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Multiplicity Results for an Eigenvalue Problem for Hemivariational Inequalities in Strip-Like Domains ⋆

ALEXANDRU KRISTÁLY
Faculty of Mathematics and Informatics, Babeş-Bolyai University, 3400 Cluj-Napoca, Romania.
e-mail: kristalysandor@email.ro

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Abstract. In this paper we study the multiplicity of solutions for a class of eigenvalue problems for hemivariational inequalities in strip-like domains. The first result is based on a recent abstract theorem of Marano and Motreanu, obtaining at least three distinct, axially symmetric solutions for certain eigenvalues. In the second result, a version of the fountain theorem of Bartsch which involves the nonsmooth Cerami compactness condition, provides not only infinitely many axially symmetric solutions but also axially nonsymmetric solutions in certain dimensions. In both cases the principle of symmetric criticality for locally Lipschitz functions plays a crucial role.

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1. Introduction

Let ω be a bounded open set in $\mathbb{R}^m$ with smooth boundary and let $\Omega = \omega \times \mathbb{R}^{N-m}$ be a strip-like domain; $m \geqslant 1$, $N - m \geqslant 2$. Let $F\colon\ \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function which is locally Lipschitz in the second variable such that

(F1) $F(x, 0) = 0$, and there exist $c_1 > 0$ and $p \in]2, 2^*[$ such that

$$|\xi| \leqslant c_1(|s| + |s|^{p-1})$$

for all $s \in \mathbb{R}$, $\xi \in \partial F(x, s)$ and a.e. $x \in \Omega$.

We denoted by $\partial F(x, s)$ the generalized gradient of $F(x, \cdot)$ at the point $s \in \mathbb{R}$, while $2^* = 2N(N-2)^{-1}$ is the Sobolev critical exponent.

In this paper we study the following *eigenvalue problem for hemivariational inequalities*. For $\lambda > 0$, denote by (EPHI$_\lambda$): *Find $u \in H_0^1(\Omega)$ such that*

$$\int_\Omega \nabla u \nabla w \,\mathrm{d}x + \lambda \int_\Omega F^0(x, u(x); -w(x)) \,\mathrm{d}x \geqslant 0 \quad \text{for all } w \in H_0^1(\Omega).$$

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The expression $F^0(x, s; t)$ stands for the generalized directional derivative of $F(x, \cdot)$ at the point $s \in \mathbb{R}$ in the direction $t \in \mathbb{R}$.

The motivation to study this type of problem comes from mathematical physics. Indeed, one often encounters problems which can be formulated in terms of some sort of *variational inequalities* which can be well analyzed in weak form (see Duvaut and Lions [10]). On the other hand, motivated also by some mechanical problems where certain nondifferentiable term perturbs the classical function, Panagiotopoulos [25] developed a more realistic approach, the so-called *theory of hemivariational inequalities* (see also the monographs [21–23]). The formulation of (EPHI_λ) is inspired by this theory. Moreover, if we particularize the form of F (see Remark 3.2), then (EPHI_λ) reduces to the following eigenvalue problem

$$-\Delta u = \lambda f(x, u) \quad \text{in } \Omega,\ u \in H_0^1(\Omega), \tag{EP_λ}$$

which is a simplified form of certain stationary waves in the nonlinear Klein–Gordon or Schrödinger equations (see, for instance, Amick [1]). Under some restrictive conditions on the nonlinear term f, (EP_λ) has been firstly studied by Esteban [11]. Further investigations, closely related to [11] can be found in the papers of Burton [6], Fan and Zhao [13], Grossinho [14], Schindler [30], Tersian [31].

Although related problems to (EP_λ) have been extensively studied on *bounded domain*s (see, for instance, the papers of Raymond [26], Guo and Webb [15], and references therein), in *unbounded domain*s the problem is more delicate, due to the lack of compactness in the Sobolev embeddings. Variational and/or topological methods are combined with different technics to overcome this difficulty: approximation by bounded sub-domains (see [11]); the use of weighted Sobolev spaces in order to obtain compact embeddings (see [5]); the use of Sobolev spaces with symmetric functions (see [11]); the use of an order-preserving operator on Hilbert space (see [6]).

In order to solve (EP_λ), Esteban [11] used a minimization procedure via axially symmetric functions. The first purpose of this paper is to give a new approach to treat eigenvalue problems on strip-like domains. This technic is based on the recent critical-point result of Marano and Motreanu (see [20]) which will be combined with the principle of symmetric criticality for locally Lipschitz functions (see [17]), establishing for certain eigenvalues the existence of at least three distinct, axially symmetric solutions to (EPHI_λ). Actually, the abstract result of [20] is an extension of Theorem 1 from the paper [27] of Ricceri (see also [28]) to the Motreanu–Panagiotopoulos type functionals (see [21, Chapter 3]). Marano and Motreanu applied their abstract result to elliptic eigenvalue problems with highly discontinuous nonlinearities on bounded domains. We mention that there are several further papers dedicated to different applications of Ricceri's result (see [29], and the references cited there). However, as far as I know, the present paper gives the first application of the Ricceri type results on unbounded domains.

In the case of strip-like domains, the space of axially symmetric functions has been the main tool in the investigations, due to its 'good behavior' concerning the compact embeddings (note that $N \geqslant m+2$, see [12]); this is the reason why many authors used this space in their works (see [11, 13, 14, 31]). On the other hand, no attention has been paid in the literature to the existence of axially *non*symmetric solutions, even in the classical case (EP$_\lambda$). Therefore, the study of existence of axially nonsymmetric solutions for (EPHI$_\lambda$) constitutes the second task of this paper. A nonsmooth version of the fountain theorem of Bartsch [3] provides not only infinitely many axially symmetric solutions but also axially nonsymmetric solutions, when $N = m+4$ or $N \geqslant m+6$.

The paper is organized as follows. Since we will use some elements from the theory of subdifferential calculus of Clarke [8], in Section 2 some basic facts about locally Lipschitz functions are recalled. In Section 3 the statement of our main theorems are given, including also some simple examples which illustrate their applicability. In Section 4 some auxiliary results are collected while the next two sections are devoted to the proofs of the main results.

2. Basic Notions

Let $(X, \|\cdot\|)$ be a real Banach space and X^* its topological dual. A function $h\colon X \to \mathbb{R}$ is called *locally Lipschitz* if each point $u \in X$ possesses a neighborhood $\mathcal{N}_u$ such that $|h(u_1) - h(u_2)| \leqslant L\|u_1 - u_2\|$ for all $u_1, u_2 \in \mathcal{N}_u$, for a constant $L > 0$ depending on $\mathcal{N}_u$. The *generalized directional derivative* of h at the point $u \in X$ in the direction $z \in X$ is

$$h^0(u; z) = \limsup_{w \to u,\ t \to 0^+} \frac{h(w + tz) - h(w)}{t}.$$

The *generalized gradient* of h at $u \in X$ is defined by

$$\partial h(u) = \{x^* \in X^* : \langle x^*, z\rangle_X \leqslant h^0(u; z) \quad \text{for all } z \in X\},$$

which is a nonempty, convex and w^*-compact subset of X^*, where $\langle \cdot, \cdot \rangle_X$ is the duality pairing between X^* and X.

Now, we list some fundamental properties of the generalized directional derivative and gradient which will be used through the whole paper.

PROPOSITION 2.1 (see [8]). (i) $(-h)^0(u; z) = h^0(u; -z)$ *for all* $u, z \in X$.

(ii) $h^0(u; z) = \max\{\langle x^*, z\rangle_X : x^* \in \partial h(u)\}$ *for all* $u, z \in X$.

(iii) *Let* $j\colon X \to \mathbb{R}$ *be a continuously differentiable function. Then* $\partial j(u) = \{j'(u)\}$, $j^0(u; z)$ *coincides with* $\langle j'(u), z\rangle_X$ *and* $(h + j)^0(u; z) = h^0(u; z) + \langle j'(u), z\rangle_X$ *for all* $u, z \in X$. *Moreover,* $\partial(h + j)(u) = \partial h(u) + j'(u)$, $\partial(hj)(u) \subseteq j(u)\partial h(u) + h(u)j'(u)$ *and* $\partial(\lambda h)(u) = \lambda \partial h(u)$ *for all* $u \in X$ *and* $\lambda \in \mathbb{R}$.

(iv) (Lebourg's mean value theorem) *Let* u *and* v *two points in* X. *Then there exists a point* w *in the open segment between* u *and* v, *and* $x^*_w \in \partial h(w)$ *such that*

$$h(u) - h(v) = \langle x^*_w, u - v\rangle_X.$$

A point $u \in X$ is a *critical point* of h if $0 \in \partial h(u)$, that is, $h^0(u; w) \geqslant 0$ for all $w \in X$ (see Chang [7]); $c = h(u)$ is a *critical value*. We define $m_h(u) = \inf\{\|x^*\|_X : x^* \in \partial h(u)\}$ (we used the notation $\|x^*\|_X$ instead of $\|x^*\|_{X^*}$). It is clear that $m_h(u)$ is attained, since $\partial h(u)$ is w^*-compact.

