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On a class of quasilinear eigenvalue problems in R^N

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We study an eigenvalue problem in $\mathbf{R}^N$ which involves the p-Laplacian ($p > N \geq 2$) and the nonlinear term has a global $(p-1)$-sublinear growth. The existence of certain open intervals of eigenvalues is guaranteed for which the eigenvalue problem has two nonzero, radially symmetric solutions. Some stability properties of solutions with respect to the eigenvalues are also obtained.

1 Introduction

Consider the problem

$$-\triangle_p u + |u|^{p-2}u \;=\; \lambda\alpha(x)f(u)\,, \quad x \in \mathbf{R}^N\,, \quad u \in W^{1,p}(\mathbf{R}^N)\,, \tag{P$_\lambda$}$$

where λ is a positive parameter, $\alpha : \mathbf{R}^N \to \mathbf{R}$ is a measurable function and $f : \mathbf{R} \to \mathbf{R}$ is a continuous function. We assume $1 < p < \infty$ and $N \geq 1$.

In the case when $p \leq N$, closely related problems to Eq. (P_λ) have been extensively studied. Existence and multiplicity results for Eq. (P_1) can be found for instance in the papers [2, 3, 21, 22] for the semilinear case (i.e., $p = 2$), while for quasilinear problems (i.e., $p \neq 2$) important contributions to Eq. (P_λ) can be found in [11]. When f is not necessarily continuous, related problems to Eq. (P_λ) were considered in [8, 9]. The reader is referred to [14, 19, 20] in order to find multiplicity results for very general eigenvalue problems. The aforementioned works have a common feature; namely, the involved nonlinearities have some sort of superlinear growth at infinity.

The purpose of this paper is to ensure the existence of multiple solutions for Eq. (P_λ), completing the above papers from two aspects. More precisely, we will investigate Eq. (P_λ) under the following conditions:

(i) $p > N \geq 2$,

and

(ii) f has a global $(p-1)$-sublinear growth, i.e.,

(A) There exist $C > 0$ and $1 < \gamma < p$ such that

$$|f(s)| \;\leq\; C\big(1 + |s|^{\gamma-1}\big)\,, \quad s \in \mathbf{R}\,.$$

The main difficulty studying Eq. (P_λ) lies on the fact that no compact embedding is available for $W^{1,p}(\mathbf{R}^N)$. In spite of the fact that the embedding $W^{1,p}(\mathbf{R}^N) \hookrightarrow L^\infty(\mathbf{R}^N)$ $(p > N)$ is continuous, it is not compact. On the other hand, Lions [12, Théorème II. 1] proved that the subspace of radially symmetric functions of $W^{1,p}(\mathbf{R}^N)$, denoted further by $W^{1,p}_r(\mathbf{R}^N)$, may be compactly embedded into $L^q(\mathbf{R}^N)$ if $p \geq N \geq 2$ and $p < q < \infty$.

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Unfortunately, this embedding fails to be compact for $N = 1$ or $q \in \{p, \infty\}$. However, as a limit case, one can still prove that $W_r^{1,p}(\mathbf{R}^N)$ can be compactly embedded into $L^\infty(\mathbf{R}^N)$ whenever $p > N \geq 2$ (cf. Proposition 2.3); this fact will be indispensable in our arguments.

Denote by $F(s) = \int_0^s f(t)\,dt$. We introduce the energy functional $\mathcal{E}_\lambda : W^{1,p}(\mathbf{R}^N) \to \mathbf{R}$ associated to problem (P_λ) which is defined by

$$\mathcal{E}_\lambda(u) = \frac{1}{p}\,\|u\|_{W^{1,p}}^p - \lambda \int_{\mathbf{R}^N} \alpha(x) F(u(x))\,dx\,.$$

Due to the principle of symmetric criticality, the critical points of $\mathcal{R}_\lambda \stackrel{\text{def}}{=} \mathcal{E}_\lambda\big|_{W_r^{1,p}(\mathbf{R}^N)}$ become critical points of $\mathcal{E}_\lambda$ as well, so radially symmetric, weak solutions of Eq. (P_λ) (cf. Propositions 2.2 and 2.1). Beside of condition (A), we make the following two assumptions:

(B) There exists $\nu > p$ such that

$$\limsup_{s\to 0} \frac{F(s)}{|s|^\nu} < \infty\,;$$

(C) $\alpha \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, $\alpha \geq 0$, and

$$\sup_{R>0} \operatorname*{essinf}_{|x|\leq R} \alpha(x) > 0 \quad \text{and} \quad \sup_{s\in\mathbf{R}} F(s) > 0\,.$$

First of all, we emphasize that neither (B) nor (C) cannot be dropped in view to obtain multiple solutions for Eq. (P_λ). Indeed, take first $f \equiv 0$. Then (B) is clearly verified while (C) fails and Eq. (P_λ) has only the trivial solution for every real λ. On the other hand, if we take $f \equiv 1$ and $\alpha(x) = \left(1 + |x|^N\right)^{-2}$ then (C) holds while (B) fails. One can check that $\mathcal{R}_\lambda$ is bounded from below, satisfies the Palais–Smale condition (cf. Propositions 3.2 and 3.3, where only condition (A) is used), and it is strictly convex. Therefore, a unique critical point of $\mathcal{R}_\lambda$ may exist which will be exactly its global minimum.

As far as the multiplicity of solutions of Eq. (P_λ) is concerned, we are going to prove first that for enough "large" λ's the infimum of $\mathcal{R}_\lambda$ is strictly negative (which ensures the existence of a negative critical level), while another critical point of $\mathcal{R}_\lambda$ (with strictly positive energy) is obtained by the classical Mountain Pass theorem. Furthermore, we obtain information about the stability of the solutions with respect to the eigenvalues, similarly as in [1] for boundary value problems. The range of those eigenvalues for which these arguments work is described as follows.

