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INFINITELY MANY RADIAL AND NON-RADIAL SOLUTIONS FOR A CLASS OF HEMIVARIATIONAL INEQUALITIES

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ABSTRACT. This paper is concerned with the existence of infinitely many radial respective non-radial solutions for a class of hemivariational inequalities, applying the non-smooth version of the fountain theorem. The main tool used in our framework is the principle of symmetric criticality for a locally Lipschitz functional which is invariant under a group action.

1. Introduction. Let $F : \mathbf{R}^N \times \mathbf{R} \to \mathbf{R}$ be a Carathéodory function which is locally Lipschitz in the second variable, fulfilling the following condition:

(F_1) there exist $c_1 > 0$ and $p \in]2, 2^*[$ such that

$$|\xi| \leq c_1(|s| + |s|^{p-1}), \quad \forall\, \xi \in \partial F(x,s), \quad \text{for a.e. } x \in \mathbf{R}^N, \quad \forall\, s \in \mathbf{R},$$

where $N \geq 2$ and $p \in]2, 2^*[$, $2^* = 2N/(N-2)$, if $N \geq 3$ and $2^* = \infty$, if $N = 2$, and $F(x, 0) = 0$ almost everywhere $x \in \mathbf{R}^N$.

The set

$$\partial F(x,s) = \{\xi \in \mathbf{R} : \xi z \leq F_x^0(x,s;z) \text{ for all } z \in \mathbf{R}\}$$

is the generalized gradient of $F(x,\cdot)$ at $s \in \mathbf{R}$, where

$$F_x^0(x,s;z) = \limsup_{\substack{y \to s \\ t \to 0^+}} \frac{F(x, y+tz) - F(x,y)}{t},$$

is the generalized directional derivative of $F(x,\cdot)$ at the point $s \in \mathbf{R}$ in the direction $z \in \mathbf{R}$, see Clarke [**9**].

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The purpose of this paper is to study the following hemivariational inequality problem:

(P) *find* $u \in H^1(\mathbf{R}^N)$ *such that*

$$\int_{\mathbf{R}^N} (\nabla u \nabla w + uw)\, dx + \int_{\mathbf{R}^N} F_x^0(x, u(x); -w(x))\, dx \geq 0, \quad \forall\, w \in H^1(\mathbf{R}^N).$$

The development of the mathematical theory of hemivariational inequalities, as well as their applications in economics, mechanics or engineering, began with the work of Panagiotopoulos [**20, 21**]. Concerning the existence of solutions of hemivariational inequalities, one can find results by Naniewicz and Panagiotopoulos [**18**] (based on pseudomonotonicity); Motreanu and Panagiotopoulos [**15**], Motreanu and Rădulescu [**16**] (based on compactness arguments), and references therein.

Remark 1.1. In particular, if $f : \mathbf{R}^N \times \mathbf{R} \to \mathbf{R}$ is a measurable, not necessarily continuous function, and there exists $c > 0$ such that

$$(1) \qquad |f(x,s)| \leq c(|s| + |s|^{p-1}), \quad \text{for a.e. } x \in \mathbf{R}^N, \quad \forall\, s \in \mathbf{R},$$

and $F : \mathbf{R}^N \times \mathbf{R} \to \mathbf{R}$ is defined by

$$(2) \qquad F(x,s) = \int_0^s f(x,t)\, dt, \quad \text{for a.e. } x \in \mathbf{R}^N, \quad \forall\, s \in \mathbf{R}.$$

Then F is a Carathéodory function which is also locally Lipschitz in the second variable and $F(x,0) = 0$, almost everywhere $x \in \mathbf{R}^N$. Moreover, F satisfies the growth condition from (F_1). Indeed, since $f(x,\cdot) \in L^\infty_{\text{loc}}(\mathbf{R})$ almost everywhere $x \in \mathbf{R}^N$, by [**15**, Proposition 1.7, p. 13] we have

$$\partial F(x,s) = [\underline{f}(x,s), \bar{f}(x,s)], \quad \text{for a.e. } x \in \mathbf{R}^N, \quad \forall\, s \in \mathbf{R},$$

where

$$\underline{f}(x,s) = \lim_{\delta \to 0^+} \operatorname{essinf}\{f(x,t) : |t-s| < \delta\},$$

and

$$\bar{f}(x, s) = \lim_{\delta \to 0^+} \text{esssup}\{f(x, t) : |t - s| < \delta\}.$$

From the above relation and (1) the desired inequality is obtained.

Moreover, when $f \in C^0(\mathbf{R}^N \times \mathbf{R}, \mathbf{R})$, due to (2), the inequality from (P) takes the form

$$\int_{\mathbf{R}^N} (\nabla u \nabla w + uw)\, dx - \int_{\mathbf{R}^N} f(x, u(x))w(x)\, dx = 0, \quad \forall\, w \in H^1(\mathbf{R}^N),$$

i.e., $u \in H^1(\mathbf{R}^N)$ is a weak solution of

$$(\mathrm{P}') \qquad \begin{cases} -\triangle v + v = f(x, v), \\ v \in H^1(\mathbf{R}^N). \end{cases}$$

Many papers are concerned with the existence and multiplicity of solutions for problems related to (P′), see Bartsch and Willem [**5, 6**], Bartsch and Wang [**4**], Strauss [**24**] (in the autonomous case), Gidas, Ni and Nirenberg [**12**], Gazzola and Rădulescu [**11**], and the references therein. The interest in this equation comes from various problems in mathematics and physics (cosmology, constructive field theory, solitary waves, nonlinear Klein-Gordon or Schrödinger equations), see [**1, 10, 24, 33**].