The function h *satisfies the nonsmooth Palais–Smale condition at level* $c \in \mathbb{R}$ (shortly $(PS)_c$), if every sequence $\{x_n\} \subset X$ such that $h(x_n) \to c$ and $m_h(x_n) \to 0$ contains a convergent subsequence in the norm of X (see [7]).

We say that h *satisfies the nonsmooth Cerami condition at level* $c \in \mathbb{R}$ (shortly $(C)_c$), if every sequence $\{x_n\} \subset X$ such that $h(x_n) \to c$ and $(1+\|x_n\|)m_h(x_n) \to 0$ contains a convergent subsequence in the norm of X (see [16]).

3. Main Results and Examples

Throughout this paper, Ω will be a *strip-like domain*, that is $\Omega = \omega \times \mathbb{R}^{N-m}$, where ω is a bounded open set in $\mathbb{R}^m$ with smooth boundary and $m \geqslant 1$, $N - m \geqslant 2$. $H_0^1(\Omega)$ is the usual Sobolev space endowed with the inner product $\langle u, v\rangle_0 = \int_\Omega \nabla u \nabla v \,\mathrm{d}x$ and norm $\|\cdot\|_0 = \sqrt{\langle\cdot,\cdot\rangle_0}$, while the norm of $L^\alpha(\Omega)$ will be denoted by $\|\cdot\|_\alpha$. Since Ω has the cone property, we have the continuous embedding $H_0^1(\Omega) \hookrightarrow L^\alpha(\Omega)$, $\alpha \in [2, 2^*]$, that is, there exists $k_\alpha > 0$ such that $\|u\|_\alpha \leqslant k_\alpha \|u\|_0$ for all $u \in H_0^1(\Omega)$.

We say that a function $h\colon \Omega \to \mathbb{R}$ is *axially symmetric*, if $h(x, y) = h(x, gy)$ for all $x \in \omega$, $y \in \mathbb{R}^{N-m}$ and $g \in O(N - m)$. In particular, we denote by $H_{0,s}^1(\Omega)$ the closed subspace of axially symmetric functions of $H_0^1(\Omega)$. $u \in H_0^1(\Omega)$ is called *axially nonsymmetric*, if it is not axially symmetric.

For the first result, we make the following assumptions on the nonlinearity term F.

(F2) $$\lim_{s\to 0} \frac{\max\{|\xi| : \xi \in \partial F(x, s)\}}{s} = 0 \quad \text{uniformly for a.e. } x \in \Omega.$$

(F3) There exist $q \in]0, 2[$, $\nu \in [2, 2^*]$, $\alpha \in L^{\frac{\nu}{\nu-q}}(\Omega)$ and $\beta \in L^1(\Omega)$ such that
$$F(x, s) \leqslant \alpha(x)|s|^q + \beta(x)$$
for all $s \in \mathbb{R}$ and a.e. $x \in \Omega$.

(F4) There exists $u_0 \in H_{0,s}^1(\Omega)$ such that $\int_\Omega F(x, u_0(x))\,\mathrm{d}x > 0$.

THEOREM 3.1. *Let* $F\colon \Omega \times \mathbb{R} \to \mathbb{R}$ *be a function which satisfies* (F1)–(F4) *and* $F(\cdot, s)$ *is axially symmetric for all* $s \in \mathbb{R}$. *Then there exist an open interval* $\Lambda_0 \subset [0, +\infty[$ *and a number* $\sigma > 0$ *such that for every* $\lambda \in \Lambda_0$, (EPHI$_\lambda$) *has at least three distinct solutions which are axially symmetric having* $\|\cdot\|_0$*-norms less than* σ.

The following theorem can be considered as an extension of Bartsch and Willem's result (see [4]) to the case of strip-like domains. We require the following assumption on F.

(F5) There exist $\nu \geqslant 1$ and $\gamma \in L^\infty(\Omega)$ with $\operatorname{essinf}_{x\in\Omega}\gamma(x) = \gamma_0 > 0$ such that

$$2F(x,s) + F^0(x,s;-s) \leqslant -\gamma(x)|s|^\nu$$

for all $s \in \mathbb{R}$ and a.e. $x \in \Omega$.

THEOREM 3.2. *Let F: $\Omega \times \mathbb{R} \to \mathbb{R}$ be a function which satisfies* (F1), (F2), *and* (F5) *for some $\nu > \max\{2, N(p-2)/2\}$. If F is axially symmetric in the first variable and even in the second variable then* (EPHI$_\lambda$) *has infinitely many axially symmetric solutions for every $\lambda > 0$. In addition, if $N = m + 4$ or $N \geqslant m + 6$,* (EPHI$_\lambda$) *has infinitely many axially nonsymmetric solutions.*

Remark 3.1. The inequality from (F5) is a nonsmooth version of one introduced by Costa and Magalhães [9]. Let us suppose for a moment that F is autonomous. Note that (F5) is implied in many cases by the following condition (of Ambrosetti–Rabinowitz type):

$$\nu F(s) + F^0(s;-s) \leqslant 0 \quad \text{for all } s \in \mathbb{R}, \tag{1}$$

where $\nu > 2$. Indeed, from (1) and Lebourg's mean value theorem, applied to the locally Lipschitz function g: $]0, +\infty[\to \mathbb{R}$, $g(t) = t^{-\nu}F(tu)$ (with arbitrary fixed $u \in \mathbb{R}$) we obtain that $t^{-\nu}F(tu) \geqslant s^{-\nu}F(su)$ for all $t \geqslant s > 0$. If we assume in addition that $\liminf_{s\to 0} \frac{F(s)}{|s|^\nu} \geqslant a_0 > 0$, from the above relation (substituting $t = 1$) we have for $u \neq 0$ that $F(u) \geqslant \liminf_{s\to 0^+} \frac{F(su)}{|su|^\nu}|u|^\nu \geqslant a_0|u|^\nu$. So, $2F(u) + F^0(u;-u) \leqslant (2-\nu)F(u) \leqslant -\gamma_0|u|^\nu$, where $\gamma_0 = a_0(\nu - 2) > 0$.

Remark 3.2. Let f: $\Omega \times \mathbb{R} \to \mathbb{R}$ be a measurable (not necessarily continuous) function and suppose that there exists $c > 0$ such that $|f(x,s)| \leqslant c(|s| + |s|^{p-1})$ for all $s \in \mathbb{R}$ and a.e. $x \in \Omega$. Define F: $\Omega \times \mathbb{R} \to \mathbb{R}$ by $F(x,s) = \int_0^s f(x,t)\,\mathrm{d}t$. Then F is a Carathéodory function which is locally Lipschitz in the second variable which satisfies the growth condition from (F1). Indeed, since $f(x,\cdot) \in L^\infty_{loc}(\mathbb{R})$, by [21, Proposition 1.7] we have $\partial F(x,s) = [\underline{f}(x,s), \overline{f}(x,s)]$ for all $s \in \mathbb{R}$ and a.e. $x \in \Omega$, where $\underline{f}(x,s) = \lim_{\delta\to 0^+} \operatorname{essinf}_{|t-s|<\delta} f(x,t)$ and $\overline{f}(x,s) = \lim_{\delta\to 0^+} \operatorname{esssup}_{|t-s|<\delta} f(x,t)$.

Moreover, if f is continuous in the second variable, then $\partial F(x,s) = \{f(x,s)\}$ for all $s \in \mathbb{R}$ and a.e. $x \in \Omega$. Therefore, the inequality from (EPHI$_\lambda$) takes the form

$$\int_\Omega \nabla u \nabla w\,\mathrm{d}x - \lambda \int_\Omega f(x,u(x))w(x)\,\mathrm{d}x = 0 \quad \text{for all } w \in H_0^1(\Omega),$$

that is, $u \in H_0^1(\Omega)$ is a weak solution of (EP$_\lambda$) in the usual sense.

Remark 3.3. In view of Remark 3.2, under additional hypotheses on f, corresponding to (F2)–(F5), it is possible to state the smooth counterparts of Theorems 3.1 and 3.2.

In the final of this section, we give some examples.

EXAMPLE 3.1. Let $p \in]2, 2^*[$ and $a\colon \Omega \to \mathbb{R}$ be a continuous, nonnegative, not identically zero, axially symmetric function with compact support in Ω. Then there exist an open interval $\Lambda_0 \subset [0, +\infty[$ and a number $\sigma > 0$ such that for every $\lambda \in \Lambda_0$, the problem

$$-\Delta u = \lambda a(x)|u|^{p-2}u \cos|u|^p \quad \text{in } \Omega,\ u \in H_0^1(\Omega)$$

has at least three distinct, axially symmetric solutions which have norms less than σ.

