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THE DISPERSING OF GEODESICS IN BERWALD SPACES OF NON-POSITIVE FLAG CURVATURE

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Abstract. It is proved that a forward complete Berwald space of non-positive flag curvature is a generalized Busemann's geodesic space of non-positive curvature.

1. Introduction

It is well known that the non-positivity of the flag curvature of a Finsler space M implies that the geodesic rays emanating from a point $x \in M$ are spreading apart faster than the corresponding rays in T_xM, i.e. the geodesic rays are dispersing. In this paper we establish a stronger conclusion for Berwald spaces concerning this dispersion. Namely, we prove that a class of forward complete Berwald spaces of non-positive flag curvature is included in the class of generalized Busemann's geodesic spaces of non-positive curvature. This immediately follows from our main theorem (see all details in Theorem 7):

For a Berwald space of non–positive flag curvature, in a suitable neighborhood of any point, two geodesics $\gamma_1, \gamma_2 : [0,1] \to M$ emanating from the same point $\gamma_1(0) = \gamma_2(0)$ satisfy the inequality

$$2d_F(\gamma_1(\frac{1}{2}), \gamma_2(\frac{1}{2})) \leq d_F(\gamma_1(1), \gamma_2(1))$$

where d_F means the Finslerian distance in the Berwald space.

This means that the length of a median line of a geodesic triangle cannot succeed the half length of the corresponding side.

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A Berwald space is a special case of Finsler spaces where the Chern connection reduces to a linear connection of the base manifold. Z.I. Szabó proved in [6] that any Berwald space is Riemann-metrizable, i.e. there is a Riemannian space such that its Levi–Civita connection coincides the linear connection of the Berwald space. In particular, the geodesics of the Berwald space and the Riemannian space are identical. Nevertheless, the Berwaldian and the Riemannian metrics are different.

Kelly and Straus ([4]) showed that if the Hilbert metric of a convex domain satisfies the inequality above, then the domain is an ellipsoid, i.e. the Hilbert metric is Riemannian. The Hilbert metric has always negative constant flag curvature -1, proven by T. Okada ([5]). It means that the property having non-positive flag curvature is not enough to be non-positively curved in the sense of Busemann.

It is important to remark that there do exist non-Riemannian and non-flat Berwald spaces with non-positive curvature. One such example in dimension three has been pointed out to the authors by Z. Shen: Let (M, α) be a 2-dimensional Riemannian space of constant curvature $K_\alpha \leq 0$, and ϵ an arbitrary positive constant. Then the Finsler metric on $\mathbb{R} \times M$

$$F(t, x, y; \tau, u, v) = \sqrt{\tau^2 + \alpha_{(x,y)}(u, v) + \epsilon\sqrt{\tau^4 + \alpha^2_{(x,y)}(u, v)}}$$

satisfies the requirements. We have checked by standard Maple routines that F is a Berwald metric, namely the geodesic coefficients are quadratic in the tange nt vectors, and the flag curvature is non-positive, though not constant necessarily .

2. Preliminaries

Throughout the paper we use the notations and notions from [1]. We recall some of them.

Let M be a n–dimensional C^∞ manifold, $TM = \bigcup_{x \in M} T_x M$ the tangent bundle. We say that (M, F) is a Finsler manifold if the continuous function $F : TM \to \mathbb{R}_+$ satisfies the following properties:

(1) F is C^∞ on $TM \setminus \{0\}$

(2) $F(tu) = tF(u),\ \forall t \geq 0,\ u \in TM$

(3) The matrix $g_{ij}(u) := (\frac{1}{2}F^2)_{y^i y^j}(u)$ is positive definite $\forall u \in TM \setminus \{0\}$.

Let $\sigma : [0, 1] \to M$ be a C^∞ curve. Its *integral length* $L(\sigma)$ is defined as

$$L(\sigma) = \int_0^1 F(\sigma, \dot\sigma)\, dt.$$

For $x_0, x_1 \in M$ denote by $\Gamma(x_0, x_1)$ the set of all piecewise C^∞ curves $\sigma : [0,1] \to M$ such that $\sigma(0) = x_0$ and $\sigma(1) = x_1$. Define a map $d_F : M \times M \to [0, \infty)$ by

$$d_F(x_0, x_1) = \inf_{\sigma \in \Gamma(x_0, x_1)} L(\sigma).$$

Of course, we have

$$d_F(x_0, x_1) \geq 0, \text{ where equality holds if and only if } x_0 = x_1 \tag{4}$$

$$d_F(x_0, x_2) \leq d_F(x_0, x_1) + d_F(x_1, x_2). \tag{5}$$

In general, $d_F(x_0, x_1) \neq d_F(x_1, x_0)$, therefore d_F is not a metric in the classical sense.