In view of hypothesis (C), one can define the following two nonempty sets

$$I_\alpha^+ = \left\{ R > 0 : \alpha_R = \operatorname*{essinf}_{|x|\leq R} \alpha(x) > 0 \right\}$$

and

$$I_F^+ = \{ s \in \mathbf{R} : F(s) > 0 \}\,.$$

For every $(R, s) \in I_\alpha^+ \times I_F^+$ we define

$$I_{R,s} = \left] \left(1 + \frac{\alpha_R F(s)}{\|\alpha\|_{L^\infty} \max_{|t|\leq|s|} |F(t)|} \right)^{-1/N}, 1 \right[\tag{1.1}$$

and for $\sigma \in I_{R,s}$ set

$$\lambda_{\sigma,R,s} = \frac{|s|^p}{p} \frac{1 + R^{-p}\left(1 - \sigma^N\right)(1 - \sigma)^{-p}}{\alpha_R F(s) \sigma^N - \|\alpha\|_{L^\infty} \max_{|t|\leq|s|} |F(t)| \left(1 - \sigma^N\right)}\,. \tag{1.2}$$

By Eqs. (1.1) and (1.2) it is clear that for the two values of $\sigma_{R,s}^0 \in \partial I_{R,s}$ (with R, s fixed) one has $\lambda_{\sigma,R,s} \to +\infty$ whenever $\sigma \to \sigma_{R,s}^0$ and $\sigma \in I_{R,s}$. Thus, the following number is well-defined:

$$\lambda^* = \inf_{(R,s)\in I_\alpha^+ \times I_F^+} \min_{\sigma\in I_{R,s}} \lambda_{\sigma,R,s}\,. \tag{1.3}$$

We will prove

Theorem 1.1 *Let $p > N \geq 2$, $f : \mathbf{R} \to \mathbf{R}$ be a continuous function, and $\alpha : \mathbf{R}^N \to \mathbf{R}$ be a radially symmetric function, which satisfy conditions* (A), (B) *and* (C).

Then, for every $\lambda > \lambda^*$ problem (P_λ) *has two distinct, nonzero, radially symmetric weak solutions u_λ and v_λ with $\mathcal{E}_\lambda(u_\lambda) < 0 < \mathcal{E}_\lambda(v_\lambda)$. Moreover, u_λ and v_λ can be constructed such that*

$$\sup_{\lambda \in K} \max\{\|u_\lambda\|_{W^{1,p}}, \|v_\lambda\|_{W^{1,p}}\} \ < \ +\infty \tag{1.4}$$

for every bounded set $K \subset]\lambda^*, +\infty[$, assuming that the infimum in Eq. (1.3) *is achieved on $I_\alpha^+ \times I_F^+$ whenever $\lambda^* = \inf K$. In addition, if $f(s) = 0$ for all $s \leq 0$, the solutions u_λ and v_λ are nonnegative.*

A simple estimation which is based on our hypotheses shows that

$$\lambda^* \ \geq \ \frac{1}{p\,\|\alpha\|_{L^\infty}} \inf_{s \in I_F^+} \frac{|s|^p}{F(s)} \ > \ 0\,,$$

and by construction, the number λ^* seems to be the lower limit of those eigenvalues for which the proof of Theorem 1.1 can be carried out in the above-described way.

A natural question instantly arises: can one ensure the existence of further positive eigenvalues which are strictly smaller than λ^* for which Eq. (P_λ) still has two nonzero solutions? An affirmative answer is given in the following

Theorem 1.2 *Under the assumptions of Theorem* 1.1 *suppose in addition that the infimum in Eq.* (1.3) *is achieved on $I_\alpha^+ \times I_F^+$.*

Then there exists an open interval $\Lambda \subset [0, \lambda^*]$ such that for every $\lambda \in \Lambda$ problem (P_λ) *has two distinct, nonzero, radially symmetric weak solutions u_λ and v_λ with the property*

$$\sup_{\lambda \in \Lambda} \max\{\|u_\lambda\|_{W^{1,p}}, \|v_\lambda\|_{W^{1,p}}\} \ < \ +\infty\,.$$

In addition, if $f(s) = 0$ for all $s \leq 0$, the solutions u_λ and v_λ are nonnegative.

This result will be proved by means of a recent abstract critical point result of G. Bonanno [4] which is actually a refinement of a general principle of B. Ricceri [17, 18]. Various applications and extensions of this general principle are already available, see for instance [5, 6, 10, 13].

Remark 1.3 Unfortunately, if the infimum in Eq. (1.3) is not achieved on $I_\alpha^+ \times I_F^+$, then we are not able to control in Theorem 1.1 the stability of solutions (as in Eq. (1.4)) for those eigenvalues which are arbitrary close to λ^*. Moreover, the conclusion of Theorem 1.2 is also radically modified in this case; namely, only the following fact can be proved: *for every $h > 1$ there exist an open interval $\Lambda_h \subset [0, h\lambda^*]$ and a number $\mu_h > 0$ such that for every $\lambda \in \Lambda_h$ problem* (P_λ) *has two distinct, nonzero, radially symmetric weak solutions u_λ and v_λ so that* $\sup_{\lambda \in \Lambda_h} \max\{\|u_\lambda\|_{W^{1,p}}, \|v_\lambda\|_{W^{1,p}}\} \leq \mu_h$. In particular, it can happen that $\lambda^* < \inf \Lambda_h$; if so, the above thesis does not furnish any new information in comparison to Theorem 1.1. However, several problems can be encountered where the infimum in Eq. (1.3) is achieved on $I_\alpha^+ \times I_F^+$, and consequently, our results can be fully applied. We present such an example in the sequel.

