G}^{1,p}(\Omega) \times W_{0,G}^{1,q}(\Omega)$ such that

$$\mathcal{H}_G(u_n, v_n) \to c > 0, \tag{12}$$

$$\big(1 + \big\|(u_n, v_n)\big\|_{1,p,q}\big)\lambda_{\mathcal{H}_G}(u_n, v_n) \to 0, \tag{13}$$

as $n \to \infty$. Since $\partial\mathcal{H}_G(u_n, v_n)$ is w^*-compact, we can fix $z_n^* \in \partial\mathcal{H}_G(u_n, v_n)$ such that $\lambda_{\mathcal{H}_G}(u_n, v_n) = \|z_n^*\|_*$, where $\|\cdot\|_*$ denotes the norm of the dual of $W_{0,G}^{1,p}(\Omega) \times W_{0,G}^{1,q}(\Omega)$. Moreover, we have

$$\mathcal{H}_G^0\left(u_n, v_n; \frac{1}{p}u_n, \frac{1}{q}v_n\right) \geqslant \left\langle z_n^*, \left(\frac{1}{p}u_n, \frac{1}{q}v_n\right)\right\rangle_{W_{0,G}^{1,p}(\Omega)\times W_{0,G}^{1,q}(\Omega)}$$
$$\geqslant -\big(1 + \big\|(u_n, v_n)\big\|_{1,p,q}\big)\big\|z_n^*\big\|_*.$$

Applying Proposition 3 with $E_p = W_{0,G}^{1,p}(\Omega)$ and $E_q = W_{0,G}^{1,q}(\Omega)$, and using (12), (13) and (3), one has for n large enough that

$$\begin{aligned}
c+1 &\geqslant \mathcal{H}_G(u_n, v_n) - \mathcal{H}_G^0\left(u_n, v_n; \frac{1}{p}u_n, \frac{1}{q}v_n\right)\\
&= -\mathcal{F}_G(u_n, v_n) - (-\mathcal{F}_G)^0\left(u_n, v_n; \frac{1}{p}u_n, \frac{1}{q}v_n\right)\\
&= -\mathcal{F}_G(u_n, v_n) - \mathcal{F}_G^0\left(u_n, v_n; -\frac{1}{p}u_n, -\frac{1}{q}v_n\right)\\
&\geqslant -\int_\Omega \left[F(u_n, v_n) + F^0\left(u_n(x), v_n(x); -\frac{1}{p}u_n(x), -\frac{1}{q}v_n(x)\right)\right]dx\\
&\geqslant -\int_\Omega \bigg[F(u_n, v_n) + \frac{1}{p}F_1^0\big(u_n(x), v_n(x); -u_n(x)\big)\\
&\quad + \frac{1}{q}F_2^0\big(u_n(x), v_n(x); -v_n(x)\big)\bigg]dx \geqslant c_2\int_\Omega \big[|u_n|^\mu + |v_n|^\nu\big]dx.
\end{aligned}$$

From the previous inequality we obtain that

$$\big\{(u_n, v_n)\big\} \text{ is bounded in } L^\mu(\Omega)\times L^\nu(\Omega). \tag{14}$$

By (F4) we have that for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $|u|^{p-1} + |v|^{(p-1)q/p} < \delta(\varepsilon)$ then

$$|w_u| < \varepsilon\big(|u|^{p-1} + |v|^{(p-1)q/p}\big) \quad \text{for all } w_u \in \partial_1 F(u, v).$$

If $|u|^{p-1} + |v|^{(p-1)q/p} \geqslant \delta(\varepsilon)$, by using (1), we have

$$\begin{aligned}
|w_u| &\leqslant c_1\Big[\big(|u|^{p-1} + |v|^{(p-1)q/p}\big)^{(r-1)/(p-1)}\big(\delta(\varepsilon)\big)^{(p-r)/(p-1)} + |u|^{r-1}\Big]\\
&\leqslant c(\varepsilon)\big(|u|^{r-1} + |v|^{(r-1)q/p}\big).
\end{aligned}$$