Under suitable hypotheses mainly on f, Strauss [**24**], Bartsch and Willem [**6**], Berestycki and Lions [**7**], Struwe [**25**] obtained existence results concerning the radial solutions of problems closely related to (P′). Bartsch and Willem [**5**] observed that a careful choice of a subgroup of $\mathbf{O}(N)$ in certain dimensions assures the existence of infinitely many non-radial solutions of (P′). In general, a functional of class C^1 is constructed which is invariant under a subgroup action of $\mathbf{O}(N)$, whose restriction to the appropriate subspace of invariant functions admits critical points. Due to the principle of symmetric criticality of Palais [**19**], these points will be also critical points of the original functional, and they are exactly the radial, respectively non-radial, solutions of (P′), depending on the choice of the subgroup of $\mathbf{O}(N)$. We emphasize that in the above works the nonlinear term f is continuous. A good survey for these problems is the book of Willem [**26**].

In practical problems, in (P′) may appear functions f which are not continuous, see Gazzola and Rădulescu [**11**] and the very recent monograph of Motreanu and Rădulescu [**16**]. Clearly, in this case the classical framework, described above, is not working. Starting from this point of view, we propose a more general problem, i.e., to study the existence of radial, respectively non-radial, solutions of (P). To the best of my knowledge, no investigation has been devoted to establish results in this direction. Our appropriate functional will be $\mathbf{O}(N)$-invariant and only locally Lipschitz; therefore, we cannot apply the classical machinery described above. Thanks to the ingenious principle of symmetric criticality of Krawcewicz and Marzantowicz [**13**] for invariant locally Lipschitz functionals, we are able to guarantee critical points (in the sense of Chang [**8**]) of the above-mentioned functionals, applying the fountain theorem of Bartsch [**2**] in the non-smooth case, proved by Motreanu and Varga [**17**]; the corresponding critical points will be radial, respectively non-radial, solutions of (P). These existence theorems improve some results from [**5, 6, 24, 26**]. On the other hand, we emphasize that our main results can be applied to several concrete cases where the earlier results fail and they seem to be the first applications of the principle of symmetric criticality for non-smooth functionals.

The paper is divided into four sections. Basic definitions and preliminary results are collected in the second section. The main results are presented in the third section, where we establish the existence of infinitely many radial respective non-radial solutions of (P). In the last part a numerical example is presented.

2. Preliminaries and key results. Let $(X, \|\cdot\|)$ be a real Banach space and X^* its dual. A function $h : X \to \mathbf{R}$ is called locally Lipschitz if each point $u \in X$ possesses a neighborhood U_u of u such that

$$|h(u_1) - h(u_2)| \leq L\|u_1 - u_2\|, \quad \forall\, u_1, u_2 \in U_u,$$

for a constant $L > 0$ depending on U_u. The generalized gradient of h at $u \in X$ is defined as being the subset of X^*

$$\partial h(u) = \{x^* \in X^* : \langle x^*, z\rangle \leq h^0(u; z) \text{ for all } z \in X\},$$

which is nonempty, convex and w^*-compact, where $\langle \cdot, \cdot\rangle$ is the duality pairing between X^* and X, $h^0(u; z)$ being the generalized directional derivative of h at the point $u \in X$ along the direction $z \in X$, namely,

$$h^0(u;z) = \limsup_{\substack{w\to u\\ t\to 0^+}} \frac{h(w+tz)-h(w)}{t},$$

see [**9**]. Moreover, $h^0(u;z) = \max\{\langle x^*, z\rangle : x^* \in \partial h(u)\}$, for all $z \in X$. It is easy to verify that $(-h)^0(u;z) = h^0(u;-z)$, and, for locally Lipschitz functions $h_1, h_2 : X \to \mathbf{R}$, one has $(h_1+h_2)^0(u;z) \le h_1^0(u;z) + h_2^0(u;z)$, for all $u, z \in X$. The Lebourg's mean value theorem says that for every $u, v \in X$ there exist $\theta \in]0,1[$ and $x_\theta^* \in \partial h(\theta u + (1-\theta)v)$ such that $h(u) - h(v) = \langle x_\theta^*, u-v\rangle$. If h_2 is continuously Gâteaux differentiable, then $\partial h_2(u) = h_2'(u)$; $h_2^0(u;z)$ coincides with the directional derivative $h_2'(u;z)$ and the above inequality reduces to $(h_1+h_2)^0(u;z) = h_1^0(u;z) + h_2'(u;z)$, for all $u, z \in X$. A point $u \in X$ is a *critical point of* h if $0 \in \partial h(u)$, i.e., $h^0(u;w) \ge 0$, for all $w \in X$. We define $\lambda_h(u) = \inf\{\|x^*\| : x^* \in \partial h(u)\}$. Of course, this infimum is attained, since $\partial h(u)$ is w^*-compact.

