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EXISTENCE OF TWO NON-TRIVIAL SOLUTIONS FOR A CLASS OF QUASILINEAR ELLIPTIC VARIATIONAL SYSTEMS ON STRIP-LIKE DOMAINS

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Abstract In this paper we study the multiplicity of solutions of the quasilinear elliptic system

$$\left.\begin{aligned} -\Delta_p u &= \lambda F_u(x,u,v) && \text{in } \Omega,\\ -\Delta_q v &= \lambda F_v(x,u,v) && \text{in } \Omega,\\ u &= v = 0 && \text{on } \partial\Omega,\end{aligned}\right\} \qquad (\mathrm{S}_\lambda)$$

where Ω is a strip-like domain and $\lambda > 0$ is a parameter. Under some growth conditions on F, we guarantee the existence of an open interval $\Lambda \subset (0,\infty)$ such that for every $\lambda \in \Lambda$, the system (S_λ) has at least two distinct, non-trivial solutions. The proof is based on an abstract critical-point result of Ricceri and on the principle of symmetric criticality.

Keywords: strip-like domain; eigenvalue problem; principle of symmetric criticality, elliptic systems

2000 *Mathematics subject classification:* Primary 35A15; 35J65
Secondary 35P30

1. Introduction

In recent years there has been increasing interest in the study of quasilinear elliptic systems of the form

$$\left.\begin{aligned} -\Delta_p u &= F_u(x,u,v) && \text{in } \Omega,\\ -\Delta_q v &= F_v(x,u,v) && \text{in } \Omega,\end{aligned}\right\} \qquad (\mathrm{S})$$

where Ω is a smooth domain in $\mathbb{R}^N$, $F \in C^1(\Omega \times \mathbb{R}^2, \mathbb{R})$; F_z designates the partial derivative of F with respect to z, and Δ_α is the α-Laplacian operator $\Delta_\alpha u = \operatorname{div}(|\nabla u|^{\alpha-2}\nabla u)$. 'denotes'? We refer to the works of Boccardo and de Figueiredo [**4**], Felmer, Manásevich and de Thélin [**12**], de Figueiredo [**8**], and de Nápoli and Mariani [**9**]. In these works the approach is variational, the *boundedness* of the domain Ω is assumed, while (S) is subjected to the standard zero Dirichlet boundary conditions. Changes to sentence OK? Words added – OK? Usually, it is considered to be a functional (denote it by $\mathcal{H}$) on $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ whose critical points are the weak solutions of (S). Various *growth conditions* on F are required in order to guarantee nonzero critical points of $\mathcal{H}$. One of them is the celebrated Ambrosetti–Rabinowitz-type condition, adapted to the above-mentioned problem (S) (see, for example, [**4**, p. 312]), which asserts that $\mathcal{H}$ satisfies the Palais–Smale or Cerami compactness condition. This condition implies, in particular, some sort of *super-linearity* of F.

In this paper we study the eigenvalue problem related to (S), namely,

$$\left.\begin{aligned} -\Delta_p u &= \lambda F_u(x,u,v) && \text{in } \Omega,\\ -\Delta_q v &= \lambda F_v(x,u,v) && \text{in } \Omega,\\ u = v &= 0 && \text{on } \partial\Omega, \end{aligned}\right\} \tag{S_λ}$$

where $\lambda > 0$ is a parameter, Ω is a *strip-like domain*, i.e. $\Omega = \omega \times \mathbb{R}^l$, ω being a bounded open subset of $\mathbb{R}^m$ with smooth boundary and $m \geqslant 1$, $l \geqslant 2$, $1 < p, q < m + l$. On the other hand, we will treat the case when F is *sub-p,q-linear* (see (F4) below).