Let $\sigma(t), 0 \leq t \leq 1$ be a geodesic with velocity field $T = \dot{\sigma}$. A vector field J along σ is said to be a *Jacobi field* if it satisfies the equation

$$D_T D_T J + R(J, T)T = 0, \tag{6}$$

where R is the curvature tensor and

$$D_T W = \left[\frac{dW^i}{dt} + W^j T^k (\Gamma^i_{jk})_{(\sigma, T)} \right] \frac{\partial}{\partial x^i}|_{\sigma(s)},$$

Γ^i_{jk} denotes the coefficients of the Chern connection, see [1, p. 130]. A Finsler manifold is called *a Berwald space* if its Chern connection Γ^i_{jk} depends only on the position. We remark that the equation (6) concerns the reference vector T.

In the sequel, let $(x, y) \in TM \setminus 0$ and V a section of the pull back bundle $\pi^* TM$. Then

$$K(y, V) = \frac{g_{(x,y)}(R(V, y)y, V)}{g_{(x,y)}(y, y) g_{(x,y)}(V, V) - [g_{(x,y)}(y, V)]^2}, \tag{7}$$

is the flag curvature with flag y and transverse edge V. Here $g_{(x,y)} := g_{ij(x,y)} dx^i \otimes dx^j := (\frac{1}{2} F^2)_{y^i y^j} dx^i \otimes dx^j$ is the Riemannian metric on the pull back bundle $\pi^* TM$, see [1, p. 68]. Let K abbreviate the collection of flag curvatures

$$\{K(V, W) : 0 \neq V, W \in T_x M, x \in M, V \text{ and } W \text{ are not collinear}\}.$$

We say that (M, F) has non-positive flag curvature if $K \leq 0$.

Proposition 1. *Let (M, F) be a Finsler manifold with non-positive flag curvature. Then no geodesic contains a conjugate point.*

Proof. See [1, p. 229]. □

Remark 2. The above proposition states that if we have a nonzero Jacobi field along a geodesic $\sigma : [0, 1] \to M$ and $J(0) = 0$ then $J(t) \neq 0$ for all $t \in (0, 1]$.

3. Preparatory steps

Proposition 3. *Let (M,F) be a Berwald space with non-positive flag curvature, J a Jacobi field along a geodesic $\sigma : [0,1] \to M$. Assuming that $J(0)=0$, we have*

$$\frac{d^2}{dt^2}[g_J(J,J)]^{\frac{1}{2}}(t) \geq 0 \qquad \forall\, t \in (0,1]. \tag{8}$$

Proof. From the assumption and Remark 2 we have that $J(t) \neq 0, \forall\, t \in (0,1]$. Hence $g_J(J,J)(t)$ is well defined for all $t \in (0,1]$. Moreover,

$$[g_J(J,J)]^{\frac{1}{2}}(t) = F(\sigma(t),J(t)) \neq 0 \qquad \forall\, t \in (0,1]. \tag{9}$$

Let T the velocity field of σ. We have

$$\frac{d}{dt}[g_J(J,J)]^{\frac{1}{2}}(t) = \frac{g_J(D_TJ,J)}{[g_J(J,J)]^{\frac{1}{2}}}(t)$$

$$\frac{d^2}{dt^2}[g_J(J,J)]^{\frac{1}{2}}(t) = \frac{d}{dt}\left[\frac{g_J(D_TJ,J)}{[g_J(J,J)]^{\frac{1}{2}}}\right](t) =$$

$$\frac{[g_J(D_TD_TJ,J)+g_J(D_TJ,D_TJ)]\cdot[g_J(J,J)]^{\frac{1}{2}} - g_J^2(D_TJ,J)\cdot[g_J(J,J)]^{-\frac{1}{2}}}{g_J(J,J)}(t) =$$

$$\frac{g_J(D_TD_TJ,J)\cdot g_J(J,J)+g_J(D_TJ,D_TJ)\cdot g_J(J,J) - g_J^2(D_TJ,J)}{[g_J(J,J)]^{\frac{3}{2}}}(t).$$

All differentiation is made here with respect to the reference vector J, while in the Jacobi equation (6) the reference vector is T. Nevertheless, taking into account that (M,F) is a Berwald space, i.e. the Chern connection is independent of the direction, the reference vector is irrelevant. Therefore the term $g_J(D_TD_TJ,J)$ is equal to $-g_J(R(J,T)T,J)$ by the Jacobi equation (6). This was the crucial point where we used that (M,F) is a Berwald space. In general, this step is impossible.