We say that h *satisfies the* (PS) *condition at level* c *in the sense of Chang* (shortly $(PS)_c$), if every sequence $\{x_n\} \subset X$ such that $h(x_n) \to c$ and $\lambda_h(x_n) \to 0$, contains a convergent subsequence in X, see [**8**].

Now, we define the functional $\psi : H^1(\mathbf{R}^N) \to \mathbf{R}$ by

$$\psi(u) = \int_{\mathbf{R}^N} F(x, u(x))\, dx, \quad \forall\, u \in H^1(\mathbf{R}^N). \tag{3}$$

As usual, $H^1(\mathbf{R}^N)$ is the Sobolev space with the inner product $(u,v)_1 = \int_{\mathbf{R}^N} [\nabla u(x) \nabla v(x) + u(x)v(x)]\, dx$ and the corresponding norm $\|u\|_1 = \left[\int_{\mathbf{R}^N} (|\nabla u(x)|^2 + |u(x)|^2)\, dx\right]^{1/2}$.

Suppose now that (F_1) holds. Let us fix $s_1, s_2 \in \mathbf{R}$ arbitrary. By using Lebourg's mean value theorem, there exist $\theta \in \,]0,1[$ and $\xi_\theta \in \partial F(x, \theta s_1 + (1-\theta)s_2)$ such that

$$F(x,s_1) - F(x,s_2) = \xi_\theta(s_1 - s_2).$$

By (F_1) we can conclude that, for almost every $x \in \mathbf{R}^N$

$$|F(x,s_1) - F(x,s_2)| \le c_2|s_1 - s_2| \cdot \left[|s_1| + |s_2| + |s_1|^{p-1} + |s_2|^{p-1}\right], \quad \forall\, s_1, s_2 \in \mathbf{R}, \tag{4}$$

where $c_2 = c_2(c_1, p) > 0$. In particular, if $u \in H^1(\mathbf{R}^N)$, we obtain that

$$|\psi(u)| \leq \int_{\mathbf{R}^N} |F(x, u(x))| dx \leq c_2(\|u\|_1^2 + \|u\|_p^p) < \infty,$$

i.e., the functional ψ is well-defined (due to the fact that the embedding $H^1(\mathbf{R}^N) \hookrightarrow L^p(\mathbf{R}^N)$ is continuous). The norm on $L^p(\mathbf{R}^N)$ is $\|u\|_p = (\int_{\mathbf{R}^N} |u(x)|^p \, dx)^{1/p}$. Moreover, thanks to relation (4), there exists $c_3 > 0$ such that for every $u, v \in H^1(\mathbf{R}^N)$

$$|\psi(u) - \psi(v)| \leq c_3 \|u - v\|_1 \left[\|u\|_1 + \|v\|_1 + \|u\|_1^{p-1} + \|v\|_1^{p-1}\right].$$

Therefore, ψ is a locally Lipschitz functional on $H^1(\mathbf{R}^N)$ and we have the following key inequality.

Proposition 2.1. *Let E be a closed subspace of $H^1(\mathbf{R}^N)$, ψ_E the restriction of ψ to E. If (F_1) holds, we have*

$$\psi_E^0(u; v) \leq \int_{\mathbf{R}^N} F_x^0(x, u(x); v(x)) \, dx$$

for every $u, v \in E$.

Proof. Let us fix u and v in E. Since $F(x, \cdot)$ is continuous, $F_x^0(x, u(x); v(x))$ can be expressed as the upper limit of $[F(x, y + tv(x)) - F(x, y)]/t$, where $t \to 0^+$ taking rational values and $y \to u(x)$ taking values in a countable dense subset of $\mathbf{R}$. Therefore, the map $x \mapsto F_x^0(x, u(x); v(x))$ is measurable as the "countable limsup" of measurable functions of x. According to (F_1), the map $x \mapsto F_x^0(x, u(x); v(x))$ is from $L^1(\mathbf{R}^N)$.

Since E is separable (being a closed subspace of a separable Hilbert space), there are functions $w_n \in E$ and numbers $t_n \to 0^+$ such that $w_n \to u$ in E and

$$\psi_E^0(u; v) = \lim_{n \to \infty} \frac{\psi_E(w_n + t_n v) - \psi_E(w_n)}{t_n},$$

and, without loss of generality, we may assume $w_n(x) \to u(x)$ almost everywhere $x \in \mathbf{R}^N$, as $n \to \infty$.