Example 1.4 Consider the problem

$$-\triangle_3 u + |u|\,u \ = \ \lambda \frac{1}{(1+|x|^2)^2}\big(\arctan u^3 + 3u^3(1+u^6)^{-1}\big)\,, \ \ x \in \mathbf{R}^2\,, \ \ u \in W^{1,3}\big(\mathbf{R}^2\big)\,. \tag{E_λ}$$

Let

$$E \ = \ \inf_{R>0} \min_{\sigma \in I_R} \frac{1 + R^{-3}(1+\sigma)(1-\sigma)^{-2}}{\sigma^2(1+R^2)^{-2} - 1 + \sigma^2} \quad \text{where} \quad I_R \ = \ \Big]\big(1 + (1+R^2)^{-2}\big)^{-1/2}, 1\Big[\,,$$

and let s_0 be the unique positive solution of the equation $(1+s^2)\arctan s = 3s/2$. (By means of standard Matlab routines we obtained $s_0 \approx 0.928$, while the infimum in the expression E is achieved for $\overline{R} \approx 0.6575$; it is $E \approx 3030.75$.) The infimum in Eq. (1.3) is achieved for $\overline{R}$ and $\bar{s} = s_0^{1/3}$; more precisely, $\lambda^* = \big(3\arctan \bar{s}^3\big)^{-1} \bar{s}^2 E$. Then, there exists an open interval $\Lambda \subset [0, \lambda^*]$ such that (E_λ) has at least two nonzero, radially symmetric weak solutions for every $\lambda \in \Lambda \cup]\lambda^*, +\infty[$ with the properties described in Theorems 1.1 and 1.2, respectively.

2 Preliminaries

The space $W^{1,p}(\mathbf{R}^N)$ is endowed with the norm

$$\|u\|_{W^{1,p}} = \left(\|\nabla u\|_{L^p}^p + \|u\|_{L^p}^p\right)^{1/p},$$

where $\|\cdot\|_{L^p}$ is the usual norm on $L^p(\mathbf{R}^N)$, $p < \infty$. The norm on $L^\infty(\mathbf{R}^N)$ is given by

$$\|u\|_{L^\infty} = \operatorname*{esssup}_{x\in\mathbf{R}^N} |u(x)|.$$

Throughout this section, we suppose that the assumptions of Theorem 1.1 are fulfilled. Since $p > N$, the embedding $W^{1,p}(\mathbf{R}^N) \hookrightarrow L^\infty(\mathbf{R}^N)$ is continuous; denote its Sobolev embedding constant by c_∞, i.e., $\|u\|_{L^\infty} \le c_\infty \|u\|_{W^{1,p}}$ for every $u \in W^{1,p}(\mathbf{R}^N)$. Moreover, every function $u \in W^{1,p}(\mathbf{R}^N)$ admits a continuous representation, see [7, p. 166]; in the sequel, we will replace u by this element. For simplicity of notation, let $\mathcal{F} : W^{1,p}(\mathbf{R}^N) \to \mathbf{R}$ be defined by

$$\mathcal{F}(u) = \int_{\mathbf{R}^N} \alpha(x)F(u(x))\,dx.$$

Proposition 2.1 *For every $\lambda \in \mathbf{R}$, the function $\mathcal{E}_\lambda$ is continuously differentiable. Moreover, every critical point of $\mathcal{E}_\lambda$ is a weak solution of Eq.* (P_λ).

Proof. Let $u, h \in W^{1,p}(\mathbf{R}^N)$. Given $x \in \mathbf{R}^N$ and $0 < |t| < 1$, the standard mean value theorem implies the existence of a $\theta_x \in]0,1[$ such that

$$\begin{aligned}\frac{|\alpha(x)F(u(x)+th(x)) - \alpha(x)F(u(x))|}{|t|} &= \alpha(x)\,|f(u(x)+t\theta_x h(x))h(x)| \\ &\le \alpha(x)\,\max\{|f(s)| : |s| \le \|u\|_{L^\infty} + \|h\|_{L^\infty}\}\,|h(x)|.\end{aligned}$$

Since $\alpha \in L^1(\mathbf{R}^N)$, the last expression of the above inequality belongs to $L^1(\mathbf{R}^N)$. Thus, it follows from the Lebesgue theorem that

$$\langle \mathcal{F}'(u), h\rangle_{W^{1,p}} = \int_{\mathbf{R}^N} \alpha(x)f(u(x))h(x)\,dx, \tag{2.1}$$

where $\langle\cdot,\cdot\rangle_{W^{1,p}}$ denotes the duality pairing between $W^{1,p}(\mathbf{R}^N)$ and its dual.

Now, let $\{u_n\}$ be a sequence in $W^{1,p}(\mathbf{R}^N)$ which converges strongly to a $u \in W^{1,p}(\mathbf{R}^N)$. In particular, $u_n \to u$ strongly in $L^\infty(\mathbf{R}^N)$. Fix an $\varepsilon > 0$ arbitrarily. Since f is uniformly continuous on the compact interval $I_u = [-\|u\|_{L^\infty} - 1, \|u\|_{L^\infty} + 1]$, there exists a number $\delta(\varepsilon) > 0$ such that

$$|f(s) - f(t)| < \frac{\varepsilon}{2c_\infty\|\alpha\|_{L^1}} \quad \text{for every} \quad t, s \in I_u, \quad |s-t| < \delta(\varepsilon). \tag{2.2}$$

In the same time, there exists $n_\varepsilon \in \mathbf{N}$ such that

$$\|u_n - u\|_{L^\infty} < \min\{\delta(\varepsilon), 1\} \quad \text{for every} \quad n \ge n_\varepsilon. \tag{2.3}$$