Combining the above relations, we have that for all $\varepsilon > 0$ there exists $c_1(\varepsilon) > 0$ such that

$$|w_u| < \varepsilon\big(|u|^{p-1} + |v|^{(p-1)q/p}\big) + c_1(\varepsilon)\big(|u|^{r-1} + |v|^{(r-1)q/p}\big) \tag{15}$$

for all $(u, v) \in \mathbb{R}^2$ and $w_u \in \partial_1 F(u, v)$. A similar calculation shows that for all $\varepsilon > 0$ there exists $c_2(\varepsilon) > 0$ such that

$$|w_v| < \varepsilon\big(|v|^{q-1} + |u|^{(q-1)p/q}\big) + c_2(\varepsilon)\big(|v|^{s-1} + |u|^{(s-1)p/q}\big) \tag{16}$$

for all $(u, v) \in \mathbb{R}^2$ and $w_v \in \partial_2 F(u, v)$.

Similarly as in (7), but using (15) and (16) instead of (1) and (2), respectively, and keeping in mind that $F(0, 0) = 0$, for all $\varepsilon > 0$ there exists $c(\varepsilon) = c(c_1(\varepsilon), c_2(\varepsilon)) > 0$ such that

$$F(u,v) \leqslant \varepsilon\big(|u|^p + |v|^{(p-1)q/p}|u| + |v|^q + |u|^{(q-1)p/q}|v|\big) \\ + c(\varepsilon)\big(|u|^r + |v|^{(r-1)q/p}|u| + |v|^s + |u|^{(s-1)p/q}|v|\big) \tag{17}$$

for all $(u,v) \in \mathbb{R}^2$.

After integration and using relation $ps = qr$, by Young's and Holder's inequalities we obtain

$$\mathcal{F}_G(u_n, v_n) \leqslant \varepsilon\left[\left(2 + \frac{1}{p} - \frac{1}{q}\right)\|u_n\|_p^p + \left(2 + \frac{1}{q} - \frac{1}{p}\right)\|v_n\|_q^q\right] \\ + c(\varepsilon)\left[\left(2 + \frac{1}{r} - \frac{1}{s}\right)\|u_n\|_r^r + \left(2 + \frac{1}{s} - \frac{1}{r}\right)\|v_n\|_s^s\right].$$

Therefore, due to (6), one has

$$\left[\frac{1}{p} - \varepsilon\left(2 + \frac{1}{p} - \frac{1}{q}\right)c_{p,p}^p\right]\|u_n\|_{1,p}^p + \left[\frac{1}{q} - \varepsilon\left(2 + \frac{1}{q} - \frac{1}{p}\right)c_{q,q}^q\right]\|v_n\|_{1,q}^q \\ \leqslant \mathcal{H}_G(u_n, v_n) + c(\varepsilon)\left[\left(2 + \frac{1}{r} - \frac{1}{s}\right)\|u_n\|_r^r + \left(2 + \frac{1}{s} - \frac{1}{r}\right)\|v_n\|_s^s\right].$$

Choosing

$$0 < \varepsilon < \frac{1}{3}\min\left\{\frac{1}{pc_{p,p}^p}, \frac{1}{qc_{q,q}^q}\right\}, \tag{18}$$

we find $c_3(\varepsilon), c_4(\varepsilon) > 0$ such that

$$c_3(\varepsilon)\big(\|u_n\|_{1,p}^p + \|v_n\|_{1,q}^q\big) \leqslant c + 1 + c_4(\varepsilon)\big(\|u_n\|_r^r + \|v_n\|_s^s\big) \tag{19}$$

for n large enough. Now, we will examine the behaviour of the sequences $\{\|u_n\|_r^r\}$ and $\{\|v_n\|_s^s\}$, respectively. To this end, we first observe that $\mu \leqslant r$ and $\nu \leqslant s$. Indeed, keeping in mind relation $ps = qr$, letting $w = y = 0$ and $u := ut^{1/p}$, $v := vt^{1/q}$ $(t > 1)$ in (7), from (30) below yield the required relations.

We distinguish two cases.

(I) $\mu = r$. From (14) we have that $\{\|u_n\|_r^r\}$ is bounded.