Indeed, let us define $F\colon \Omega \times \mathbb{R} \to \mathbb{R}$ by $F(x,s) = (1/p)a(x)\sin|s|^p$. Clearly, $F(x,\cdot)$ is continuously differentiable and (F1)–(F2) hold immediately. For (F3) we choose $\alpha(x) = \beta(x) = a(x)/p$ $(x \in \Omega)$, and any $q \in]0, 2[$, $\nu \in [2, 2^*]$. Since a is an axially symmetric function, $\operatorname{supp} a$ will be an $id^m \times O(N-m)$-invariant set, i.e., if $(x, y) \in \operatorname{supp} a$ then $(x, gy) \in \operatorname{supp} a$ for all $g \in O(N-m)$. Therefore, it is possible to fix an element $u_0 \in H^1_{0,s}(\Omega)$ such that $u_0(x) = (\pi/2)^{1/p}$ for all $x \in \operatorname{supp} a$. One has that

$$\int_\Omega F(x, u_0(x))\, \mathrm{d}x = \frac{1}{p}\int_{\operatorname{supp} a} a(x)\sin|u_0(x)|^p\, \mathrm{d}x = \frac{1}{p}\int_{\operatorname{supp} a} a(x)\, \mathrm{d}x > 0.$$

The conclusion follows from Theorem 3.1 and Remark 3.2.

EXAMPLE 3.2. Let $a\colon \Omega \to \mathbb{R}$ be as in Example 3.1 and let $F\colon \Omega \times \mathbb{R} \to \mathbb{R}$ defined by $F(x,s) = a(x)\min\{s^3, |s|^q\}$, where $q \in]0, 2[$ is a fixed number. The conclusion of Theorem 3.1 holds in dimensions $N \in \{3, 4, 5\}$.

Indeed, we can verify (F1)–(F4), choosing $p = 3$, $\alpha = a$, $\beta = 0$ and u_0 as in Example 3.1.

EXAMPLE 3.3. Let $p \in]2, 2^*[$. Then, for all $\lambda > 0$, the problem

$$-\Delta u = \lambda|u|^{p-2}u \quad \text{in } \Omega,\ u \in H_0^1(\Omega),$$

has infinitely many axially symmetric solutions. Moreover, if $N = m + 4$ or $N \geqslant m + 6$, the problem has infinitely many axially nonsymmetric solutions.

Indeed, consider the (continuously differentiable) function $F(x,s) = F(s) = |s|^p$, which verifies obviously the assumptions of Theorem 3.2 (choose $\nu = p$).

EXAMPLE 3.4. We denote by $\lfloor u \rfloor$ the nearest integer to $u \in \mathbb{R}$, if $u + (1/2) \notin \mathbb{Z}$; otherwise we put $\lfloor u \rfloor = u$. Let $N \in \{3, 4, 5\}$ and let $F\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x,s) = F(s) = \int_0^s \lfloor t|t| \rfloor\, \mathrm{d}t + |s|^3.$$

It is clear that F is a locally Lipschitz, even function. Due to the first part of Remark 3.2, F verifies (F1) with the choice $p = 3$ while (F2) follows from the fact that $\lfloor t|t| \rfloor = 0$ if $t \in]-2^{-1/2}, 2^{-1/2}[$. Since F is even (in particular,

$F^0(s;-s) = F^0(-s;s)$ for all $s \in \mathbb{R}$), it is enough to very (F5) for $s \geqslant 0$. We have that $F(s) = s^3$ if $s \in [0, a_1]$, and $F(s) = s^3 + ns - \sum_{k=1}^{n} \sqrt{2k-1}/\sqrt{2}$ if $s \in]a_n, a_{n+1}]$, where $a_n = (2n-1)^{1/2}2^{-1/2}$, $n \in \mathbb{N} \setminus \{0\}$. Moreover, $F^0(s;-s) = -3s^3 - ns$ when $s \in]a_n, a_{n+1}[$ while $F^0(a_n;-a_n) = -3a_n^3 - (n-1)a_n$, since $\partial F(a_n) = [3a_n^2 + n - 1, 3a_n^2 + n]$, $n \in \mathbb{N} \setminus \{0\}$. Choosing $\gamma(x) = \gamma_0 = 1/3$ and $\nu = 3$, from the above expressions the required inequality yields. Therefore (EPHI$_\lambda$) has infinitely many axially symmetric solutions for every $\lambda > 0$. Moreover, if $\Omega = \omega \times \mathbb{R}^4$, where ω is an open bounded interval in $\mathbb{R}$, then (EPHI$_\lambda$) has infinitely many axially nonsymmetric solutions for every $\lambda > 0$.

4. Some Auxiliary Results

LEMMA 4.1. *If $F\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies* (F1) *and* (F2)*, for every $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that*

(i) $|\xi| \leqslant \varepsilon|s| + c(\varepsilon)|s|^{p-1}$ *for all $s \in \mathbb{R}$, $\xi \in \partial F(x,s)$ and a.e. $x \in \Omega$.*
(ii) $|F(x,s)| \leqslant \varepsilon s^2 + c(\varepsilon)|s|^p$ *for all $s \in \mathbb{R}$ and a.e. $x \in \Omega$.*

Proof. (i) Let $\varepsilon > 0$. Condition (F2) implies that there exists a $\delta := \delta(\varepsilon) > 0$ such that $|\xi| \leqslant \varepsilon|s|$ for $|s| < \delta$, $\xi \in \partial F(x,s)$ and a.e. $x \in \Omega$. If $|s| \geqslant \delta$, (F1) implies that $|\xi| \leqslant c_1(|s|^{2-p} + 1)|s|^{p-1} \leqslant c(\delta)|s|^{p-1}$ for all $\xi \in \partial F(x,s)$ and a.e. $x \in \Omega$. Combining the above relations, the required inequality yields.

(ii) We use Lebourg's mean value theorem, obtaining $|F(x,s)| = |F(x,s) - F(x,0)| = |\xi_{\theta s}s|$ for some $\xi_{\theta s} \in \partial F(x,\theta s)$ where $\theta \in]0,1[$. Now, we apply (i). □

Define $\mathcal{F}\colon H_0^1(\Omega) \to \mathbb{R}$ by

$$\mathcal{F}(u) = \int_\Omega F(x, u(x))\,\mathrm{d}x.$$

The following result plays a crucial role in the study of (EPHI$_\lambda$).

LEMMA 4.2. *Let $F\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function which satisfies* (F1). *Then $\mathcal{F}$ is well-defined and it is locally Lipschitz. Moreover, let E be a closed subspace of $H_0^1(\Omega)$ and $\mathcal{F}_E$ the restriction of $\mathcal{F}$ to E. Then*

$$\mathcal{F}_E^0(u;w) \leqslant \int_\Omega F^0(x, u(x); w(x))\,\mathrm{d}x \quad \textit{for all } u, w \in E. \tag{2}$$

Proof. Let us fix $s_1, s_2 \in \mathbb{R}$ arbitrary. By using Lebourg's mean value theorem, there exist $\theta \in]0,1[$ and $\xi_\theta \in \partial F(x, \theta s_1 + (1-\theta)s_2)$ such that $F(x,s_1) - F(x,s_2) = \xi_\theta(s_1 - s_2)$. By (F1) we conclude that

$$|F(x,s_1) - F(x,s_2)| \leqslant d|s_1 - s_2| \cdot [|s_1| + |s_2| + |s_1|^{p-1} + |s_2|^{p-1}] \tag{3}$$

for all $s_1, s_2 \in \mathbb{R}$ and a.e. $x \in \Omega$, where $d = d(c_1, p) > 0$. In particular, if $u \in H_0^1(\Omega)$, we obtain that

$$\begin{aligned}|\mathcal{F}(u)| &\leqslant \int_\Omega |F(x, u(x))|\, \mathrm{d}x \\ &\leqslant d(\|u\|_2^2 + \|u\|_p^p) \leqslant d(k_2^2\|u\|_0^2 + k_p^p\|u\|_0^p) < +\infty,\end{aligned}$$

that is, the function $\mathcal{F}$ is well-defined. Moreover, by (3), there exists $d_0 > 0$ such that for every $u, v \in H_0^1(\Omega)$

$$|\mathcal{F}(u) - \mathcal{F}(v)| \leqslant d_0\|u - v\|_0[\|u\|_0 + \|v\|_0 + \|u\|_0^{p-1} + \|v\|_0^{p-1}].$$

Therefore, $\mathcal{F}$ is a locally Lipschitz function on $H_0^1(\Omega)$.

Let us fix u and w in E. Since $F(x, \cdot)$ is continuous, $F^0(x, u(x); w(x))$ can be expressed as the upper limit of $(F(x, y + tw(x)) - F(x, y))/t$, where $t \to 0^+$ takes rational values and $y \to u(x)$ takes values in a countable dense subset of $\mathbb{R}$. Therefore, the map $x \mapsto F^0(x, u(x); w(x))$ is measurable as the 'countable limsup' of measurable functions of x. According to (F1) and Proposition 2.1(ii), the map $x \mapsto F^0(x, u(x); w(x))$ belongs to $L^1(\Omega)$.