Combining Eq. (2.2) with Eq. (2.3), for every $n \ge n_\varepsilon$ and $h \in W^{1,p}(\mathbf{R}^N)$ one has

$$\left|\langle \mathcal{F}'(u_n) - \mathcal{F}'(u), h\rangle_{W^{1,p}}\right| \le \frac{\varepsilon}{2}\,\|h\|_{W^{1,p}},$$

thus, $\|\mathcal{F}'(u_n) - \mathcal{F}'(u)\|_{(W^{1,p})^*} < \varepsilon$. Using Eq. (2.1), we obtain at once the last part of the proposition. □

The action of $O(N)$ on $W^{1,p}(\mathbf{R}^N)$, defined by $(gu)(x) = u(g^{-1}x)$ for every $g \in O(N)$, $u \in W^{1,p}(\mathbf{R}^N)$, $x \in \mathbf{R}^N$, is linear and isometric; in particular $\|gu\|_{W^{1,p}} = \|u\|_{W^{1,p}}$, for every $g \in O(N)$ and $u \in W^{1,p}(\mathbf{R}^N)$. We say that a function $h : \mathbf{R}^N \to \mathbf{R}$ is *radially symmetric* if $h(gx) = h(x)$ for every $g \in O(N)$, and $x \in \mathbf{R}^N$. One can define the subspace of radially symmetric functions of $W^{1,p}(\mathbf{R}^N)$ by

$$W_r^{1,p}(\mathbf{R}^N) \ = \ \{u \in W^{1,p}(\mathbf{R}^N) : gu = u \text{ for every } g \in O(N)\} .$$

Taking into account that $\alpha : \mathbf{R}^N \to \mathbf{R}$ is radially symmetric, then $\mathcal{F}$ is $O(N)$-invariant, so is $\mathcal{E}_\lambda$. In particular, the principle of symmetric criticality of Palais [15, Theorem 5.4] realizes as follows.

Proposition 2.2 *Every critical point of $\mathcal{R}_\lambda = \mathcal{E}_\lambda|_{W_r^{1,p}(\mathbf{R}^N)}$ will be also a critical point of $\mathcal{E}_\lambda$.*

The next compactness result plays a vital role in our arguments.

Proposition 2.3 *If $p > N \geq 2$, the embedding $W_r^{1,p}(\mathbf{R}^N) \hookrightarrow L^\infty(\mathbf{R}^N)$ is compact.*

Proof. It is well-know that for every $u \in W_r^{1,p}(\mathbf{R}^N)$ we have

$$|u(x)| \ \leq \ C(p,N)\, \|u\|_{W^{1,p}}\, |x|^{(1-N)/p} \quad \text{a.e.} \quad x \in \mathbf{R}^N , \tag{2.4}$$

where $C(p,N) > 0$, see Lions [12, Lemme II.1].

Let $\{u_n\}$ be a sequence in $W_r^{1,p}(\mathbf{R}^N)$ which converges weakly to some $u \in W_r^{1,p}(\mathbf{R}^N)$. By applying inequality (2.4) for $(u_n - u)$, and taking into account that the sequence $\{u_n - u\}$ is bounded in $W_r^{1,p}(\mathbf{R}^N)$, and $N \geq 2$, then for every $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that

$$\|u_n - u\|_{L^\infty(|x| \geq R_\varepsilon)} \ \leq \ C'\, |R_\varepsilon|^{(1-N)/p} \ < \ \varepsilon \quad \text{for every} \quad n \in \mathbf{N} , \tag{2.5}$$

where $C' > 0$ does not depend on n.

On the other hand, by Rellich theorem it follows that $u_n \to u$ strongly in $C^0(B_N[0,R_\varepsilon])$, i.e., there exists $n_\varepsilon \in \mathbf{N}$ such that

$$\|u_n - u\|_{C^0(B_N[0,R_\varepsilon])} \ < \ \varepsilon \quad \text{for every} \quad n \ \geq \ n_\varepsilon . \tag{2.6}$$

(Here, $B_N[0,r]$ denotes the N-dimensional closed ball with center 0 and radius $r > 0$.) Combining Eq. (2.5) with Eq. (2.6), one concludes that $\|u_n - u\|_{L^\infty} < \varepsilon$ for every $n \geq n_\varepsilon$, i.e., $u_n \to u$ strongly in $L^\infty(\mathbf{R}^N)$. □

In the proof of our results the following inequality will be useful.

Lemma 2.4 *There exists $C'' > 0$ such that $F(s) \leq C''\, |s|^\nu$ for every $s \in \mathbf{R}$.*

Proof. By hypothesis (B) we may fix two numbers, namely $\beta \in]0,1]$ and $M > 0$, such that

$$F(s) \ \leq \ M\, |s|^\nu , \quad |s| \ < \ \beta .$$

By condition (A), we have

$$|F(s)| \ \leq \ C\big(1 + |s|^{\gamma-1}\big)\, |s| \ \leq \ C\, \frac{1 + \beta^{\gamma-1}}{\beta^{\nu-1}}\, |s|^\nu , \quad |s| \ \geq \ \beta .$$

By choosing $C'' = \max\left\{M, C\, \frac{1+\beta^{\gamma-1}}{\beta^{\nu-1}}\right\}$, the required relation follows. □

3 Proofs

We assume the hypotheses of Theorem 1.1 are fulfilled. For simplicity of notations, denote further by $\mathcal{R}_\lambda$, $\|\cdot\|_r$ and $\mathcal{F}_r$ the restriction of $\mathcal{E}_\lambda$, $\|\cdot\|_{W^{1,p}}$ and $\mathcal{F}$ to the space $W_r^{1,p}(\mathbf{R}^N)$, respectively.