We define $g_n : \mathbf{R}^N \to \mathbf{R} \cup \{+\infty\}$ by

$$g_n(x) = -\frac{F(x, w_n(x) + t_n v(x)) - F(x, w_n(x))}{t_n} + c_2 |v(x)| \cdot \big[|w_n(x)+t_n v(x)|\,|w_n(x)|+|w_n(x)+t_n v(x)|^{p-1}+|w_n(x)|^{p-1}\big].$$

The maps g_n are measurable and non-negative, see (4). From Fatou's lemma we have

$$J = \int_{\mathbf{R}^N} \limsup_{n\to\infty} [-g_n(x)] dx \geq \limsup_{n\to\infty} \int_{\mathbf{R}^N} [-g_n(x)]\, dx = I.$$

Let $g_n = -A_n + B_n$, where

$$A_n(x) = \frac{F(x, w_n(x) + t_n v(x)) - F(x, w_n(x))}{t_n},$$

and

$$B_n(x) = c_2 |v(x)| \big[|w_n(x) + t_n v(x)| + |w_n(x)| + |w_n(x) + t_n v(x)|^{p-1} + |w_n(x)|^{p-1}\big].$$

Introducing the notation $b_n = \int_{\mathbf{R}_N} B_n(x)\, dx$, we have

$$I = \limsup_{n\to\infty} \left(\int_{\mathbf{R}^N} A_n(x) dx - b_n \right). \tag{5}$$

Denote by $\|\cdot\|_E$ the restriction of $\|\cdot\|_1$ to E. After an elementary calculation we obtain the following estimation:

$$\begin{aligned}
&\left| b_n - 2c_2 \int_{\mathbf{R}^N} |v(x)| \left(|u(x)| + |u(x)|^{p-1}\right)\, dx \right| \\
&\quad \leq c_2 \big\{ 2\|v\|_E \|w_n - u\|_E + t_n \|v\|_E^2 + (p-1) 2^{p-2} \|v\|_p \\
&\qquad \times \big[\|w_n - u\|_p \left(\|u\|_p^{p-2} + \|w_n\|_p^{p-2}\right) \\
&\qquad + (\|w_n - u\|_p + t_n \|v\|_p) \left(\|u\|_p^{p-2} + (\|w_n\|_p + t_n\|v\|_p)^{p-2}\right)\big]\big\}.
\end{aligned}$$

Since the embedding $E \subseteq H^1(\mathbf{R}^N) \hookrightarrow L^p(\mathbf{R}^N)$ is continuous, $2 < p < 2^*$, and $\|w_n - u\|_E \to 0$, $t_n \to 0^+$, we obtain that the sequence $\{b_n\}$ is convergent, its limit being

$$\lim_{n\to\infty} b_n = 2c_2 \int_{\mathbf{R}^N} |v(x)| \left(|u(x)| + |u(x)|^{p-1}\right)\, dx.$$

From (3) and (5), we have

$$\begin{aligned} I &= \limsup_{n\to\infty} \frac{\psi_E(w_n + t_n v) - \psi_E(w_n)}{t_n} - \lim_{n\to\infty} b_n \\ &= \psi_E^0(u; v) - 2c_2 \int_{\mathbf{R}^N} |v(x)| \left(|u(x)| + |u(x)|^{p-1}\right) dx. \end{aligned}$$

Now we estimate J. Denoting by

$$J_A = \int_{\mathbf{R}^N} \limsup_{n\to\infty} A_n(x)\, dx \quad \text{and} \quad J_B = \int_{\mathbf{R}^N} \liminf_{n\to\infty} B_n(x)\, dx,$$

we have $J \le J_A - J_B$.

Since $w_n(x) \to u(x)$ almost everywhere on $\mathbf{R}^N$ and $t_n \to 0^+$, we have

$$J_B = 2c_2 \int_{\mathbf{R}^N} |v(x)| \left(|u(x)| + |u(x)|^{p-1}\right) dx.$$

On the other hand, we obtain

$$\begin{aligned} J_A &= \int_{R^N} \limsup_{n\to\infty} \frac{F(x, w_n(x) + t_n v(x)) - F(x, w_n(x))}{t_n}\, dx \\ &\le \int_{R^N} \limsup_{\substack{y\to u(x)\\ t\to 0^+}} \frac{F(x, y + tv(x)) - F(x, y)}{t}\, dx \\ &= \int_{\mathbf{R}^N} F_x^0(x, u(x); v(x))\, dx. \end{aligned}$$

From the above estimations we obtain the desired relation. $\square$

Remark 2.1. The above inequality has been proved only for bounded domains of $\mathbf{R}^N$ by Clarke [**9**], Motreanu and Panagiotopoulos [**15**], where the growth conditions are different than in our situation.

Let G be a compact Lie group which acts linear isometrically on the real Banach space $(X, \|\cdot\|)$, i.e., the action $G \times X \to X : [g, u] \mapsto gu$ is continuous and for every $g \in G$, the map $u \mapsto gu$ is linear such that $\|gu\| = \|u\|$, for every $u \in X$. A function $h : X \to \mathbf{R}$ is *G-invariant* if

$h(gu) = h(u)$, for all $g \in G$, $u \in X$. The action on X induces an action of the same type on the dual space X^* defined by $(gx^*)(u) = x^*(gu)$, for all $g \in G$, $u \in X$ and $x^* \in X^*$. We have $\|gx^*\| = \|x^*\|$, for all $g \in G$, $x^* \in X^*$. Supposing that $h : X \to \mathbf{R}$ is a G-invariant, locally Lipschitz functional, then $g\partial h(u) = \partial h(gu) = \partial h(u)$, for all $g \in G$, $u \in X$. Therefore the function $u \mapsto \lambda_h(u)$ is G-invariant.