Since E is separable, there are functions $u_n \in E$ and numbers $t_n \to 0^+$ such that $u_n \to u$ in E and

$$\mathcal{F}_E^0(u; w) = \lim_{n\to+\infty} \frac{\mathcal{F}_E(u_n + t_n w) - \mathcal{F}_E(u_n)}{t_n},$$

and without loss of generality, we may assume that there exist $h_2 \in L^2(\Omega, \mathbb{R}_+)$ and $h_p \in L^p(\Omega, \mathbb{R}_+)$ such that $|u_n(x)| \leqslant \min\{h_2(x), h_p(x)\}$ and $u_n(x) \to u(x)$ a.e. in Ω, as $n \to +\infty$.

We define $g_n\colon \Omega \to \mathbb{R} \cup \{+\infty\}$ by

$$\begin{aligned}g_n(x) = {}&-\frac{F(x, u_n(x) + t_n w(x)) - F(x, u_n(x))}{t_n} + \\ &+ d|w(x)|[|u_n(x) + t_n w(x)| + \\ &\qquad + |u_n(x)| + |u_n(x) + t_n w(x)|^{p-1} + |u_n(x)|^{p-1}].\end{aligned}$$

The maps g_n are measurable and nonnegative, see (3). From Fatou's lemma we have

$$I = \limsup_{n\to+\infty} \int_\Omega [-g_n(x)]\, \mathrm{d}x \leqslant \int_\Omega \limsup_{n\to+\infty}[-g_n(x)]\, \mathrm{d}x = J.$$

Let $B_n = A_n + g_n$, where

$$A_n(x) = \frac{F(x, u_n(x) + t_n w(x)) - F(x, u_n(x))}{t_n}.$$

By the Lebesgue dominated convergence theorem, we have

$$\lim_{n\to+\infty} \int_\Omega B_n\, \mathrm{d}x = 2d \int_\Omega |w(x)|(|u(x)| + |u(x)|^{p-1})\, \mathrm{d}x.$$

Therefore, we have

$$I = \limsup_{n\to+\infty} \frac{\mathcal{F}_E(u_n + t_n w) - \mathcal{F}_E(u_n)}{t_n} - \lim_{n\to+\infty} \int_\Omega B_n \, \mathrm{d}x$$
$$= \mathcal{F}_E^0(u; w) - 2d \int_\Omega |w(x)|(|u(x)| + |u(x)|^{p-1}) \, \mathrm{d}x.$$

Now, we estimate J. We have $J \leqslant J_A - J_B$, where

$$J_A = \int_\Omega \limsup_{n\to+\infty} A_n(x) \, \mathrm{d}x \quad \text{and} \quad J_B = \int_\Omega \liminf_{n\to+\infty} B_n(x) \, \mathrm{d}x.$$

Since $u_n(x) \to u(x)$ a.e. in Ω and $t_n \to 0^+$, we have

$$J_B = 2d \int_\Omega |w(x)|(|u(x)| + |u(x)|^{p-1}) \, \mathrm{d}x.$$

On the other hand,

$$J_A = \int_\Omega \limsup_{n\to+\infty} \frac{F(x, u_n(x) + t_n w(x)) - F(x, u_n(x))}{t_n} \, \mathrm{d}x$$
$$\leqslant \int_\Omega \limsup_{y\to u(x),\ t\to 0^+} \frac{F(x, y + tw(x)) - F(x, y)}{t} \, \mathrm{d}x$$
$$= \int_\Omega F^0(x, u(x); w(x)) \, \mathrm{d}x.$$

From the above estimations we obtain (2), which concludes the proof. □

As we already mentioned in the Introduction, the proof of Theorem 3.1 is based on a nonsmooth version of Ricceri's result (see [27, Theorem 1]), proved by Marano and Motreanu [20].

PROPOSITION 4.1 ([20, Theorem B]). *Let X be a separable and reflexive Banach space, let Ψ_1, Ψ_2: $X \to \mathbb{R}$ two locally Lipschitz functions and Λ be a real interval. Suppose that:*

(i) *Ψ_1 is weakly sequentially lower semicontinuous while Ψ_2 is weakly sequentially continuous.*
(ii) *For every $\lambda \in \Lambda$, the function $\Psi_1 + \lambda\Psi_2$ satisfies $(PS)_c$, $c \in \mathbb{R}$, together with $\lim_{\|u\|\to+\infty}(\Psi_1(u) + \lambda\Psi_2(u)) = +\infty$.*
(iii) *There exists a continuous concave function h: $\Lambda \to \mathbb{R}$ such that*

$$\sup_{\lambda\in\Lambda} \inf_{u\in X} (\Psi_1(u) + \lambda\Psi_2(u) + h(\lambda)) < \inf_{u\in X} \sup_{\lambda\in\Lambda} (\Psi_1(u) + \lambda\Psi_2(u) + h(\lambda)).$$

Then there is an open interval $\Lambda_0 \subseteq \Lambda$ and a number $\sigma > 0$ such that for each $\lambda \in \Lambda_0$ the function $\Psi_1 + \lambda\Psi_2$ has at least three critical points in X (with different critical values), having norm less than σ.

Remark 4.1. Note that Proposition 4.1 is a particular form of [20, Theorem B]. Indeed, in [20], the authors work in a very general framework, considering instead of Ψ_1 functions of the form $\Psi_1 + \psi$, where $\psi\colon X \to]-\infty, +\infty]$ is convex, proper, and lower semicontinuous, i.e. functions of Motreanu–Panagiotopoulos type (see [21, Chapter 3]). We emphasize that in our case (i.e. $\psi = 0$), the Palais–Smale condition and critical point notion of Motreanu and Panagiotopoulos coincide with those of Chang [7] (see also [21, Remark 3.1]).

In order to prove Theorem 3.2, we recall the following result.

PROPOSITION 4.2. *Let E be a Hilbert space, $\{e_j : j \in \mathbb{N}\}$ an orthonormal basis of E and set $E_k = \mathrm{span}\{e_1, \dots, e_k\}$. Let $h\colon E \to \mathbb{R}$ be an even, locally Lipschitz function such that:*

(i) *h satisfies $(C)_c$ for all $c > h(0)$.*
(ii) *For all $k \geqslant 1$ there exists $R_k > 0$ such that $h(u) \leqslant h(0)$, for all $u \in E_k$ with $\|u\| \geqslant R_k$.*
(iii) *There exist $k_0 \geqslant 1$, $b > h(0)$ and $\rho > 0$ such that $h(u) \geqslant b$ for every $u \in E_{k_0}^{\perp}$ with $\|u\| = \rho$.*

Then h possesses a sequence of critical values $\{c_k\}$ such that $c_k \to +\infty$ as $k \to +\infty$.

Remark 4.2. The above result is a nonsmooth version of the fountain theorem of Bartsch (see [3, Theorem 2.25]). Note that although the original result involves the Palais–Smale condition and not the Cerami one, this extension can be made by means of a suitable deformation lemma. For other nonsmooth extension, even for continuous functions, the reader can consult the paper of Arioli and Gazzola [2, Theorems 4.3 and 4.4]. See also [21, Corollary 2.7].

Let G be a compact Lie group which acts *linear isometrically* on the real Banach space $(X, \|\cdot\|)$, that is, the action $G \times X \to X\colon [g, u] \mapsto gu$ is continuous and for every $g \in G$, the map $u \mapsto gu$ is linear such that $\|gu\| = \|u\|$ for every $u \in X$. A function $h\colon X \to \mathbb{R}$ is *$G-$invariant* if $h(gu) = h(u)$ for all $g \in G$ and $u \in X$. Denoting by $X^G = \{u \in X : gu = u \text{ for all } g \in G\}$, we have the *principle of symmetric criticality* for locally Lipschitz functions, proved by Krawcewicz and Marzantowicz [17, p. 1045] (see also [18]).

PROPOSITION 4.3. *Let $h\colon X \to \mathbb{R}$ be a G-invariant, locally Lipschitz functional. If h_G denotes the restriction of h to X^G and $u \in X^G$ is a critical point of h_G then u is a critical point of h.*

Remark 4.3. For differentiable functions, the above principle has been proved by Palais [24].