(II) $\mu \in \,]\max\{p, N(r-p)/p\}, r[$. We have the interpolation inequality

$$\|u\|_r \leqslant \|u\|_\mu^{1-\delta}\|u\|_{p^*}^\delta \quad \text{for all } u \in L^\mu(\Omega) \cap L^{p^*}(\Omega)$$

with

$$\delta = \frac{p^*}{r}\frac{r-\mu}{p^*-\mu}.$$

From (14) and the continuous embedding $W_{0,G}^{1,p}(\Omega) \subset W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, we have $c_5 > 0$ such that $\|u_n\|_r^r \leqslant c_5\|u_n\|_{1,p}^{\delta r}$, with $\delta r < p$.

Taking into consideration the similar relations for the sequence $\{\|v_n\|_s^s\}$, we can conclude from (19) that the sequences $\{\|u_n\|_{1,p}\}$ and $\{\|v_n\|_{1,q}\}$ are bounded. Since the embeddings $W_{0,G}^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$, $W_{0,G}^{1,q}(\Omega) \hookrightarrow L^s(\Omega)$ are compact, up to a subsequence, we have

$$(u_n, v_n) \to (u, v) \quad \text{weakly in } W_{0,G}^{1,p}(\Omega) \times W_{0,G}^{1,q}(\Omega), \tag{20}$$
$$u_n \to u \quad \text{strongly in } L^r(\Omega), \tag{21}$$
$$v_n \to v \quad \text{strongly in } L^s(\Omega). \tag{22}$$

Moreover, we have

$$\begin{aligned}
&\mathcal{H}_G^0(u_n, v_n; u - u_n, v - v_n) \\
&\quad = \int_\Omega |\nabla u_n|^{p-2} \nabla u_n (\nabla u - \nabla u_n) + \int_\Omega |\nabla v_n|^{q-2} \nabla v_n (\nabla v - \nabla v_n) \\
&\qquad + (-\mathcal{F}_G)^0(u_n, v_n; u - u_n, v - v_n)
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{H}_G^0(u, v; u_n - u, v_n - v) \\
&\quad = \int_\Omega |\nabla u|^{p-2} \nabla u (\nabla u_n - \nabla u) + \int_\Omega |\nabla v|^{q-2} \nabla v (\nabla v_n - \nabla v) \\
&\qquad + (-\mathcal{F}_G)^0(u, v; u_n - u, v_n - v).
\end{aligned}$$

Adding the above two relations, we obtain

$$\begin{aligned}
J_n :=& \int_\Omega \big(|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u\big)(\nabla u_n - \nabla u) \\
&+ \int_\Omega \big(|\nabla v_n|^{q-2} \nabla v_n - |\nabla v|^{q-2} \nabla v\big)(\nabla v_n - \nabla v) = J_n^1 - J_n^2 - J_n^3,
\end{aligned} \tag{23}$$

where

$$J_n^1 = \mathcal{F}_G^0(u_n, v_n; u_n - u, v_n - v) + \mathcal{F}_G^0(u, v; u - u_n, v - v_n),$$
$$J_n^2 = \mathcal{H}_G^0(u_n, v_n; u - u_n, v - v_n)$$

and

$$J_n^3 = \mathcal{H}_G^0(u, v; u_n - u, v_n - v).$$

In the sequel, we will estimate J_n^i ($i \in \{1, 2, 3\}$). Using Proposition 3, (15), (16) and $ps = qr$, one has