Proposition 3.1 *Let $\lambda \in \mathbf{R}$ be arbitraryly fixed. Then every bounded sequence $\{u_n\}$ in $W_r^{1,p}(\mathbf{R}^N)$ such that $\|\mathcal{R}'_\lambda(u_n)\|_{(W_r^{1,p})^*} \to 0$, contains a strongly convergent subsequence.*

Proof. Taking a subsequence if necessary, thanks to Proposition 2.3, we may assume that

$$u_n \longrightarrow u \quad \text{weakly in} \quad W_r^{1,p}\big(\mathbf{R}^N\big)\,, \tag{3.1}$$

$$u_n \longrightarrow u \quad \text{strongly in} \quad L^\infty\big(\mathbf{R}^N\big)\,. \tag{3.2}$$

One the other hand, we have

$$\begin{aligned} I_n \ &\overset{\text{not.}}{=}\ \int_{\mathbf{R}^N} \big(|\nabla u_n|^{p-2}\,\nabla u_n - |\nabla u|^{p-2}\,\nabla u\big)(\nabla u_n - \nabla u) \\ &\quad + \int_{\mathbf{R}^N} \big(|u_n|^{p-2}\,u_n - |u|^{p-2}\,u\big)(u_n - u) \\ &= \ \langle \mathcal{R}'_\lambda(u_n), u_n - u\rangle_{W^{1,p}} + \langle \mathcal{R}'_\lambda(u), u - u_n\rangle_{W^{1,p}} \\ &\quad + \lambda \int_{\mathbf{R}^N} \alpha(x)[f(u_n(x)) - f(u(x))](u_n(x) - u(x))\,dx\,. \end{aligned}$$

Since $\|\mathcal{R}'_\lambda(u_n)\|_{(W_r^{1,p})^*} \to 0$, the first term tends to 0. By Eq. (3.1) it follows that the second term tends to 0 as well. Finally, for $n \in \mathbf{N}$ large enough one has

$$\begin{aligned} &\left|\int_{\mathbf{R}^N} \alpha(x)[f(u_n(x)) - f(u(x))](u_n(x) - u(x))\,dx\right| \\ &\quad \le\ 2\,\|\alpha\|_{L^1} \max\{|f(s)| : |s| \le \|u\|_{L^\infty} + 1\}\,\|u_n - u\|_{L^\infty}\,, \end{aligned}$$

and the last term tends to 0, due to Eq. (3.2). In conclusion,

$$\lim_{n\to\infty} I_n \ =\ 0\,. \tag{3.3}$$

Since we have the general inequality $|t-s|^p \le \big(|t|^{p-2}\,t - |s|^{p-2}\,s\big)(t-s)$ for every $t, s \in \mathbf{R}^m$ ($m \in \mathbf{N}$) we infer that $\|u_n - u\|_r^p \le I_n$. The last inequality combined with Eq. (3.3) leads to the fact that $u_n \to u$ strongly in $W_r^{1,p}\big(\mathbf{R}^N\big)$, as claimed. □

Proposition 3.2 *For every $\lambda \ge 0$, $\mathcal{R}_\lambda$ is coercive and bounded below.*

Proof. By condition (A) and the continuous imbedding $W^{1,p}\big(\mathbf{R}^N\big) \hookrightarrow L^\infty\big(\mathbf{R}^N\big)$ one readily has for every $u \in W_r^{1,p}\big(\mathbf{R}^N\big)$ that

$$\mathcal{R}_\lambda(u) \ \ge\ \frac{1}{p}\,\|u\|_r^p - \lambda C\,\|\alpha\|_{L^1}\big(c_\infty\,\|u\|_r + c_\infty^\gamma\,\|u\|_r^\gamma\big)\,. \tag{3.4}$$

The assertion clearly follows, since $\gamma < p$. □

Proposition 3.3 *For every $\lambda \ge 0$, $\mathcal{R}_\lambda$ satisfies the Palais–Smale condition.*

Proof. Let $\{u_n\}$ be a sequence in $W_r^{1,p}\big(\mathbf{R}^N\big)$ such that $\{\mathcal{R}_\lambda(u_n)\}$ is bounded and $\|\mathcal{R}'_\lambda(u_n)\|_{(W_r^{1,p})^*} \to 0$ as $n \to +\infty$. The coercivity of $\mathcal{R}_\lambda$ (Proposition 3.2) implies the boundedness of the sequence $\{u_n\}$. Therefore, from Proposition 3.1 the claim follows. □

For $(R, s) \in I_\alpha^+ \times I_F^+$ and $\sigma \in I_{R,s}$ define

$$w_{\sigma,R,s}(x) \ =\ \begin{cases} 0\,, & \text{if} \quad x \in \mathbf{R}^N \setminus B_N(0,R)\,, \\ s\,, & \text{if} \quad x \in B_N(0,\sigma R)\,, \\ \dfrac{s}{R(1-\sigma)}\,(R - |x|)\,, & \text{if} \quad x \in B_N(0,R) \setminus B_N(0,\sigma R)\,, \end{cases} \tag{3.5}$$

where $B_N(0,r)$ denotes the N-dimensional open ball with center 0 and radius $r > 0$.