5. Proof of Theorem 3.1

Let $\Psi\colon H_0^1(\Omega) \times [0, +\infty[\to \mathbb{R}$ defined by

$$\Psi(u, \lambda) = \frac{1}{2}\|u\|_0^2 - \lambda \mathcal{F}(u). \tag{4}$$

By virtue of Lemma 4.2, Ψ is well-defined and $\Psi(\cdot, \lambda)$ is a locally Lipschitz function for all $\lambda \in [0, +\infty[$. Let us denote by Ψ_s the restriction of Ψ to $H_{0,s}^1(\Omega) \times [0, +\infty[$, while $\mathcal{F}_s$, $\langle \cdot, \cdot \rangle_s$ and $\|\cdot\|_s$ the restrictions of $\mathcal{F}$, $\langle \cdot, \cdot \rangle_0$ and $\|\cdot\|_0$ to $H_{0,s}^1(\Omega)$, respectively. We will show that the assumptions of Proposition 4.1 are fulfilled with the following choice: $X = H_{0,s}^1(\Omega)$, $\Lambda = [0, +\infty[$, $\Psi_1 = (1/2)\|\cdot\|_s^2$, $\Psi_2 = -\mathcal{F}_s$ and $h(\lambda) = \rho_0\lambda$, $\lambda \in \Lambda$ ($\rho_0 > 0$ will be specified later).

Clearly, Ψ_1 and Ψ_2 are locally Lipschitz functions. Moreover, $\Psi_1 \in C^1(X, \mathbb{R})$. The weakly sequentially lower semicontinuity of Ψ_1 is obvious. Now, let $\{u_n\}$ be a sequence from X which converges weakly to $u \in X$. In particular, $\{u_n\}$ is bounded in X and by virtue of Lemma 4.1, $F(x, t) = o(t^2)$ as $t \to 0$ and $F(x, t) = o(t^{2^*})$ as $t \to +\infty$, for a.e. $x \in \Omega$. From [11, Lemma 4, p. 368] it follows $\mathcal{F}_s(u_n) \to \mathcal{F}_s(u)$, that is, Ψ_2 is weakly sequentially continuous. This proves (i) from Proposition 4.1.

Now, let us fix $\lambda \in \Lambda$. Firstly, we will show that

$$\lim_{\|u\|_s \to +\infty} \Psi_s(u, \lambda) = +\infty. \tag{5}$$

Indeed, due to (F3), by Hölder's inequality we have

$$\begin{aligned}\Psi_s(u, \lambda) &= \Psi_1(u) + \lambda\Psi_2(u)\\ &\geqslant \frac{1}{2}\|u\|_s^2 - \lambda \int_\Omega \alpha(x)|u(x)|^q \,\mathrm{d}x - \lambda \int_\Omega \beta(x)\,\mathrm{d}x\\ &\geqslant \frac{1}{2}\|u\|_s^2 - \lambda\|\alpha\|_{\frac{\nu}{\nu-q}}\|u\|_\nu^q - \lambda\|\beta\|_1.\end{aligned}$$

Since $X \hookrightarrow L^\nu(\Omega)$ is continuous and $q < 2$, it is clear that $\|u\|_s \to +\infty$ implies $\Psi_s(u, \lambda) \to +\infty$.

In the sequel, let us consider a sequence $\{u_n\}$ from X such that

$$\Psi_s(u_n, \lambda) \longrightarrow c \in \mathbb{R}, \tag{6}$$

$$m_{\Psi_s(\cdot,\lambda)}(u_n) \longrightarrow 0 \tag{7}$$

as $n \to +\infty$. It is clear from (5) and (6) that $\{u_n\}$ should be bounded in X. Since $N - m \geqslant 2$, the embedding $X \hookrightarrow L^p(\Omega)$ is compact (see [12, Theorem 1] or [19, Théorème III.2.]). Therefore, up to a subsequence, $u_n \rightharpoonup u$ in X and $u_n \to u$ in $L^p(\Omega)$. From Proposition 2.1(iii) we have

$$\Psi_s(\cdot, \lambda)^0(u_n; u - u_n) = \langle u_n, u - u_n\rangle_s + \lambda(-\mathcal{F}_s)^0(u_n; u - u_n),$$
$$\Psi_s(\cdot, \lambda)^0(u; u_n - u) = \langle u, u_n - u\rangle_s + \lambda(-\mathcal{F}_s)^0(u; u_n - u).$$

Adding these two relations and using Proposition 2.1(i), we have

$$\begin{aligned}\|u_n - u\|_s^2 = {}&\lambda[\mathcal{F}_s{}^0(u_n; -u + u_n) + \mathcal{F}_s{}^0(u; -u_n + u)] -\\ &- \Psi_s(\cdot, \lambda)^0(u_n; u - u_n) - \Psi_s(\cdot, \lambda)^0(u; u_n - u).\end{aligned}$$

On the other hand, there exists $z_n \in \partial\Psi_s(\cdot,\lambda)(u_n)$ such that $\|z_n\|_s = m_{\Psi_s(\cdot,\lambda)}(u_n)$. Here, we used the Riesz representation theorem. By (7), one has $\|z_n\|_s \to 0$ as $n \to +\infty$. Since $u_n \rightharpoonup u$ in X, fixing an element $z \in \partial\Psi_s(\cdot,\lambda)(u)$, we have $\langle z, u_n - u\rangle_s \to 0$. Therefore, by the inequality (2) (with $E = X$), Proposition 2.1(ii) and Lemma 4.1(i), we have

$$\begin{aligned}
\|u_n - u\|_s^2 &\leqslant \lambda \int_\Omega [F^0(x, u_n(x); -u(x) + u_n(x)) + F^0(x, u(x); -u_n(x) + \\
&\qquad + u(x))]\,\mathrm{d}x - \langle z_n, u - u_n\rangle_s - \langle z, u_n - u\rangle_s \\
&= \lambda \int_\Omega \max\{\xi_n(x)(-u(x) + u_n(x)) : \xi_n(x) \in \partial F(x, u_n(x))\}\,\mathrm{d}x + \\
&\quad + \lambda \int_\Omega \max\{\xi(x)(-u_n(x) + u(x)) : \xi(x) \in \partial F(x, u(x))\}\,\mathrm{d}x - \\
&\quad - \langle z_n, u - u_n\rangle_s - \langle z, u_n - u\rangle_s \\
&\leqslant \lambda \int_\Omega [\varepsilon(|u_n(x)| + |u(x)|) + c(\varepsilon)(|u_n(x)|^{p-1} + \\
&\qquad + |u(x)|^{p-1})]|u_n(x) - u(x)|\,\mathrm{d}x + \|z_n\|_s\|u - u_n\|_s - \\
&\quad - \langle z, u_n - u\rangle_s \\
&\leqslant 2\varepsilon\lambda k_2^2(\|u_n\|_s^2 + \|u\|_s^2) + \lambda c(\varepsilon)(\|u_n\|_p^{p-1} + \\
&\quad + \|u\|_p^{p-1})\|u_n - u\|_p + \|z_n\|_s\|u - u_n\|_s - \langle z, u_n - u\rangle_s.
\end{aligned}$$

Due to the arbitrariness of $\varepsilon > 0$, we have that $\|u_n - u\|_s^2 \to 0$ as $n \to +\infty$. Thus, $\{u_n\}$ converges strongly to u in X. This concludes (ii) from Proposition 4.1.

The proof of (iii) is similar to that of Ricceri [28]. By virtue of Lemma 4.1(ii), we have that for

$$g(r) := \sup\{\mathcal{F}_s(u) : \|u\|_s^2 \leqslant 2r\},$$

$\lim_{r\to 0^+}(g(r)/r) = 0$. Since $u_0 \in X$ from (F4) cannot be 0 (note that $F(x,0) = 0$ for a.e. $x \in \Omega$), choose η such that

$$0 < \eta < \frac{2}{\|u_0\|_s^2}\mathcal{F}_s(u_0).$$

Therefore, there exists $r_0 \in]0, \|u_0\|_s^2/2[$ such that $g(r_0) < \eta r_0$. Thus, $g(r_0) < (2r_0/\|u_0\|_s^2)\mathcal{F}_s(u_0)$. Let $\rho_0 > 0$ such that

$$g(r_0) < \rho_0 < \frac{2r_0}{\|u_0\|_s^2}\mathcal{F}_s(u_0). \tag{8}$$

We shall prove that $h \colon \Lambda \to \mathbb{R}$, defined by $h(\lambda) = \rho_0\lambda$ fulfils the inequality (iii) from Proposition 4.1.