$$\begin{aligned}
J_n^1 \leqslant& \int_\Omega \Big[F^0\big(u_n(x), v_n(x); u_n(x) - u(x), v_n(x) - v(x)\big) \\
&+ F^0\big(u(x), v(x); u(x) - u_n(x), v(x) - v_n(x)\big)\Big]\, dx \\
\leqslant& \int_\Omega \Big[\big|F_1^0\big(u_n(x), v_n(x); u_n(x) - u(x)\big)\big| + \big|F_2^0\big(u_n(x), v_n(x); v_n(x) - v(x)\big)\big| \\
&+ \big|F_1^0\big(u(x), v(x); u(x) - u_n(x)\big)\big| + \big|F_2^0\big(u(x), v(x); v(x) - v_n(x)\big)\big|\Big]\, dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega \Big|\max\big\{w_n^1(x)\big(u_n(x)-u(x)\big):\ w_n^1(x)\in\partial_1 F\big(u_n(x),v_n(x)\big)\big\}\Big|\,dx\\
&\quad + \int_\Omega \Big|\max\big\{w_n^2(x)\big(v_n(x)-v(x)\big):\ w_n^2(x)\in\partial_2 F\big(u_n(x),v_n(x)\big)\big\}\Big|\,dx\\
&\quad + \int_\Omega \Big|\max\big\{w^1(x)\big(u(x)-u_n(x)\big):\ w^1(x)\in\partial_1 F\big(u(x),v(x)\big)\big\}\Big|\,dx\\
&\quad + \int_\Omega \Big|\max\big\{w^2(x)\big(v(x)-v_n(x)\big):\ w^2(x)\in\partial_2 F\big(u(x),v(x)\big)\big\}\Big|\,dx\\
&\leqslant \varepsilon\Big[\big(\|u_n\|_p^{p-1}+\|u\|_p^{p-1}+\|v_n\|_q^{(p-1)q/p}+\|v\|_q^{(p-1)q/p}\big)\|u-u_n\|_p\\
&\quad + \big(\|v_n\|_q^{q-1}+\|v\|_q^{q-1}+\|u_n\|_p^{(q-1)p/q}+\|u\|_p^{(q-1)p/q}\big)\|v-v_n\|_q\Big]\\
&\quad + c_1(\varepsilon)\big(\|u_n\|_r^{r-1}+\|u\|_r^{r-1}+\|v_n\|_s^{(r-1)s/r}+\|v\|_s^{(r-1)s/r}\big)\|u-u_n\|_r\\
&\quad + c_2(\varepsilon)\big(\|v_n\|_s^{s-1}+\|v\|_s^{s-1}+\|u_n\|_r^{(s-1)r/s}+\|u\|_r^{(s-1)r/s}\big)\|v-v_n\|_s.
\end{aligned}
$$

Since the sequences $\{u_n\}$ and $\{v_n\}$ are bounded in $W_{0,G}^{1,p}(\Omega)$ ($\hookrightarrow L^p(\Omega)\cap L^r(\Omega)$) and $W_{0,G}^{1,q}(\Omega)$ ($\hookrightarrow L^q(\Omega)\cap L^s(\Omega)$), respectively, and using relations (21) and (22), from the arbitrariness of $\varepsilon>0$ one has

$$\limsup_{n\to\infty} J_n^1 \leqslant 0. \tag{24}$$

Since

$$
\begin{aligned}
J_n^2 &\geqslant \big\langle z_n^*, (u-u_n, v-v_n)\big\rangle_{W_{0,G}^{1,p}(\Omega)\times W_{0,G}^{1,q}(\Omega)}\\
&\geqslant -\big\|z_n^*\big\|_*\big(\|u-u_n\|_{1,p}+\|v-v_n\|_{1,q}\big),
\end{aligned}
$$

due to (13), we have

$$\liminf_{n\to\infty} J_n^2 \geqslant 0. \tag{25}$$

Now, we fix an element $z^*\in\partial\mathcal{H}_G(u,v)$. Clearly,

$$J_n^3 \geqslant \big\langle z^*, (u_n-u, v_n-v)\big\rangle_{W_{0,G}^{1,p}(\Omega)\times W_{0,G}^{1,q}(\Omega)}$$

and from (20) we have

$$\liminf_{n\to\infty} J_n^3 \geqslant 0. \tag{26}$$

Therefore, from relations (23)–(26) we obtain

$$\limsup_{n\to\infty} J_n \leqslant 0. \tag{27}$$

On the other hand, from the inequality

$$
|t-s|^\alpha \leqslant \begin{cases} (|t|^{\alpha-2}t-|s|^{\alpha-2}s)(t-s), & \text{if } \alpha\geqslant 2,\\ ((|t|^{\alpha-2}t-|s|^{\alpha-2}s)(t-s))^{\alpha/2}(|t|^\alpha+|s|^\alpha)^{(2-\alpha)/2}, & \text{if } 1<\alpha<2, \end{cases}
$$

for all $t, s \in \mathbb{R}^N$ (see [16]) and (27), we obtain that

$$\lim_{n\to\infty} \int_\Omega \left(|\nabla u_n - \nabla u|^p + |\nabla v_n - \nabla v|^q\right) = 0,$$

that is, the sequences $\{u_n\}$ and $\{v_n\}$ are strongly convergent in $W_{0,G}^{1,p}(\Omega)$ and $W_{0,G}^{1,q}(\Omega)$, respectively. □