It is clear that $w_{\sigma,R,s}$ belongs to $W_r^{1,p}(\mathbf{R}^N)$. Denoting by ω_N the volume of the N-dimensional unit ball, an elementary calculation shows that

$$\begin{aligned}\|w_{\sigma,R,s}\|_r^p &= |s|^p\,\omega_N R^N\big(\sigma^N + R^{-p}\big(1-\sigma^N\big)(1-\sigma)^{-p}\big)\\ &\quad + |s|^p\,R^{-p}(1-\sigma)^{-p}\int_{\sigma R\le|x|\le R}(R-|x|)^p\,dx\\ &< |s|^p\,\omega_N R^N\big(1+R^{-p}\big(1-\sigma^N\big)(1-\sigma)^{-p}\big)\end{aligned}\tag{3.6}$$

and

$$\mathcal{F}_r(w_{\sigma,R,s}) \;\ge\; \omega_N R^N\left[\alpha_R F(s)\sigma^N - \|\alpha\|_{L^\infty}\max_{|t|\le|s|}|F(t)|\big(1-\sigma^N\big)\right].\tag{3.7}$$

Since $\sigma\in I_{R,s}$ one clearly has $\mathcal{F}(w_{\sigma,R,s})>0$ (cf. Eq. (1.1)), thus we may define

$$\tilde\lambda_{\sigma,R,s} \;=\; \frac{\|w_{\sigma,R,s}\|_r^p}{p\mathcal{F}_r(w_{\sigma,R,s})}\,.\tag{3.8}$$

P r o o f o f T h e o r e m 1.1. Let $K\subset\,]\lambda^*,+\infty[$ be a bounded set as in the hypothesis. We shall prove that for every $\lambda\in K$ there exist two elements u_λ and v_λ in $W_r^{1,p}(\mathbf{R}^N)$ which verify $\mathcal{R}_\lambda(u_\lambda)<0<\mathcal{R}_\lambda(v_\lambda)$, they are critical points of $\mathcal{R}_\lambda$ (consequently, radially symmetric, weak solutions of Eq. (P_λ) in view of Propositions 2.2 and 2.1), and the sets $\{\|u_\lambda\|_{W^{1,p}}\}_{\lambda\in K}$ and $\{\|v_\lambda\|_{W^{1,p}}\}_{\lambda\in K}$, respectively, are uniformly bounded. In particular, the first part of Theorem 1.1 will be answered as well, choosing $K=\{\lambda\}$ with $\lambda>\lambda^*$.

Thus, let us start with a set $K\subset\,]\lambda^*,+\infty[$ as was specified above. Either in the case when $\lambda^*=\inf K$ (thus the infimum in Eq. (1.3) is achieved on $I_\alpha^+\times I_F^+$), or in the case when $\lambda^*<\inf K$, we are able to fix $(\overline{R},\bar s)\in I_\alpha^+\times I_F^+$ as well as $\overline{\sigma}=\sigma(\overline{R},\bar s)\in I_{\overline{R},\bar s}$ such that

$$\lambda_{\overline{\sigma},\overline{R},\bar s} \;=\; \min_{\sigma\in I_{\overline{R},\bar s}}\lambda_{\sigma,\overline{R},\bar s}\;\le\;\lambda\quad\text{for all}\quad\lambda\in K\,.\tag{3.9}$$

By Eqs. (1.2) and (3.6)–(3.8) it is clear that

$$\tilde\lambda_{\overline{\sigma},\overline{R},\bar s}\;<\;\lambda_{\overline{\sigma},\overline{R},\bar s}\,.\tag{3.10}$$

First, we prove that for every $\lambda\in K$ we have $\inf_{u\in W_r^{1,p}(\mathbf{R}^N)}\mathcal{R}_\lambda(u)<0$. To this end, it is enough to prove that $\mathcal{R}_\lambda\big(w_{\overline{\sigma},\overline{R},\bar s}\big)<0$, where $w_{\overline{\sigma},\overline{R},\bar s}$ is defined in Eq. (3.5). Thanks to Eqs. (3.8)–(3.10), for every $\lambda\in K$ we have

$$\mathcal{R}_\lambda\big(w_{\overline{\sigma},\overline{R},\bar s}\big) \;=\; \frac{1}{p}\big\|w_{\overline{\sigma},\overline{R},\bar s}\big\|_r^p - \lambda\mathcal{F}_r\big(w_{\overline{\sigma},\overline{R},\bar s}\big) \;=\; \frac{1}{p}\big\|w_{\overline{\sigma},\overline{R},\bar s}\big\|_r^p\left(1-\frac{\lambda}{\tilde\lambda_{\overline{\sigma},\overline{R},\bar s}}\right)\;<\;0\,.$$

Taking into account Propositions 3.2 and 3.3 as well as [16, Theorem 2.7], for every $\lambda\in K$ we find an element $u_\lambda\in W_r^{1,p}(\mathbf{R}^N)$ such that $\mathcal{R}'_\lambda(u_\lambda)=0$ and

$$\mathcal{R}_\lambda(u_\lambda)\;=\;\inf_{u\in W_r^{1,p}}\mathcal{R}_\lambda(u)\;<\;0\,.\tag{3.11}$$

Now, we prove that for every $\lambda\in K$ the functional $\mathcal{R}_\lambda$ has the Mountain Pass geometry. Since $\alpha\ge0$, then Lemma 2.4 gives

$$\mathcal{F}(u)\;\le\;\|\alpha\|_{L^1}\,C''c_\infty^\nu\,\|u\|_{W^{1,p}}^\nu\,,\quad u\in W^{1,p}\big(\mathbf{R}^N\big)\,.\tag{3.12}$$

We take $\rho>0$ to satisfy

$$\rho\;<\;\min\left\{\big((\sup K)p\,\|\alpha\|_{L^1}C''c_\infty^\nu\big)^{\frac{1}{p-\nu}},\;\big\|w_{\overline{\sigma},\overline{R},\bar s}\big\|_r\right\}.$$

Consequently, by Eq. (3.12), for every $\lambda \in K$ we have

$$\begin{aligned}\mathcal{R}_\lambda(u) &\geq \frac{1}{p}\|u\|_r^p - \lambda\,\|\alpha\|_{L^1} C'' c_\infty^\nu \,\|u\|_r^\nu \\ &\geq \left(\frac{1}{p} - (\sup K)\,\|\alpha\|_{L^1}\, C'' c_\infty^\nu \rho^{\nu-p}\right)\rho^p \;\equiv\; \eta \;>\; 0\,, \quad \|u\|_r \;=\; \rho\,.\end{aligned}$$

By construction, one has $\left\|w_{\overline{\sigma},\overline{R},\bar{s}}\right\|_r > \rho$ and from above $\mathcal{R}_\lambda\big(w_{\overline{\sigma},\overline{R},\bar{s}}\big) < 0 = \mathcal{R}_\lambda(0)$. Beside of these facts, Propositions 3.3 allows us to apply the Mountain Pass theorem (see for instance [16, Theorem 2.2]). Namely, for every $\lambda \in K$ there exists an element $v_\lambda \in W_r^{1,p}\big(\mathbf{R}^N\big)$ such that $\mathcal{R}'_\lambda(v_\lambda) = 0$ and $\mathcal{R}_\lambda(v_\lambda) \geq \eta > 0$.