Since $r_0 < \|u_0\|_s^2/2$, by (8) we have

$$\rho_0 < \mathcal{F}_s(u_0). \tag{9}$$

Clearly, the function $\lambda \mapsto \inf_{u\in X}((1/2)\|u\|_s^2 + \lambda(\rho_0 - \mathcal{F}_s(u)))$ is upper semicontinuous on Λ. Due to (9), we have

$$\lim_{\lambda\to+\infty} \inf_{u\in X}(\Psi_s(u,\lambda) + \rho_0\lambda) \leqslant \lim_{\lambda\to+\infty}(\tfrac{1}{2}\|u_0\|_s^2 + \lambda(\rho_0 - \mathcal{F}_s(u_0))) = -\infty.$$

Thus we find an element $\overline{\lambda} \in \Lambda$ such that

$$\sup_{\lambda\in\Lambda}\inf_{u\in X}(\Psi_s(u,\lambda) + \rho_0\lambda) = \inf_{u\in X}(\tfrac{1}{2}\|u\|_s^2 + \overline{\lambda}(\rho_0 - \mathcal{F}_s(u))). \tag{10}$$

Since $g(r_0) < \rho_0$, for all $u \in X$ such that $\|u\|_s^2 \leqslant 2r_0$, we have $\mathcal{F}_s(u) < \rho_0$. Therefore,

$$r_0 \leqslant \inf\left\{\frac{\|u\|_s^2}{2} : \mathcal{F}_s(u) \geqslant \rho_0\right\}.$$

On the other hand,

$$\begin{aligned}\inf_{u\in X}\sup_{\lambda\in\Lambda}(\Psi_s(u,\lambda) + \rho_0\lambda) &= \inf_{u\in X}\left(\frac{\|u\|_s^2}{2} + \sup_{\lambda\in\Lambda}(\lambda(\rho_0 - \mathcal{F}_s(u)))\right)\\ &= \inf_{u\in X}\left\{\frac{\|u\|_s^2}{2} : \mathcal{F}_s(u) \geqslant \rho_0\right\}.\end{aligned}$$

Combining with the above inequality, yields

$$r_0 \leqslant \inf_{u\in X}\sup_{\lambda\in\Lambda}(\Psi_s(u,\lambda) + \rho_0\lambda). \tag{11}$$

We distinguish two cases.

(I) If $0 \leqslant \overline{\lambda} < r_0/\rho_0$, we have

$$\begin{aligned}\sup_{\lambda\in\Lambda}\inf_{u\in X}(\Psi_s(u,\lambda) + \rho_0\lambda) &\overset{(10)}{=} \inf_{u\in X}(\tfrac{1}{2}\|u\|_s^2 + \overline{\lambda}(\rho_0 - \mathcal{F}_s(u)))\\ &\leqslant \overline{\lambda}\rho_0 < r_0 \overset{(11)}{\leqslant} \inf_{u\in X}\sup_{\lambda\in\Lambda}(\Psi_s(u,\lambda) + \rho_0\lambda).\end{aligned}$$

(II) If $r_0/\rho_0 \leqslant \overline{\lambda}$, we obtain

$$\begin{aligned}\sup_{\lambda\in\Lambda}\inf_{u\in X}(\Psi_s(u,\lambda) + \rho_0\lambda) &\overset{(10)}{=} \inf_{u\in X}(\tfrac{1}{2}\|u\|_s^2 + \overline{\lambda}(\rho_0 - \mathcal{F}_s(u)))\\ &\leqslant \tfrac{1}{2}\|u_0\|_s^2 + \overline{\lambda}(\rho_0 - \mathcal{F}_s(u_0))\\ &\overset{(9)}{\leqslant} \tfrac{1}{2}\|u_0\|_s^2 + \frac{r_0}{\rho_0}(\rho_0 - \mathcal{F}_s(u_0))\\ &= \tfrac{1}{2}\|u_0\|_s^2 - \frac{r_0}{\rho_0}\mathcal{F}_s(u_0) + r_0 \overset{(8)}{<} r_0\\ &\overset{(11)}{\leqslant} \inf_{u\in X}\sup_{\lambda\in\Lambda}(\Psi_s(u,\lambda) + \rho_0\lambda).\end{aligned}$$

Therefore, the assumptions of Proposition 4.1 are fulfilled. So, there exist an open interval $\Lambda_0 \subset [0, +\infty[$ and $\sigma > 0$ such that for all $\lambda \in \Lambda_0$ the function $\Psi_s(\cdot, \lambda)$ has at least three critical points in $H^1_{0,s}(\Omega)$ (with different critical values), having norms less than σ; denote them by u^i_λ $(i \in \{1, 2, 3\})$.

Let $G = id^m \times O(N-m)$ be the subgroup of $O(N)$. The action of G on $H^1_0(\Omega)$ can be defined by

$$gu(x, y) = u(x, g_0 y)$$

for all $(x, y) \in \omega \times \mathbb{R}^{N-m} = \Omega$, $g = id^m \times g_0 \in G$ and $u \in H^1_0(\Omega)$. G acts linear isometrically on $H^1_0(\Omega)$, and $\Psi(\cdot, \lambda)$ is G-invariant since $F(\cdot, s)$ is axially symmetric for all $s \in \mathbb{R}$. Moreover, we observe that

$$H^1_0(\Omega)^G \overset{\text{df.}}{=} \{u \in H^1_0(\Omega) : gu = u \text{ for all } g \in G\} = H^1_{0,s}(\Omega).$$

By Proposition 4.3 it follows that u^i_λ $(i \in \{1, 2, 3\})$ are also critical points of $\Psi(\cdot, \lambda)$, that is

$$\Psi(\cdot, \lambda)^0(u^i_\lambda; w) \geqslant 0 \quad \text{for all } w \in H^1_0(\Omega).$$

Using again Lemma 4.2 (now, with $E = H^1_0(\Omega)$), we obtain that

$$\begin{aligned} 0 &\leqslant \langle u^i_\lambda, w\rangle_0 + \lambda(-\mathcal{F})^0(u^i_\lambda; w) \\ &= \langle u^i_\lambda, w\rangle_0 + \lambda\mathcal{F}^0(u^i_\lambda; -w) \\ &\leqslant \int_\Omega \nabla u^i_\lambda \nabla w \, \mathrm{d}x + \lambda \int_\Omega F^0(x, u^i_\lambda(x); -w(x)) \, \mathrm{d}x \end{aligned}$$

for all $w \in H^1_0(\Omega)$. This means that u^i_λ $(i \in \{1, 2, 3\})$ are solutions for (EPHI$_\lambda$), which completes the proof.

6. Proof of Theorem 3.2

Throughout this section, we suppose that assumptions of Theorem 3.2 are fulfilled. Let Ψ from (4), and let us denote by $\mathcal{F}_E$, $\Psi_E(\cdot, \lambda)$, $\langle \cdot, \cdot\rangle_E$ and $\|\cdot\|_E$ the restrictions of $\mathcal{F}$, $\Psi(\cdot, \lambda)$, $\langle \cdot, \cdot\rangle_0$ and $\|\cdot\|_0$, respectively, to a closed subspace E of $H^1_0(\Omega)$, $(\lambda > 0)$.

LEMMA 6.1. *If the embedding $E \hookrightarrow L^p(\Omega)$ is compact then $\Psi_E(\cdot, \lambda)$ satisfies $(C)_c$ for all $\lambda, c > 0$.*

Proof. Let us fix a $\lambda > 0$ and a sequence $\{u_n\}$ from E such that $\Psi_E(u_n, \lambda) \to c > 0$ and

$$(1 + \|u_n\|_E) m_{\Psi_E(\cdot,\lambda)}(u_n) \longrightarrow 0 \tag{12}$$

as $n \to +\infty$. We shall prove firstly that $\{u_n\}$ is bounded in E. Let $z_n \in \partial\Psi_E(\cdot, \lambda)(u_n)$ such that $\|z_n\|_E = m_{\Psi_E(\cdot,\lambda)}(u_n)$; it is clear that $\|z_n\|_E \to 0$ as

$n \to +\infty$. Moreover, $\Psi_E(\cdot,\lambda)^0(u_n; u_n) \geqslant \langle z_n, u_n\rangle_E \geqslant -\|z_n\|_E\|u_n\|_E \geqslant -(1+\|u_n\|_E)m_{\Psi_E(\cdot,\lambda)}(u_n)$. Therefore, by Lemma 4.2, for n large enough

$$\begin{aligned}2c+1 &\geqslant 2\Psi_E(u_n,\lambda) - \Psi_E(\cdot,\lambda)^0(u_n; u_n)\\ &= -2\lambda\mathcal{F}_E(u_n) - \lambda(-\mathcal{F}_E)^0(u_n; u_n)\\ &\geqslant -\lambda\int_\Omega [2F(x,u_n(x)) + F^0(x,u_n(x); -u_n(x))]\,\mathrm{d}x\\ &\geqslant \lambda\gamma_0\|u_n\|_\nu^\nu.\end{aligned}$$

Thus,

$$\{u_n\} \text{ is bounded in } L^\nu(\Omega). \tag{13}$$

After integration in Lemma 4.1(ii), we obtain that for all $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that $\mathcal{F}_E(u_n) \leqslant \varepsilon\|u_n\|_E^2 + c(\varepsilon)\|u_n\|_p^p$ (note that $\|u\|_2^2 \leqslant k_2^2\|u\|_0^2$). For n large, one has

$$c+1 \geqslant \Psi_E(u_n,\lambda) = \tfrac{1}{2}\|u_n\|_E^2 - \lambda\mathcal{F}_E(u_n) \geqslant (\tfrac{1}{2}-\varepsilon\lambda)\|u_n\|_E^2 - \lambda c(\varepsilon)\|u_n\|_p^p.$$

Choosing $\varepsilon > 0$ small enough, we will find $c_2, c_3 > 0$ such that

$$c_2\|u_n\|_E^2 \leqslant c+1+c_3\|u_n\|_p^p. \tag{14}$$

Since $\nu \leqslant p$ (compare Lemma 4.1(ii) and (15) below), we distinguish two cases.