5. Mountain pass geometry

Proposition 6. *Under the conditions of Theorem* 1, *there exist* $\eta, \rho > 0$ *and* $(e_p, e_q) \in W_{0,G}^{1,p}(\Omega) \times W_{0,G}^{1,q}(\Omega)$ *such that for all* $(u, v) \in W_{0,G}^{1,p}(\Omega) \times W_{0,G}^{1,q}(\Omega)$

$$\mathcal{H}_G(u, v) \geqslant \eta \quad \text{with } \left\|(u, v)\right\|_{1,p,q} = \rho, \tag{28}$$

and

$$\left\|(e_p, e_q)\right\|_{1,p,q} > \rho, \qquad \mathcal{H}_G(e_p, e_q) \leqslant 0. \tag{29}$$

Proof. By using (17), we obtain

$$\begin{aligned}
\mathcal{H}_G(u, v) &= \frac{1}{p}\|u\|_{1,p}^p + \frac{1}{q}\|v\|_{1,q}^q - \int_\Omega F(u, v)\,dx \\
&\geqslant \left[\frac{1}{p} - \varepsilon\left(2 + \frac{1}{p} - \frac{1}{q}\right)c_{p,p}^p\right]\|u\|_{1,p}^p + \left[\frac{1}{q} - \varepsilon\left(2 + \frac{1}{q} - \frac{1}{p}\right)c_{q,q}^q\right]\|v\|_{1,q}^q \\
&\quad - c(\varepsilon)\left[\left(2 + \frac{1}{r} - \frac{1}{s}\right)\|u\|_r^r + \left(2 + \frac{1}{s} - \frac{1}{r}\right)\|v\|_s^s\right].
\end{aligned}$$

Choosing ε as in (18), we can fix $c_5(\varepsilon), c_6(\varepsilon) > 0$ such that

$$\mathcal{H}_G(u, v) \geqslant c_5(\varepsilon)\left(\|u\|_{1,p}^p + \|v\|_{1,q}^q\right) - c_6(\varepsilon)\left(\|u\|_{1,p}^r + \|v\|_{1,q}^s\right).$$

Since the function $t \mapsto (x^t + y^t)^{1/t},\ t > 0$, is nonincreasing ($x, y \geqslant 0$), using again $ps = qr$, we have

$$\|u\|_{1,p}^r + \|v\|_{1,q}^s \leqslant \left[\|u\|_{1,p}^p + \|v\|_{1,q}^q\right]^{r/p(=s/q)}.$$

Therefore,

$$\mathcal{H}_G(u, v) \geqslant \left[c_5(\varepsilon) - c_6(\varepsilon)\left(\|u\|_{1,p}^p + \|v\|_{1,q}^q\right)^{r/p-1}\right]\left(\|u\|_{1,p}^p + \|v\|_{1,q}^q\right).$$

Let $0 < \rho < 1$ and denote

$$B_\rho = \left\{(u, v) \in W_{0,G}^{1,p}(\Omega) \times W_{0,G}^{1,q}(\Omega):\ \left\|(u, v)\right\|_{1,p,q} = \rho\right\}.$$

Then, we have $(\rho/2)^{\max\{p,q\}} \leqslant \|u\|_{1,p}^p + \|v\|_{1,q}^q \leqslant \rho$ for all $(u, v) \in B_\rho$. Choosing ρ small enough, there exists $\eta > 0$ such that $\mathcal{H}_G(u, v) \geqslant \eta$ for all $(u, v) \in B_\rho$, due to the fact that $r > p$. This is exactly the relation (28).