Let $X^G = \{u \in X : gu = u, \ \forall\, g \in G\}$. We recall the Principle of Symmetric Criticality of Krawcewicz and Marzantowicz [**13**, p. 1045], which will be crucial in the proof of our main theorems. This result was established for Banach G-manifolds of class C^2; we will use only a particular form of this one which works on Banach spaces.

Proposition 2.2. *Let $u \in X^G$ and $h : X \to \mathbf{R}$ be a G-invariant, locally Lipschitz functional. Then u is a critical point of h if and only if u is a critical point of $h^G := h|_{X^G} : X^G \to \mathbf{R}$.*

At the end of this section we recall the non-smooth version of the fountain theorem, due to Motreanu and Varga [**17**, Corollary 3.4]. This result was obtained for a compact Lie group G acting on finite dimensional vector space satisfying the admissibility condition in the sense of Bartsch [**3**]; here we recall this one for $G := \mathbf{Z}_2$. First, the fountain theorem was established by Bartsch [**2**] for functionals of class C^1.

Proposition 2.3. *Let E be a Hilbert space, $\{e_j : j \in \mathbf{N}\}$ an orthonormal basis of E, and set $E_k = \operatorname{span}\{e_1, \dots, e_k\}$. Let $h : E \to \mathbf{R}$ be a locally Lipschitz functional which satisfies the following hypotheses*:

(i) *$h(-u) = h(u)$, for all $u \in E$;*

(ii) *for every $k \geq 1$, there exists $R_k > 0$ such that $h(u) \leq h(0)$, for all $u \in E_k$ with $\|u\| \geq R_k$;*

(iii) *there exist $k_0 \geq 1$, $b > h(0)$ and $\rho > 0$ such that $h(u) \geq b$ for every $u \in E_{k_0}^{\perp}$ with $\|u\| = \rho$;*

(iv) *h satisfies the $(PS)_c$ condition for every $c \in \mathbf{R}$.*

Then h possesses a sequence of critical values $\{c_k\}$ such that $c_k \to \infty$ as $k \to \infty$.

3. Main results.

Lemma 3.1. *Suppose that* (F_1) *holds. Let* $\varphi : H^1(\mathbf{R}^N) \to \mathbf{R}$ *defined by*

$$\varphi(u) = \frac{1}{2}\,\|u\|_1^2 - \psi(u), \quad \forall\, u \in H^1(\mathbf{R}^N),$$

where ψ *is defined by* (3). *Then the critical points of* φ *are solutions of* (P).

Proof. Clearly, $u \mapsto \|u\|_1^2/2$ is of class C^1; therefore, the map φ is locally Lipschitz. Let u be a critical point of φ. Applying Proposition 2.1 for $E := H^1(\mathbf{R}^N)$, we have for every $w \in H^1(\mathbf{R}^N)$

$$\begin{aligned} 0 \le \varphi^0(u;w) &= (u,w)_1 + (-\psi)^0(u;w) \\ &= (u,w)_1 + \psi^0(u;-w) \\ &\le (u,w)_1 + \int_{\mathbf{R}^N} F_x^0(x,u(x);-w(x))\,dx, \end{aligned}$$

i.e., u is a solution of (P). □

In order to obtain existence results, we impose further assumptions on F:

(F_2) $F(x,-s) = F(x,s)$ for almost every $x \in \mathbf{R}^N$, for all $s \in \mathbf{R}$;

(F_3) $F(gx,s) = F(x,s)$, for almost every $x \in \mathbf{R}^N$, for all $g \in \mathbf{O}(N)$, for all $s \in \mathbf{R}$;

(F_4) there exist $\alpha > 2$, $\lambda \in [0,(\alpha-2)/2[$ and $c_4 > 0$ such that for almost every $x \in \mathbf{R}^N$ and all $s \in \mathbf{R}$

$$(F_4 - a) \qquad \alpha F(x,s) + F_x^0(x,s;-s) - \lambda s^2 \le 0,$$

and

$$(F_4 - b) \qquad c_4(|s|^\alpha - |s|^2) \le F(x,s);$$

(F_5) $\lim_{s\to 0} \max\{|\xi| : \xi \in \partial F(x,s)\}/s = 0$ uniformly for almost every $x \in \mathbf{R}^N$.

Lemma 3.2. *Let E be a closed subspace of $H^1(\mathbf{R}^N)$ which is compactly embedded in $L^p(\mathbf{R}^N)$. Denoting by $\widehat{\varphi}$ the restriction of φ to E and assuming that (F_1), (F_4-a) and (F_5) hold, then $\widehat{\varphi}$ satisfies the $(PS)_c$ condition, $c\in\mathbf{R}$.*