(I) If $\nu = p$ it is clear from (14) and (13) that $\{u_n\}$ is bounded in E.

(II) If $\nu < p$, we have the interpolation inequality

$$\|u_n\|_p \leqslant \|u_n\|_\nu^{1-\delta}\|u_n\|_{2^*}^\delta \leqslant k_{2^*}^\delta\|u_n\|_\nu^{1-\delta}\|u_n\|_E^\delta$$

(since $u_n \in E \hookrightarrow L^\nu(\Omega)\cap L^{2^*}(\Omega)$), where $1/p = (1-\delta)/\nu + \delta/2^*$. Since $\nu > N(p-2)/2$, we have $\delta p < 2$. According again to (13) and (14), we conclude that $\{u_n\}$ should be bounded in E.

Now, we will proceed as in the proof of Theorem 3.1, changing $\Psi_s(\cdot,\lambda)$, $X = H^1_{0,s}(\Omega)$, $\langle\cdot,\cdot\rangle_s$ and $\|\cdot\|_s$ to $\Psi_E(\cdot,\lambda)$, E, $\langle\cdot,\cdot\rangle_E$ and $\|\cdot\|_E$, respectively; the only minor modification will be in the estimation of $\Psi_E(\cdot,\lambda)^0(u_n; u-u_n)$. In fact, there we used (7) which is clearly implied by (12). This concludes the proof. □

Proof of Theorem 3.2. For the first part, we shall verify the assumptions of Proposition 4.2, choosing $E = H^1_{0,s}(\Omega)$ and $h = \Psi_E(\cdot,\lambda)$, where $\Psi_E(\cdot,\lambda)$ denotes the restriction of $\Psi(\cdot,\lambda)$ to E, $\lambda > 0$ being arbitrary fixed. By assumption, F is even in the second variable, so $\Psi_E(\cdot,\lambda)$ is also even, and by Lemma 4.2 it is a locally Lipschitz function.

Since $\Psi_E(0,\lambda) = 0$, the assumption (i) from Proposition 4.2 follows from Lemma 6.1, due to the compact embedding $E \hookrightarrow L^p(\Omega)$.

To prove (ii), we consider firstly the function $g\colon \Omega \times]0, +\infty[\to \mathbb{R}$ defined by

$$g(x,t) = t^{-2}F(x,t) - \frac{\gamma(x)}{\nu-2}t^{\nu-2}.$$

Let us fix $x \in \Omega$. Clearly, $g(x,\cdot)$ is a locally Lipschitz function and by Proposition 2.1(iii) we have

$$\partial g(x,t) \subseteq -2t^{-3}F(x,t) + t^{-2}\partial F(x,t) - \gamma(x)t^{\nu-3}, \quad t > 0.$$

Let $t > s > 0$. By Lebourg's mean value theorem, there exist $\tau = \tau(x) \in]s,t[$ and $w_\tau = w_\tau(x) \in \partial g(x,\tau)$ such that $g(x,t) - g(x,s) = w_\tau(t-s)$. Therefore, there exists $\xi_\tau = \xi_\tau(x) \in \partial F(x,\tau)$ such that $w_\tau = -2\tau^{-3}F(x,\tau) + \tau^{-2}\xi_\tau - \gamma(x)\tau^{\nu-3}$ and

$$g(x,t) - g(x,s) \geqslant -\tau^{-3}[2F(x,\tau) + F^0(x,\tau;-\tau) + \gamma(x)\tau^\nu](t-s).$$

By (F5) one has $g(x,t) \geqslant g(x,s)$. On the other hand, by Lemma 4.1 we have that $F(x,s) = \mathrm{o}(s^2)$ as $s \to 0$ for a.e. $x \in \Omega$. If $s \to 0^+$ in the above inequality, we have that $F(x,t) \geqslant \gamma(x)t^\nu/(\nu-2)$ for all $t > 0$ and a.e. $x \in \Omega$. Since $F(x,0) = 0$ and $F(x,\cdot)$ is even, we obtain that

$$F(x,t) \geqslant \frac{\gamma(x)}{\nu-2}|t|^\nu \quad \text{for all } t \in \mathbb{R} \text{ and a.e. } x \in \Omega. \tag{15}$$

Now, let $\{e_i\}$ be a fixed orthonormal basis of E and $E_k = \{e_1, \ldots, e_k\}$, $k \geqslant 1$. Denoting by $\|\cdot\|_E$ the restriction of $\|\cdot\|_0$ to E, from (15) one has

$$\Psi_E(u,\lambda) \leqslant \tfrac{1}{2}\|u\|_E^2 - \frac{\lambda\gamma_0}{\nu-2}\|u\|_\nu^\nu \quad \text{for all } u \in E.$$

Let us fix $k \geqslant 1$ arbitrary. Since $\nu > 2$ and on the finite-dimensional space E_k all norms are equivalent (in particular $\|\cdot\|_0$ and $\|\cdot\|_\nu$), choosing a large $R_k > 0$, we have $\Psi_E(u,\lambda) \leqslant \Psi_E(0,\lambda) = 0$ if $\|u\|_E \geqslant R_k$, $u \in E_k$. This proves (ii).

Again, from Lemma 4.1(ii) we have that for all $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that $\mathcal{F}_E(u) \leqslant \varepsilon\|u\|_E^2 + c(\varepsilon)\|u\|_p^2$ for all $u \in E$. Let $\beta_k = \sup\{\|u\|_p/\|u\|_E : u \in E_k^\perp, u \neq 0\}$. As in [4, Lemma 3.3], it can be proved that $\beta_k \to 0$ as $k \to +\infty$. For $u \in E_k^\perp$, one has

$$\Psi_E(u,\lambda) \geqslant (\tfrac{1}{2} - \varepsilon\lambda)\|u\|_E^2 - \lambda c(\varepsilon)\|u\|_p^p \geqslant (\tfrac{1}{2} - \varepsilon\lambda)\|u\|_E^2 - \lambda c(\varepsilon)\beta_k^p\|u\|_E^p.$$

Choosing $\varepsilon < (p-2)(2p\lambda)^{-1}$ and $\rho_k = (p\lambda c(\varepsilon)\beta_k^p)^{1/(2-p)}$, we have

$$\Psi_E(u,\lambda) \geqslant \left(\frac{1}{2} - \frac{1}{p} - \varepsilon\lambda\right)\rho_k^2$$

for every $u \in E_k^\perp$ with $\|u\|_E = \rho_k$. Since $\beta_k \to 0$, then $\rho_k \to +\infty$ as $k \to +\infty$. The assumption (iii) from Proposition 4.2 is concluded.

Hence, $\Psi_E(\cdot, \lambda)$ has infinitely many critical points on $E = H^1_{0,s}(\Omega)$. Using Proposition 4.3 and Lemma 4.2 as in the proof of Theorem 3.1, the above points will be solutions for (EPHI$_\lambda$).

Now, we deal with the second part. The following construction is inspired by [4]. Let $N = m + 4$ or $N \geqslant m + 6$. In both cases we find at least a number $k \in [2, (N-m)/2] \cap \mathbb{N} \setminus \{(N-m-1)/2\}$. For a such $k \in \mathbb{N}$, we have $\Omega = \omega \times \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^{N-2k-m}$. Let $H = id^m \times O(k) \times O(k) \times O(N-2k-m)$ and define

$$G_\tau = \langle H \cup \{\tau\}\rangle,$$

where $\tau(x_1, x_2, x_3, x_4) = (x_1, x_3, x_2, x_4)$, for every $x_1 \in \omega$, $x_2, x_3 \in \mathbb{R}^k$, $x_4 \in \mathbb{R}^{N-2k-m}$. G_τ will be a subgroup of $O(N)$ and its elements can be written uniquely as h or $h\tau$ with $h \in H$. The action of G_τ on $H^1_0(\Omega)$ is defined by

$$gu(x_1, x_2, x_3, x_4) = \pi(g)u(x_1, g_2x_2, g_3x_3, g_4x_4) \tag{16}$$

for all $g = id^m \times g_2 \times g_3 \times g_4 \in G_\tau$, $(x_1, x_2, x_3, x_4) \in \omega \times \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^{N-2k-m}$, where $\pi\colon G_\tau \to \{\pm 1\}$ is the canonical epimorphism, that is, $\pi(h) = 1$ and $\pi(h\tau) = -1$ for all $h \in H$. The group G_τ acts linear isometrically on $H^1_0(\Omega)$, and $\Psi(\cdot, \lambda)$ is G_τ-invariant, since F is axially symmetric in the first variable and even in the second variable. Let

$$H^1_{0,ns}(\Omega) = \{u \in H^1_0(\Omega) : gu = u \text{ for all } g \in G_\tau\}.$$

Clearly, $H^1_{0,ns}(\Omega)$ is a closed subspace of $H^1_0(\Omega)$ and

$$H^1_{0,ns}(\Omega) \subset H^1_0(\Omega)^H \overset{\text{df.}}{=} \{u \in H^1_0(\Omega) : hu = u \text{ for all } h \in H\}.$$

On the other hand, $H^1_0(\Omega)^H \hookrightarrow L^p(\Omega)$ is compact (see [19, Théorème III.2]), hence $H^1_{0,ns}(\Omega) \hookrightarrow L^p(\Omega)$ is also compact.