To prove (29), we fix arbitrary an element $(u,v)\in\mathbb{R}^2$. We define the function $g:]0,\infty[\to\mathbb{R}$ by

$$g(t)=t^{-1}F(t^{1/p}u,t^{1/q}v)-c_2\frac{p}{\mu-p}t^{\mu/p-1}|u|^{\mu}-c_2\frac{q}{\nu-q}t^{\nu/q-1}|v|^{\nu}.$$

Since g is locally Lipschitz, due to Lebourg's mean value theorem, for a fixed $t>1$ there exists $\tau=\tau(t,u,v)\in\,]1,t[$ such that

$$g(t)-g(1)\in\partial_t g(\tau)(t-1),$$

where ∂_t stands for the generalized gradient with respect to $t\in\mathbb{R}$. From the Second Chain Rule and Proposition 2(i), we have

$$\partial_t F(t^{1/p}u,t^{1/q}v)\subseteq\frac{1}{p}\partial_1 F(t^{1/p}u,t^{1/q}v)t^{1/p-1}u+\frac{1}{q}\partial_2 F(t^{1/p}u,t^{1/q}v)t^{1/q-1}v.$$

Hence, by Proposition 1(iii) one has

$$\begin{aligned}\partial_t g(t)\subseteq&-t^{-2}F(t^{1/p}u,t^{1/q}v)\\&+t^{-1}\left[\frac{1}{p}\partial_1 F(t^{1/p}u,t^{1/q}v)t^{1/p-1}u+\frac{1}{q}\partial_2 F(t^{1/p}u,t^{1/q}v)t^{1/q-1}v\right]\\&-c_2\big[t^{\mu/p-2}|u|^{\mu}+t^{\nu/q-2}|v|^{\nu}\big].\end{aligned}$$

Let $w^\tau\in\partial_t g(\tau)$ such that $g(t)-g(1)=w^\tau(t-1)$. There exist $w_i^\tau\in\partial_i F(\tau^{1/p}u,\tau^{1/q}v)$ $(i\in\{1,2\})$ such that

$$\begin{aligned}g(t)-g(1)=&-\tau^{-2}\bigg[F(\tau^{1/p}u,\tau^{1/q}v)+\frac{1}{p}w_1^\tau(-\tau^{1/p}u)+\frac{1}{q}w_2^\tau(-\tau^{1/q}v)\\&+c_2\big(|\tau^{1/p}u|^{\mu}+|\tau^{1/q}v|^{\nu}\big)\bigg](t-1)\\\geqslant&-\tau^{-2}\bigg[F(\tau^{1/p}u,\tau^{1/q}v)+\frac{1}{p}F_1^0(\tau^{1/p}u,\tau^{1/q}v;-\tau^{1/p}u)\\&+\frac{1}{q}F_2^0(\tau^{1/p}u,\tau^{1/q}v;-\tau^{1/q}v)+c_2\big(|\tau^{1/p}u|^{\mu}+|\tau^{1/q}v|^{\nu}\big)\bigg](t-1).\end{aligned}$$

Due to (3), we have $g(t)\geqslant g(1)$. Thus,

$$F(t^{1/p}u,t^{1/q}v)\geqslant tF(u,v)+c_2\left[\frac{p}{\mu-p}(t^{\mu/p}-t)|u|^{\mu}+\frac{q}{\nu-q}(t^{\nu/q}-t)|v|^{\nu}\right]\tag{30}$$

for all $t>1$ and $(u,v)\in\mathbb{R}^2$.