Now, we will prove Eq. (1.4). As far as the v_λ is concerned, for every $\lambda \in K$ the mountain pass level $\mathcal{R}_\lambda(v_\lambda)$ is characterized as

$$\mathcal{R}_\lambda(v_\lambda) \;=\; \inf_{g\in\Gamma}\max_{t\in[0,1]} \mathcal{R}_\lambda(g(t))\,, \tag{3.13}$$

where

$$\Gamma \;=\; \left\{g \in C\big([0,1]; W_r^{1,p}\big(\mathbf{R}^N\big)\big) : g(0) = 0,\; g(1) = w_{\overline{\sigma},\overline{R},\bar{s}}\right\}.$$

Let $g_0 : [0,1] \to W_r^{1,p}\big(\mathbf{R}^N\big)$ be defined by $g_0(t) = t w_{\overline{\sigma},\overline{R},\bar{s}}$. Since $g_0 \in \Gamma$, by using Eq. (3.13), one has for every $\lambda \in K$

$$\mathcal{R}_\lambda(v_\lambda) \;\leq\; \max_{t\in[0,1]} \mathcal{R}_\lambda(g_0(t)) \;\leq\; \frac{1}{p}\left\|w_{\overline{\sigma},\overline{R},\bar{s}}\right\|_r^p + (\sup K)\max_{t\in[0,1]}\left|\mathcal{F}_r\big(t w_{\overline{\sigma},\overline{R},\bar{s}}\big)\right| \;\equiv\; C'''\,.$$

($C''' > 0$ does not depend on $\lambda \in K$.) Combining the above inequality with Eq. (3.4), for every $\lambda \in K$ it follows

$$\|v_\lambda\|_r^p \;\leq\; p(\sup K) C\,\|\alpha\|_{L^1}\big(c_\infty\,\|v_\lambda\|_r + c_\infty^\gamma\,\|v_\lambda\|_r^\gamma\big) + p\,C'''\,.$$

Since $1 < \gamma < p$, we have at once that $\sup_{\lambda\in K}\|v_\lambda\|_r < +\infty$. On the other hand, since $\mathcal{R}_\lambda(u_\lambda) < 0$ for every $\lambda \in K$ (see Eq. (3.11)), a direct application of Eq. (3.4) shows that $\sup_{\lambda\in K}\|u_\lambda\|_r < +\infty$ as well. Thus, relation (1.4) is completely proved.

Suppose now that $f(s) = 0$ for every $s \leq 0$, and let u be a weak solution of Eq. (P_λ) for some $\lambda > 0$, i.e., for every $v \in W^{1,p}\big(\mathbf{R}^N\big)$

$$\int_{\mathbf{R}^N}\big(|\nabla u|^{p-2}\,\nabla u\,\nabla v + |u|^{p-2}uv\big)\,dx \;=\; \lambda\int_{\mathbf{R}^N}\alpha(x) f(u(x)) v(x)\,dx\,. \tag{3.14}$$

Set $S = \big\{x \in \mathbf{R}^N : u(x) < 0\big\}$, and assume that $S \neq \emptyset$. Since (the representation of) u is continuous, the set S is open. Applying Eq. (3.14) with $v \equiv u_S = \min\{u, 0\} \in W^{1,p}\big(\mathbf{R}^N\big)$ one has

$$0 \;=\; \int_{\mathbf{R}^N}\big(|\nabla u|^{p-2}\,\nabla u\,\nabla u_S + |u|^{p-2}\,u u_S\big)\,dx \;=\; \int_S\big(|\nabla u|^p + |u|^p\big)\,dx \;=\; \|u\|_{W^{1,p}(S)}^p\,,$$

which contradicts the choice of the set S. This completes the proof. □

In the sequel, we are dealing with the proof of Theorem 1.2. In order to do this, we recall the following recent critical point result.

Theorem 3.4 [4, Theorem 2.1] *Let X be a separable and reflexive real Banach space, and let $\Phi, J : X \to \mathbf{R}$ be two continuously Gâteaux differentiable functionals. Assume that there exists $x_0 \in X$ such that $\Phi(x_0) = J(x_0) = 0$ and $\Phi(x) \geq 0$ for every $x \in X$ and that there exists $x_1 \in X$, $\rho > 0$ such that*

(i) $\rho < \Phi(x_1)$,

(ii) $\sup_{\Phi(x)<\rho} J(x) < \rho\,\frac{J(x_1)}{\Phi(x_1)}$.

Further, put

$$\overline{a} \;=\; \zeta\rho\left(\rho\,\frac{J(x_1)}{\Phi(x_1)} - \sup_{\Phi(x)<\rho} J(x)\right)^{-1} \quad \textit{where} \quad \zeta \;>\; 1\,,$$

assume that the functional $\Phi - \lambda J$ is sequentially weakly lower semicontinuous, satisfies the Palais-Smale condition and

(iii) $\lim_{\|x\|\to+\infty}(\Phi(x) - \lambda J(x)) = +\infty$, *for every* $\lambda \in [0, \overline{a}]$.