Now, repeating the proof of the first part for $E = H^1_{0,ns}(\Omega)$ instead of $H^1_{0,s}(\Omega)$, we obtain infinitely many solutions for (EPHI$_\lambda$), which belong to $H^1_{0,ns}(\Omega)$. But we observe that 0 is the only axially symmetric function of $H^1_{0,ns}(\Omega)$. Indeed, let $u \in H^1_{0,ns}(\Omega) \cap H^1_{0,s}(\Omega)$. Since $gu = u$ for all $g \in G_\tau$, choosing in particular $\tau \in G_\tau$ and using (16), we have that $u(x_1, x_2, x_3, x_4) = -u(x_1, x_3, x_2, x_4)$ for all $(x_1, x_2, x_3, x_4) \in \omega \times \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^{N-2k-m}$. Since u is axially symmetric and $|(x_2, x_3, x_4)| = |(x_3, x_2, x_4)|$, ($|\cdot|$ being the norm on $\mathbb{R}^{N-m}$), it follows that u must be 0. Therefore, the above solutions are axially nonsymmetric functions. This concludes the proof. $\square$

7. Final Remarks

1. Theorem 3.2 complements Theorem 3.1 in the sense that in the first case F is subquadratic (see (F3)) while in the second one F is superquadratic (see relation (15) and note that $\nu > 2$).

2. The reader can observe that we considered only $N \geqslant m+2$. In fact, in this case $H^1_{0,s}(\Omega)$ can be embedded compactly to $L^p(\Omega)$, $p \in]2, 2^*[$ which was crucial in the verification of the Palais–Smale and Cerami conditions. When $N = m+1$ the above embedding is *no* longer compact. In the latter case it is recommended to construct the convex closed cone (see [12, Theorem 2]), defined by

$$\begin{aligned}\mathcal{K} = \{u \in H^1_0(\omega \times \mathbb{R}) : u \geqslant 0,\ u(x, y) \text{ is nonincreasing in } y \text{ for } x \in \omega,\\ y \geqslant 0,\ \text{and } u(x, y) \text{ is nondecreasing in } y \text{ for } x \in \omega,\ y \leqslant 0\},\end{aligned}$$

because the Sobolev embedding from $H^1_0(\omega \times \mathbb{R})$ into $L^p(\omega \times \mathbb{R})$ transforms the bounded closed sets of $\mathcal{K}$ into relatively compact sets of $L^p(\omega \times \mathbb{R})$, $p \in]2, 2^*[$ (note that $2^* = +\infty$, if $m = 1$). Since $\mathcal{K}$ is not a subspace of $H^1_0(\omega \times \mathbb{R})$, the above described machinery (in Sections 5 and 6) does not work; however, we believe that this inconvenience can be handled by the Motreanu–Panagiotopoulos type functional (see Remark 4.1, choosing ψ as the indicator function of $\mathcal{K}$). Since this approach differs substantially to the above, we will treat it in a forthcoming paper.

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References

1. Amick, C. J.: Semilinear elliptic eigenvalue problems on an infinite strip with an application to stratified fluids, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **11** (1984), 441–499.
2. Arioli, G. and Gazzola, F.: Existence and multiplicity results for quasilinear elliptic differential systems, *Comm. Partial Differential Equations* **25** (2000), 125–153.
3. Bartsch, T.: *Topological Methods for Variational Problems with Symmetries*, Springer, Berlin, 1993.
4. Bartsch, T. and Willem, M.: Infinitely many nonradial solutions of an Euclidean scalar field equation, *J. Funct. Anal.* **117** (1993), 447–460.
5. Bongers, A., Heinz, H. P. and Küpper, T.: Existence and bifurcation theorems for nonlinear elliptic eigenvalue problems on unbounded domains, *J. Differential Equations* **47** (1983), 327–357.
6. Burton, G. R.: Semilinear elliptic equations on unbounded domains, *Math. Z.* **190** (1985), 519–525.
7. Chang, K. C.: Variational methods for nondifferentiable functionals and their applications to partial differential equations, *J. Math. Anal. Appl.* **80** (1981), 102–129.
8. Clarke, F. H.: *Optimization and Nonsmooth Analysis*, Wiley, New York, 1983.
9. Costa, D. G. and Magalhães, C. A.: A unified approach to a class of strongly indefinite functionals, *J. Differential Equations* **125** (1996), 521–548.

10. Duvaut, G. and Lions, J.-L.: *Inequalities in Mechanics and Physics*, Springer, Berlin, 1976.
11. Esteban, M. J.: Nonlinear elliptic problems in strip-like domains: Symmetry of positive vortex rings, *Nonlinear Anal.* **7** (1983), 365–379.
12. Esteban, M. J. and Lions, P. L.: A compactness lemma, *Nonlinear Anal.* **7** (1983), 381–385.
13. Fan, X. L. and Zhao, Y. Z.: Linking and multiplicity results for the p-Laplacian on unbounded cylinders, *J. Math. Anal. Appl.* **260** (2001), 479–489.
14. Grossinho, M. R.: Some existence and bifurcation results for nonlinear elliptic problems in strip-like domains, *Ricerche Mat.* **36** (1987), 127–138.
15. Guo, Z. and Webb, J. R. L.: Large and small solutions of a class of quasilinear elliptic eigenvalue problems, *J. Differential Equations* **180** (2002), 1–50.
16. Kourogenis, N. C. and Papageorgiou, N. S.: Nonsmooth critical point theory and nonlinear elliptic equations at resonance, *J. Austral. Math. Soc.* **69** (2000), 245–271.
17. Krawcewicz, W. and Marzantowicz, W.: Some remarks on the Lusternik–Schnirelman method for nondifferentiable functionals invariant with respect to a finite group action, *Rocky Mountain J. Math.* **20** (1990), 1041–1049.
18. Kristály, A.: Infinitely many radial and nonradial solutions for a class of hemivariational inequalities, *Rocky Mountain J. Math.*, to appear.
19. Lions, P. L.: Symétrie et compacité dans les espaces Sobolev, *J. Funct. Anal.* **49** (1982), 315–334.
20. Marano, S. A. and Motreanu, D.: On a three critical point theorem for nondifferentiable functions and applications to nonlinear boundary value problems, *Nonlinear Anal.* **48** (2002), 37–52.
21. Motreanu, D. and Panagiotopoulos, P. D.: *Minimax Theorems and Qualitative Properties of the Solutions of Hemivariational Inequalities*, Kluwer Acad. Publ., Dordrecht, 1999.
22. Motreanu, D. and Rădulescu, V.: *Variational and nonvariational methods in nonlinear analysis and boundary value problems*, Kluwer Acad. Publ., Dordrecht, 2003.
23. Naniewicz, Z. and Panagiotopoulos, P. D.: *Mathematical Theory of Hemivariational Inequalities and Applications*, Marcel Dekker, New York, 1995.
24. Palais, R. S.: The principle of symmetric criticality, *Comm. Math. Phys.* **69** (1979), 19–30.
25. Panagiotopoulos, P. D.: *Hemivariational Inequalities: Applications to Mechanics and Engineering*, Springer, New York, 1993.
26. Raymond, J. S.: On the multiplicity of solutions of the equation $-\Delta u = \lambda \cdot f(u)$, *J. Differential Equations* **180** (2002), 65–88.
27. Ricceri, B.: On a three critical points theorem, *Arch. Math.* **75** (2000), 220–226.
28. Ricceri, B.: Existence of three solutions for a class of elliptic eigenvalue problems, *Math. Comput. Modelling* **32** (2000), 1485–1494.
29. Ricceri, B.: Three solutions for a Neumann problem, *Topol. Methods Nonlinear Anal.* **20** (2002), 275–281.
30. Schindler, I.: Quasilinear elliptic boundary value problems on unbounded cylinders and a related mountain-pass lemma, *Arch. Rational. Mech. Anal.* **120** (1992), 363–374.
31. Tersian, S. A.: On the solvability of semilinear Schrödinger equations in strip-like domains, *C.R. Acad. Bulgare Sci.* **51** (1998), 17–20.