Now, fix $u_p^0\in W_{0,G}^{1,p}(\Omega)$ and $v_q^0\in W_{0,G}^{1,q}(\Omega)$ such that $\|u_p^0\|_{1,p}=\|v_q^0\|_{1,q}=1$. Then, for every $t>1$ we have

$$\begin{aligned}\mathcal{H}_G\big(t^{1/p}u_p^0,t^{1/q}v_q^0\big)&=\left(\frac{1}{p}+\frac{1}{q}\right)t-\int_\Omega F\big(t^{1/p}u_p^0,t^{1/q}v_q^0\big)\,dx\\&\leqslant\left(\frac{1}{p}+\frac{1}{q}-\int_\Omega F\big(u_p^0,v_q^0\big)\,dx\right)t\end{aligned}$$

$$-c_2\left[\frac{p}{\mu-p}(t^{\mu/p}-t)\left\|u_p^0\right\|_\mu^\mu+\frac{q}{\nu-q}(t^{\nu/q}-t)\left\|v_q^0\right\|_\nu^\nu\right].$$

Due to Sobolev embeddings, $\|u_p^0\|_\mu \neq 0 \neq \|v_q^0\|_\nu$. Therefore, $\mathcal{H}_G(t^{1/p}u_p^0, t^{1/q}v_q^0) \to -\infty$ as $t \to \infty$ (recall that $\mu > p$ and $\nu > q$). Choosing $t = t_0$ large enough and denoting by $e_p = t_0^{1/p}u_p^0$ and $e_q = t_0^{1/q}v_q^0$, we are led to (29). This completes the proof. □

6. Proof of Theorem 1

We recall a version of the Mountain Pass Theorem, proved by Kourogenis and Papageorgiou [13, Theorem 6].

Proposition 7. *Let X be a Banach space, $h : X \to \mathbb{R}$ be a locally Lipschitz function with $h(0) = 0$. Suppose that there exist an element $e \in X$ and constants $\rho, \eta > 0$ such that*

(i) *$h(u) \geqslant \eta$ for all $u \in X$ with $\|u\| = \rho$;*
(ii) *$\|e\| > \rho$ and $h(e) \leqslant 0$;*
(iii) *h satisfies $(C)_c$, with $c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} h(\gamma(t))$, where $\Gamma = \{\gamma \in C([0,1], X)$: $\gamma(0) = 0,\ \gamma(1) = e\}$.*

Then $c \geqslant \eta$ and $c \in \mathbb{R}$ is a critical value of h.

Proof of Theorem 1 completed. Let us choose $X = W_{0,G}^{1,p}(\Omega) \times W_{0,G}^{1,q}(\Omega)$ and $h = \mathcal{H}_G$ in Proposition 7. Conditions (i) and (ii) are verified due to Proposition 6. Defining $c_e \in \mathbb{R}$ as in Proposition 7 for the element $e = (e_p, e_q)$ (e_p, e_q from Proposition 6), we have $c_e \geqslant \eta > 0$. By Proposition 5, $\mathcal{H}_G$ satisfies $(C)_{c_e}$. Hence, there exists a critical point $(u_0, v_0) \in W_{0,G}^{1,p}(\Omega) \times W_{0,G}^{1,q}(\Omega)$ of $\mathcal{H}_G$, the critical value $c_e = \mathcal{H}_G(u_0, v_0)$ being strictly positive. Since $\mathcal{H}_G(0,0) = 0$, (u_0, v_0) cannot be $(0,0)$.

On the other hand, the action G on $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ can be defined by

$$g(u,v) = (gu, gv) \quad \text{for all } g \in G,\ (u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega).$$

Moreover, G acts isometrically on $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ and $\mathcal{H}$ is G-invariant, that is

$$\left\|g(u,v)\right\|_{1,p,q} = \left\|(u,v)\right\|_{1,p,q} \quad \text{and} \quad \mathcal{H}\big(g(u,v)\big) = \mathcal{H}(u,v)$$

for all $g \in G$, $(u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$.

Now, we are in the position to apply the Principle of Symmetric Criticality for locally Lipschitz functions, proved by Krawcewicz and Marzantowicz [14, p. 1045] (see also [15]). This means that the critical point $(u_0, v_0) \in W_{0,G}^{1,p}(\Omega) \times W_{0,G}^{1,q}(\Omega)$ of $\mathcal{H}_G$ will be a critical point of $\mathcal{H}$ on the whole space $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, due to the fact that

$$W_{0,G}^{1,p}(\Omega) \times W_{0,G}^{1,q}(\Omega) = \big\{(u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega):\ g(u,v) = (u,v) \text{ for all } g \in G\big\}.$$

Therefore, it remains to apply Proposition 4. □

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