Proof. Let $\{u_n\}\subset E$ be a sequence such that $\widehat{\varphi}(u_n)\to c$ and $\lambda_{\widehat{\varphi}}(u_n)\to 0$ as $n\to\infty$. Therefore, for every $n\in\mathbf{N}$, there exists $z_n^*\in\partial\widehat{\varphi}(u_n)$ such that $\|z_n^*\|=\lambda_{\widehat{\varphi}}(u_n)$. Since E is a Hilbert space, from the Riesz's representation theorem, for every $n\in\mathbf{N}$, there exists $z_n\in E$ such that $\|z_n\|_E=\|z_n^*\|$ and $(z_n,w)_E=\langle z_n^*,w\rangle$, for all $w\in E$. Let us denote by $\widehat{\psi}$ the restriction of ψ to E. Using Proposition 2.1 and (F_4-a), for n large enough, we obtain

$$\begin{aligned}
c+1+\|u_n\|_E &\ge \widehat{\varphi}(u_n)-\frac{1}{\alpha}\,\widehat{\varphi}^0(u_n;u_n)\\
&=\frac{1}{2}\,\|u_n\|_E^2-\widehat{\psi}(u_n)-\frac{1}{\alpha}\Big(\|u_n\|_E^2+\widehat{\psi}^0(u_n;-u_n)\Big)\\
&\ge\left(\frac{1}{2}-\frac{1}{\alpha}\right)\|u_n\|_E^2\\
&\qquad-\int_{\mathbf{R}^N}\left[F(x,u_n(x))+\frac{1}{\alpha}\,F_x^0(x,u_n(x);-u_n(x))\right]dx\\
&\ge\left(\frac{1}{2}-\frac{1}{\alpha}\right)\|u_n\|_E^2-\frac{\lambda}{\alpha}\int_{\mathbf{R}^N}|u_n(x)|^2\,dx\\
&\ge\left(\frac{1}{2}-\frac{1}{\alpha}\right)\|u_n\|_E^2-\frac{\lambda}{\alpha}\,\|u_n\|_E^2\\
&=\frac{1}{\alpha}\left(\frac{\alpha-2}{2}-\lambda\right)\|u_n\|_E^2.
\end{aligned}$$

Therefore, the sequence $\{u_n\}$ is bounded in E.

Since the embedding $E\hookrightarrow L^p(\mathbf{R}^N)$ is compact, passing to a subsequence if necessary, we may suppose that $u_n\rightharpoonup u$ in E and $u_n\to u$ in $L^p(\mathbf{R}^N)$.

On the other hand, we have

$$\begin{aligned}
\widehat{\varphi}^0(u_n;u-u_n)&=(u_n,u-u_n)_E+\widehat{\psi}^0(u_n;u_n-u),\\
\widehat{\varphi}^0(u;u_n-u)&=(u,u_n-u)_E+\widehat{\psi}^0(u;u-u_n).
\end{aligned}$$

Adding these relations, we obtain

$$\begin{aligned}\|u_n - u\|_E^2 \\ &= \left[\widehat{\psi}^0(u_n; u_n - u) + \widehat{\psi}^0(u; u - u_n)\right] - \widehat{\varphi}^0(u_n; u - u_n) - \widehat{\varphi}^0(u; u_n - u) \\ &= I_n^1 - I_n^2 - I_n^3.\end{aligned}$$

Now we will estimate I_n^i, $i = \overline{1,3}$. To this end, from (F_1) and (F_5), we have that for all $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$(6)\quad |\xi| \le \varepsilon|s| + c_\varepsilon |s|^{p-1}, \quad \forall \xi \in \partial F(x,s), \quad \text{for a.e. } x \in \mathbf{R}^N, \quad \forall s \in \mathbf{R}.$$

From Proposition 2.1 and (6) we have

$$\begin{aligned}I_n^1 &= \widehat{\psi}^0(u_n; u_n - u) + \widehat{\psi}^0(u; u - u_n) \\ &\le \int_{\mathbf{R}^N} F_x^0(x, u_n(x); u_n(x) - u(x))\, dx \\ &\quad + \int_{\mathbf{R}^N} F_x^0(x, u(x); u(x) - u_n(x))\, dx \\ &= \int_{\mathbf{R}^N} \max\{\xi_n(x)(u_n(x) - u(x)) : \xi_n(x) \in \partial F(x, u_n(x))\}\, dx \\ &\quad + \int_{\mathbf{R}^N} \max\{\xi(x)(u(x) - u_n(x)) : \xi(x) \in \partial F(x, u(x))\}\, dx \\ &\le \int_{\mathbf{R}^N} [\varepsilon\,(|u_n(x)| + |u(x)|) \\ &\quad + c_\varepsilon \left(|u_n(x)|^{p-1} + |u(x)|^{p-1}\right)\big]\, |u_n(x) - u(x)|\, dx \\ &\le 2\varepsilon(\|u_n\|_E^2 + \|u\|_E^2) + c_\varepsilon \|u_n - u\|_p \left(\|u_n\|_p^{p-1} + \|u\|_p^{p-1}\right).\end{aligned}$$

On the other hand,

$$I_n^2 = \widehat{\varphi}^0(u_n; u - u_n) \ge \langle z_n^*, u - u_n\rangle = (z_n, u - u_n)_E \ge -\|z_n\|_E \|u - u_n\|_E.$$

Moreover, let us fix a $z^* \in \partial\widehat{\varphi}(u)$. We have $I_n^3 = \widehat{\varphi}^0(u; u_n - u) \ge \langle z^*, u_n - u\rangle$ and therefore we have a $z \in E$ such that $I_n^3 \ge (z, u_n - u)_E$.