Using the symmetry property of the curvature, see [1, Exercise 3.9.6, p. 73], the formula (7) of the flag curvature, and the Schwarz inequality we have

$$\begin{aligned}-g_J(R(J,T)T,J) &= -g_J(R(T,J)J,T) = \\ &= -K(J,T)\cdot[g_J(J,J)g_J(T,T) - g_J^2(J,T)] \geq 0.\end{aligned}$$

For the second and third terms of the numerator we apply the Schwarz inequality again, and consequently, we obtain (8). ☐

Proposition 4. *Under the conditions of Proposition 3 we have*

$$[g_J(J,J)]^{\frac{1}{2}}(s) + (t-s)\frac{d}{dt}[g_J(J,J)]^{\frac{1}{2}}(s) \leq F(\sigma(t),J(t)) \quad \forall\, t\in(0,1], s\in(0,1].$$

Proof. Since $J(t) \neq 0, \forall\, t \in (0,1]$, the mapping $t \mapsto F(\sigma(t), J(t))$ is C^∞ on $(0,1]$. Using Proposition 3 and the second order Taylor formula about s we get the assertion. $\square$

Corollary 5. *Under the conditions of Proposition 3 we have*

$$2F(\sigma(\frac{1}{2}), J(\frac{1}{2})) \leq F(\sigma(1), J(1)). \tag{10}$$

Proof. First, let $s = \frac{1}{2}$ and $t = 1$. We have

$$[g_J(J,J)]^{\frac{1}{2}}(\frac{1}{2}) + \frac{1}{2}\frac{d}{dt}[g_J(J,J)]^{\frac{1}{2}}(\frac{1}{2}) \leq F(\sigma(1), J(1)).$$

Secondly, let $s = \frac{1}{2}$ and $t \to 0$. Since F is continuous, J is C^∞ and using $J(0) = 0$, we get

$$[g_J(J,J)]^{\frac{1}{2}}(\frac{1}{2}) - \frac{1}{2}\frac{d}{dt}[g_J(J,J)]^{\frac{1}{2}}(\frac{1}{2}) \leq 0.$$

Adding the two inequalities above and using the relation (9) we get the desired relation (10). $\square$

4. Main theorem

Let (M, F) be a Finsler manifold, where F is positively (but perhaps not absolutely) homogeneous of degree one.

Let $p \in M$, $r > 0$. Let $B_p(r) := \{y \in T_pM : F(p,y) < r\}$ be the open tangent ball, $\mathcal{B}_p^+(r) := \{x \in M : d_F(p,x) < r\}$ be the forward and $\mathcal{B}_p^-(r) := \{x \in M : d_F(x,p) < r\}$ be the backward metric ball respectively.

It is well-known that for every point $p \in M$ there exists a small $r > 0$ (depending only on p) such that for all points q in $\mathcal{B}_p^+(r) \cap \mathcal{B}_p^-(r)$ the mapping $\exp_q$ is C^1-diffeomorphism from $B_q(2r)$ onto $\mathcal{B}_q^+(2r)$, see [1, p. 160], and every pair of points q_0, q_1 in $\mathcal{B}_p^+(r) \cap \mathcal{B}_p^-(r)$ can be joined by a unique minimizing geodesic from q_0 to q_1. Since $\mathcal{B}_p^-(r)$ is an open set in the manifold topology, see [1, p. 155], we have a small positive number $r_1 > 0$ (and $r_1 \leq r$) such that $\mathcal{B}_p^+(r_1) \subset \mathcal{B}_p^-(r)$, see [1, p. 150]. Therefore, from above, for all $p \in M$ there exists a $r_1 > 0$ such that for all points q in $\mathcal{B}_p^+(r_1)$ the mapping $\exp_q$ is C^1-diffeomorphism from $B_q(2r_1)$ onto $\mathcal{B}_q^+(2r_1)$, and every pair of points q_0, q_1 in $\mathcal{B}_p^+(r_1)$ can be joined by a unique minimizing geodesic from q_0 to q_1.