Then there is an open interval $\Lambda \subseteq [0, \overline{a}]$ and a number $\mu > 0$ such that for each $\lambda \in \Lambda$, the equation $\Phi'(x) - \lambda J'(x) = 0$ admits at least three solutions in X having norm less than μ.

Proposition 3.5 *For every $\lambda \in \mathbf{R}$, $\mathcal{R}_\lambda$ is sequentially weakly lower semicontinuous.*

Proof. Since the function $u \mapsto \|u\|^p_{W^{1,p}_r}$ is sequentially weakly lower semicontinuous (see [7, Proposition III.5]), it suffices to prove that the functional $\mathcal{F}_r$ is sequentially weakly continuous. To this end, let $\{u_n\}$ be a sequence in $W^{1,p}_r(\mathbf{R}^N)$ which converges weakly to $u \in W^{1,p}_r(\mathbf{R}^N)$ and suppose that the sequence $\{\mathcal{F}_r(u_n)\}$ does not converge to $\mathcal{F}_r(u)$ as $n \to \infty$. Therefore, there exist $\overline{\varepsilon} > 0$ and a subsequence of $\{u_n\}$, denoted again by $\{u_n\}$, such that $0 < \overline{\varepsilon} \le |\mathcal{F}_r(u_n) - \mathcal{F}_r(u)|$ for every $n \in \mathbf{N}$, and $u_n \to u$ strongly in $L^\infty(\mathbf{R}^N)$ (see Proposition 2.3). By the mean value theorem and Eq. (2.1), for some $\theta_n \in]0,1[$ with n large enough, one has

$$\begin{aligned} 0 < \overline{\varepsilon} &\le \left|\langle \mathcal{F}'_r(u + \theta_n(u_n - u)), u_n - u\rangle_{W^{1,p}}\right| \\ &\le \int_{\mathbf{R}^N} \alpha(x)\,|f(u(x) + \theta_n(u_n(x) - u(x)))|\,|u_n(x) - u(x)|\,dx \\ &\le \|\alpha\|_{L^1} \max\{|f(s)| : |s| \le \|u\|_{L^\infty} + 1\}\,\|u_n - u\|_{L^\infty}\,. \end{aligned}$$

But the last term tends to 0, which is a contradiction. □

Proof of Theorem 1.2. Let $X = W^{1,p}_r(\mathbf{R}^N)$ and for every $u \in W^{1,p}_r(\mathbf{R}^N)$ put $\Phi(u) = \frac{1}{p}\|u\|^p_r$ and $J(u) = \mathcal{F}_r(u)$ in Theorem 3.4. Thus, $\mathcal{R}_\lambda = \Phi - \lambda J$.

By hypotheses, there exists $(\overline{R}, \bar{s}) \in I^+_\alpha \times I^+_F$ as well as $\overline{\sigma} = \sigma(\overline{R}, \bar{s}) \in I_{\overline{R},\bar{s}}$ such that

$$\lambda_{\overline{\sigma},\overline{R},\bar{s}} = \min_{\sigma\in I_{\overline{R},\bar{s}}} \lambda_{\sigma,\overline{R},\bar{s}} = \lambda^*\,.$$

Similarly as in Eq. (3.10), we have

$$\tilde{\lambda}_{\overline{\sigma},\overline{R},\bar{s}} < \lambda_{\overline{\sigma},\overline{R},\bar{s}}\,. \tag{3.15}$$

By Eq. (3.12) and $\nu > p$ one readily has that

$$\lim_{\rho\to0^+} \frac{\sup\{\mathcal{F}_r(u) : \|u\|^p_r < p\rho\}}{\rho} = 0\,.$$

In view of the above limit, as well as Eqs. (3.5), (3.8) and (3.15), we may fix a number $\rho > 0$ so small that

$$\begin{cases} p\rho < \left\|w_{\overline{\sigma},\overline{R},\bar{s}}\right\|^p_r\,, \\ \dfrac{\sup\left\{\mathcal{F}_r(u) : \|u\|^p_r < p\rho\right\}}{\rho} < \dfrac{1}{\tilde{\lambda}_{\overline{\sigma},\overline{R},\bar{s}}}\,, \\ (1+\rho)\left(\dfrac{1}{\tilde{\lambda}_{\overline{\sigma},\overline{R},\bar{s}}} - \dfrac{\sup\{\mathcal{F}_r(u) : \|u\|^p_r < p\rho\}}{\rho}\right)^{-1} < \lambda_{\overline{\sigma},\overline{R},\bar{s}}\,. \end{cases}$$

Moreover, in Theorem 3.4 set $x_0 = 0$, $x_1 = w_{\overline{\sigma},\overline{R},\bar{s}}$, $\zeta = 1 + \rho$ and

$$\overline{a} = (1+\rho)\left(\frac{1}{\tilde{\lambda}_{\overline{\sigma},\overline{R},\bar{s}}} - \frac{\sup\{\mathcal{F}_r(u) : \|u\|^p_r < p\rho\}}{\rho}\right)^{-1}.$$

On account of Propositions 3.2, 3.3 and 3.5, the assumptions of Theorem 3.4 are fulfilled.

Thus, there exists an open interval $\Lambda \subseteq [0, \overline{a}] \subset \left[0, \lambda_{\overline{\sigma},\overline{R},\bar{s}}\right] = [0, \lambda^*]$ and a number $\mu > 0$ such that for each $\lambda \in \Lambda$ the functional $\mathcal{R}_\lambda = \Phi - \lambda J$ admits at least three critical points in $W^{1,p}_r(\mathbf{R}^N)$ having norm less than μ. Thus, it remains to apply again Propositions 2.2 and 2.1, respectively. □

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