Since $\{u_n\}$ is bounded in E, letting $\varepsilon \to 0^+$ and keeping in mind that $u_n \to u$ in $L^p(\mathbf{R}^N)$, $\|z_n\|_E \to 0$ and $u_n \rightharpoonup u$ in E, from the above estimations we obtain that $\|u_n - u\|_E \to 0$ as $n \to \infty$. $\square$

Let

$$H^1_{\mathbf{O}(N)}(\mathbf{R}^N) := H^1(\mathbf{R}^N)^{\mathbf{O}(N)} = \{u \in H^1(\mathbf{R}^N) : gu = u,\ \forall\, g \in \mathbf{O}(N)\}.$$

The action of $\mathbf{O}(N)$ on $H^1(\mathbf{R}^N)$ is $gu(x) = u(g^{-1}x)$ for every $g \in \mathbf{O}(N)$, $u \in H^1(\mathbf{R}^N)$ and for almost every $x \in \mathbf{R}^N$. The elements of $H^1_{\mathbf{O}(N)}(\mathbf{R}^N)$ are exactly the radial functions of $H^1(\mathbf{R}^N)$.

Theorem 3.1. *If the assumptions* (F_1)–(F_5) *hold, then the problem* (P) *has infinitely many radial solutions.*

Proof. For abbreviation, we choose $E := H^1_{\mathbf{O}(N)}(\mathbf{R}^N)$ and $G := \mathbf{O}(N)$. Denote by $(\cdot,\cdot)_E$ the restriction of $(\cdot,\cdot)_1$ to E.

By assumption (F_3) and from the fact that G acts linear isometrically on $H^1(\mathbf{R}^N)$, the functional φ is G-invariant. Let $\widehat{\varphi}$ again be the restriction of φ to E. Due to Proposition 2.2 and Lemma 3.1, any critical point of $\widehat{\varphi}$ is a radial solution of (P). Therefore, it suffices to prove the existence of an unbounded sequence of critical points $\{u_n\} \subset E$ of $\widehat{\varphi}$. To this end, we will verify the requirements from Proposition 2.3 for E and $h := \widehat{\varphi}$.

By assumption (F_2), $\widehat{\varphi}$ is an even function. Let us choose an orthonormal basis $\{e_j\}$ of E and set $E_k = \operatorname{span}\{e_1, \dots, e_k\}$, $k \geq 1$.

Let us fix $k \geq 1$. From $(F_4 - b)$, we obtain

$$\widehat{\varphi}(u) \leq \frac{1}{2}\,\|u\|_E^2 - c_4\|u\|_\alpha^\alpha + c_4\|u\|_2^2.$$

The requirement (ii) from Proposition 2.3 follows for $R_k > 0$ large enough, since $\alpha > 2$, $\widehat{\varphi}(0) = 0$ and all norms on the finite dimension space E_k are equivalent.

Using (6) instead of (F_1), a similar calculation as in (4) shows that

$$\psi(u) \leq \varepsilon\|u\|_1^2 + c_\varepsilon\|u\|_p^p, \quad \forall\, u \in H^1(\mathbf{R}^N).$$

If $u \in E_k^\perp$, then

$$\widehat{\varphi}(u) \geq \left(\frac{1}{2} - \varepsilon\right)\|u\|_E^2 - c_\varepsilon\|u\|_p^p \geq \left(\frac{1}{2} - \varepsilon\right)\|u\|_E^2 - c_\varepsilon\mu_k^p\|u\|_E^p,$$

where

$$\mu_k = \sup_{\substack{u \in E_k^\perp \\ u \neq 0}} \frac{\|u\|_p}{\|u\|_E}.$$

It is well known that $\mu_k \to 0$ as $k \to \infty$, see [**5**, Lemma 3.3]. Choosing $\varepsilon < (p-2)/2p$ and $r_k = (pc_\varepsilon \mu_k^p)^{1/(2-p)}$, we have

$$\widehat{\varphi}(u) \geq \left(\frac{1}{2} - \varepsilon - \frac{1}{p}\right) r_k^2,$$

for every $u \in E_k^\perp$ with $\|u\|_E = r_k$. Due to the choice of ε and since $\mu_k \to 0$ as $k \to \infty$, the assumption (iii) from Proposition 2.3 is verified.

Since the embedding $H^1_{\mathbf{O}(N)}(\mathbf{R}^N) \hookrightarrow L^p(\mathbf{R}^N)$ is compact, $N \geq 2$ and $p \in]2, 2^*[$, see [**24**] or [**26**, Corollary 1.26], the $(PS)_c$ condition follows from Lemma 3.2.

Now, Proposition 2.3 guarantees the existence of an unbounded sequence of critical points of $\widehat{\varphi}$, which completes the proof. □

The purpose of the following result is to establish the existence of non-radial solutions of (P). This is a non-smooth version of the result of Bartsch and Willem, see [**26**, Theorem 3.13, p. 63].