Following the Whitehead's theorem, see [7] or [1, Exercise 6.4.3, p. 164], there exists $\varepsilon > 0$ (and $\varepsilon \leq r_1$) such that $\mathcal{B}_p^+(\varepsilon)$ is strictly convex, i.e. any geodesic segment with endpoints in $\mathcal{B}_p^+(\varepsilon)$, it must entirely stay in $\mathcal{B}_p^+(\varepsilon)$. Hence we can summarize the above argument in the following

Lemma 6. *Let (M,F) be a Finsler manifold, where F is positively (but perhaps not absolutely) homogeneous of degree one. For every point $p \in M$ there exists a small $\varrho > 0$ (depending only on p) such that for all $q \in \mathcal{B}_p^+(\varrho)$ the mapping $\exp_q$ is C^1-diffeomorphism from $B_q(2\varrho)$ onto $\mathcal{B}_q^+(2\varrho)$ and every pair of points q_0, q_1 in $\mathcal{B}_p^+(\varrho)$ can be joined by a unique minimizing geodesic segment from q_0 to q_1 lying entirely in $\mathcal{B}_p^+(\varrho)$.*

Now, we give the main result of this paper.

Theorem 7. *Let (M,F) be a Berwald space with non-positive flag curvature, where F is positively (but perhaps not absolutely) homogeneous of degree one. Let $p \in M$ be an arbitrary fixed point. Let $\varrho > 0$ be as in Lemma 6 and let $\gamma_1, \gamma_2 : [0,1] \to M$ be two geodesics with $\gamma_1(0) = \gamma_2(0) = x \in \mathcal{B}_p^+(\varrho)$. Supposing that $\gamma_1(1), \gamma_2(1) \in \mathcal{B}_p^+(\varrho)$, we have*

$$2d_F(\gamma_1(\frac{1}{2}), \gamma_2(\frac{1}{2})) \leq d_F(\gamma_1(1), \gamma_2(1)). \tag{11}$$

Proof. From Lemma 6 we get a unique geodesic $\gamma : [0,1] \to M$ joining $\gamma_1(1)$ with $\gamma_2(1)$ and $d_F(\gamma_1(1), \gamma_2(1)) = L(\gamma)$. Of course, $\gamma(s) \in \mathcal{B}_p^+(\varrho), \forall s \in [0,1]$, and $\exp_{\gamma(s)}^{-1}$ is well defined on $\mathcal{B}_{\gamma(s)}^+(2\varrho)$.

We define a variation $\Sigma(t,s) = \exp_{\gamma(s)}((1-t)\cdot\exp_{\gamma(s)}^{-1}(x)), \Sigma : [0,1]\times[0,1] \to M$. We observe that $\Sigma(\cdot\,, s)$ is geodesic, for all $s \in [0,1]$. Moreover $\Sigma(0,0) = x = \gamma_1(0)$, $\Sigma(1,0) = \gamma(0) = \gamma_1(1)$. From the uniqueness of the geodesic between x and $\gamma_1(1)$, we have $\Sigma(\cdot\,,0) = \gamma_1$. Similarly $\Sigma(\cdot\,,1) = \gamma_2$.

Since Σ is a geodesic variation, the vector field J_s defined by $J_s(t) = \frac{\partial}{\partial s}\Sigma(t,s) \in T_{\Sigma(t,s)}M$ is a Jacobi field along $\Sigma(\cdot\,,s)$, $s \in [0,1]$, see [1, p. 130].

We have $J_s(0) = 0, J_s(1) = \frac{\partial}{\partial s}\Sigma(1,s) = \frac{d\gamma}{ds}$, $J_s(\frac{1}{2}) = \frac{\partial}{\partial s}\Sigma(\frac{1}{2}, s)$ and $\Sigma(1,s) = \gamma(s)$. From (10) we get

$$2F(\Sigma(\frac{1}{2}, s), \frac{\partial}{\partial s}\Sigma(\frac{1}{2}, s)) \leq F(\gamma(s), \frac{d\gamma}{ds})$$

for all fixed $s \in [0,1]$. Integrating the above relation with respect to s from 0 to 1 we get

$$2L(\Sigma(\frac{1}{2}, \cdot)) = 2\int_0^1 F(\Sigma(\frac{1}{2}, s), \frac{\partial}{\partial s}\Sigma(\frac{1}{2}, s))\, ds \leq$$

$$\leq \int_0^1 F(\gamma(s), \frac{d\gamma}{ds})\, ds = L(\gamma) = d_F(\gamma_1(1), \gamma_2(1)).$$