Theorem 3.2. *If the assumptions* (F_1)–(F_5) *hold and* $N = 4$ *or* $N \geq 6$*, then the problem* (P) *has infinitely many non-radial solutions.*

Proof. It suffices to adapt the argument of [**26**, Theorem 1.31, p. 20], where the space (denoted by E) of non-radial functions is constructed. Using the result of Lions [**14**. Theorem 4.1], the embedding $E \hookrightarrow L^p(\mathbf{R}^N)$ is compact. Applying Lemma 3.2 to our restricted functional to E, the $(PS)_c$ holds, $c \in \mathbf{R}$. The rest of the proof is similar to that of Theorem 3.1. □

Remark 3.1. When $f \in C^0(\mathbf{R}^N \times \mathbf{R}, \mathbf{R})$ is a function which fulfills (1), and F is defined as in (2), $(F_4 - a)$ takes the form

$$\alpha F(x,s) - f(x,s)s - \lambda s^2 \leq 0, \tag{7}$$

which is a weaker condition than those in [**4, 24, 26**], due to the third member involving λ. We will see in the next section that in some applications the presence of this member is essential. Moreover, if there exists $R > 0$ such that

$$\inf_{\substack{x \in \mathbf{R}^N \\ |s| \geq R}} F(x,s) > 0,$$

see [**26**, p. 62] and [**24**, Rel. (42)], then from (1) and (7) for $\lambda = 0$, we can deduce after an integration the relation $(F_4 - b)$. If f satisfies further hypotheses as in [**26**, Theorem 3.12], our theorems improve the results from [**24**], [**26**, Theorem 3.12], [**5**, Theorem 2.1] and [**26**, Theorem 3.13]. Moreover, in the next section we give a concrete example where the earlier results fail and we can apply Theorem 3.1 and Theorem 3.2.

4. An example. We denote by $\lfloor u \rceil$ the nearest integer to $u \in \mathbf{R}$, if $u + 1/2 \notin \mathbf{Z}$; otherwise, we put $\lfloor u \rceil = u$.

Example. Let $F : \mathbf{R} \to \mathbf{R}$ be defined by

$$F(s) = \int_0^s \lfloor |t| t \rceil \, dt.$$

Then the conclusion of Theorem 3.1 holds for $N \in \{2, 3, 4, 5\}$. Moreover, if $N = 4$, the conclusion of Theorem 3.2 holds too.

Proof. We verify the hypotheses for $p := 3$. To have $p < 2^*$, we need $N \in \{2, 3, 4, 5\}$. It is easy to show that F is an even function; therefore, (F_2) holds. Moreover, according to Remark 1.1, (F_1) holds too. Since $F(s) = 0$ for every $s \in [-1/\sqrt{2}, 1/\sqrt{2}]$, (F_5) holds. Since F is even, it is enough to verify (F_4) only for nonnegative numbers, choosing $\alpha := 3$, $\lambda := 1/4$ and $c_4 := 1/3$.

One has

$$F(s) = \begin{cases} 0, & s \in [0, (1/\sqrt{2})], \\ F_n(s), & s \in I_n, \end{cases} \tag{8}$$

where $I_n = \left(\sqrt{(2n-1)/2}, \sqrt{(2n+1)/2}\right]$, $n \in \mathbf{N}^*$ and $F_n(s) = ns - (1 + \sqrt{3} + \cdots + \sqrt{2n-1})/\sqrt{2}$, $s \in I_n$. Now, we use the following inequalities for every $n \in \mathbf{N}^*$:

$$(I_{\leq}^n) \qquad 2n\sqrt{\frac{2n+1}{2}} - 3 \cdot \frac{1+\sqrt{3}+\cdots+\sqrt{2n-1}}{\sqrt{2}} - \frac{2n+1}{8} \leq 0,$$

and

$$(I_{\geq}^n) \qquad \frac{4n+1}{2}\sqrt{\frac{2n-1}{2}} - 3 \cdot \frac{1+\sqrt{3}+\cdots+\sqrt{2n-1}}{\sqrt{2}} + \frac{2n-1}{2} \geq 0.$$

Let us fix $s \geq 0$. If $s \in \left[0, 1/\sqrt{2}\right]$, then the two inequalities from (F_4) are trivial. Otherwise, there exists a unique $n \in \mathbf{N}^*$ such that $s \in I_n$. If $s \in \operatorname{int} I_n$, then $F^0(s; -s) = -ns$ and due to (8), we need

$$3\left(ns - \frac{1+\sqrt{3}+\cdots+\sqrt{2n-1}}{\sqrt{2}}\right) - ns - \frac{s^2}{4} \leq 0,$$

which follows from $(I_{\leq}^n)$. If $s_n = \sqrt{(2n+1)/2}$, then $F^0(s_n; -s_n) = -n\sqrt{(2n+1)/2}$. In this case, $(F_4 - a)$ reduces exactly to $(I_{\leq}^n)$.

Since the function $x \mapsto (x^3 - x^2)/3 - nx$ is decreasing in I_n, $n \in \mathbf{N}^*$, to show $(F_4 - b)$, it is enough to verify that

$$\frac{1}{3}\left(\left(\frac{2n-1}{2}\right)^{3/2} - \frac{2n-1}{2}\right) \leq n\sqrt{\frac{2n-1}{2}} - \frac{1+\sqrt{3}+\cdots+\sqrt{2n-1}}{\sqrt{2}},$$

which is exactly $(I_{\geq}^n)$. This completes the proof. $\square$

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