Since $\Sigma(\frac{1}{2}, 0) = \gamma_1(\frac{1}{2}), \Sigma(\frac{1}{2}, 1) = \gamma_2(\frac{1}{2})$ and $\Sigma(\frac{1}{2}, \cdot)$ is a C^∞ curve, using the definition of d_F, we get the required relation (11). $\square$

Remark 8. From above it follows that if we take two geodesics $\gamma_1, \gamma_2 : [0,1] \to M$ from $\mathcal{B}_p^+(\varrho)$, we obtain that the function $t \mapsto d_F(\gamma_1(t), \gamma_2(t))$ is convex.

5. Generalized Busemann NPC spaces

Now we define the notion of generalized Busemann's non-positive curvature spaces.

Let (M,d) be a non-reversible metric space. (M,d) is called a *geodesic space* if for every two points $x, y \in M$ there exists a shortest geodesic arc joining them, i.e. a continuous curve $\gamma : [0,1] \to M$ with $\gamma(0) = x,\ \ \gamma(1) = y$ and $\ell(\gamma) = d(x,y)$, where $\ell(\gamma)$ denotes the length of γ and it is defined by

$$\ell(\gamma) = \sup\{\sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) : \quad 0 = t_0 < t_1 < \cdots < t_n = 1, n \in \mathbb{N}\}.$$

A geodesic space (M,d) is said to be a *generalized Busemann non-positive curvature (NPC)* space if for every $p \in M$ there exists $\delta_p > 0$ such that for any two shortest geodesics $\gamma_1, \gamma_2 : [0,1] \to M$ with $\gamma_1(0) = \gamma_2(0) = x \in B^+(p, \delta_p)$ and with points $\gamma_1(1), \gamma_2(1) \in B^+(p, \delta_p)$ we have

$$d(\gamma_1(\frac{1}{2}), \gamma_2(\frac{1}{2})) \leq \frac{1}{2} d(\gamma_1(1), \gamma_2(1)),$$

where $B^+(p,r) := \{x \in M : d(p,x) < r\}$.
We remark that if (M,d) is a metric space, then the above notion coincide with the classical Busemann NPC notion, see [3, Section 36].

We say that (M,F) is *forward complete* if every geodesic $\gamma : [a,b] \to M$ can be extended to a geodesic defined on $[a, \infty)$.

Theorem 9. *Let (M,F) be a connected forward complete Berwald space of non-positive flag curvature. Then (M, d_F) is a generalized Busemann NPC space.*

Proof. Using the assumption that (M,F) is forward complete, and applying the Hopf-Rinow theorem, see [1, Theorem 6.6.1, p. 168], we obtain that (M, d_F) is a geodesic space. Moreover, in our case the two notions of length coincide, see [2, Theorem 2]. Using the definition of generalized Busemann NPC space and Theorem 7, the assertion holds. $\square$

Remark 10. If F is absolutely homogeneous, we get that a connected complete Berwald space with non-positive flag curvature is a Busemann NPC space in the classical sense. In fact, the notion of forward completeness reduce to the classical completeness.

References

[1] Bao, D. – Chern, S.-S – Shen. Z., *Introduction to Riemann–Finsler Geometry,* Graduate Texts in Mathematics, 200, Springer Verlag, 2000.

[2] Busemann, H. – Mayer, W. On the foundations of the calculus of variations, *Trans. AMS.* **49**, (1941), 173–198.

[3] Busemann, H., *The Geometry of Geodesics,* Academic Press, 1955.

[4] Kelly, P. – Straus, E., Curvature in Hilbert geometry, *Pacific J. Math.* **8** (1958), 119–125.

[5] Okada, T., On models of projectively flat Finsler spaces of constant negative curvature, *Tensor, N.S.* **40**, (1983), 117–124.

[6] Szabó, Z.I., Positive definite Berwald spaces (structure theorems on Berwald spaces), *Tensor, N.S.* **35**, (1981), 25–39.

[7] Whitehead, J.H.C., Convex region in the geometry of paths, *Quarterly J. Math. Oxford,* Ser. **3**, (1932), 33–42.

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