The motivation to investigate elliptic eigenvalue problems on strip-like domains arises from mathematical physics (see, for example, [**1**, **2**]). The mathematical development of these kinds of problem (in the *scalar* case) was initiated by Esteban [**10**]; for further related works we refer the reader to [**13**], [**14**], [**11**] and [**20**]. Recently, Carrião and Miyagaki [**7**] guaranteed the existence of at least one positive non-trivial solution of a related problem to (S) (namely, $p = q$) on strip-like domains and on domains which are situated between to infinite cylinders. They assumed that the nonlinear term F has some sort of homogeneity and, in addition, the right-hand side of (S) is perturbed by a gradient-type derivative of a p^*-homogeneous term (p^* is the critical exponent). Their approach is based on a suitable version of the concentration compactness principle. Although we do not treat the critical case in the present paper, we allow $p \neq q$ and we do not assume any homogeneity property on F.

'two'? Otherwise, clarify sentence?

The main result of this paper guarantees the existence of an open interval $\Lambda \subset (0,\infty)$ such that for every $\lambda \in \Lambda$, the system (S_λ) has at least two distinct, non-trivial weak solutions $(u^i_\lambda, v^i_\lambda)$, $i \in \{1,2\}$. Moreover, u^i_λ, v^i_λ are axially symmetric functions and the families $\{u^1_\lambda, u^2_\lambda\}_{\lambda\in\Lambda}$ and $\{v^1_\lambda, v^2_\lambda\}_{\lambda\in\Lambda}$ are *uniformly bounded* with respect to the $W^{1,p}_0(\Omega)$- and $W^{1,q}_0(\Omega)$-norms, respectively. The proof is based on a recent abstract critical-point result of Ricceri [**18**] and on the well-known principle of symmetric criticality of Palais [**17**].

The paper is organized as follows. In § 2 we will give the hypotheses on F and the statement of the main result (Theorem 2.2). Here, we also include a simple example, illustrating the applicability of our theorem. The proof of Theorem 2.2 is given in § 3.

2. The main result

Let Ω be a strip-like domain, i.e. $\Omega = \omega \times \mathbb{R}^l$, ω is a bounded open subset of $\mathbb{R}^m$ with smooth boundary and $m \geqslant 1$, $l \geqslant 2$, $1 < p, q < N = m + l$. Denoting by α^* the Sobolev critical exponent, i.e. $\alpha^* = \alpha N/(N - \alpha)$ ($\alpha \in \{p, q\}$), we require the following hypotheses on the nonlinear term F.

(F1) $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function, $(s,t) \mapsto F(x,s,t)$ is of class C^1 and $F(x,0,0) = 0$ for every $x \in \Omega$.

(F2) There exist $c_1 > 0$ and $r \in (p, p^*)$, $s \in (q, q^*)$ such that

$$|F_u(x,u,v)| \leqslant c_1(|u|^{p-1} + |v|^{(p-1)q/p} + |u|^{r-1}), \tag{2.1}$$
$$|F_v(x,u,v)| \leqslant c_1(|v|^{q-1} + |u|^{(q-1)p/q} + |v|^{s-1}) \tag{2.2}$$

for every $x \in \Omega$ and $(u,v) \in \mathbb{R}^2$.

The space $W_0^{1,\alpha}(\Omega)$ can be endowed with the norm

$$\|u\|_{1,\alpha} = \left(\int_\Omega |\nabla u|^\alpha\right)^{1/\alpha}, \quad \alpha \in \{p,q\},$$

and for $\beta \in [\alpha, \alpha^*]$ we have the Sobolev embeddings $W_0^{1,\alpha}(\Omega) \hookrightarrow L^\beta(\Omega)$. In view of (F1) and (F2), the energy functional $\mathcal{H} : W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) \times [0,\infty) \to \mathbb{R}$,

$$\mathcal{H}(u,v,\lambda) = \frac{1}{p}\|u\|_{1,p}^p + \frac{1}{q}\|v\|_{1,q}^q - \lambda \int_\Omega F(x,u,v)\,\mathrm{d}x,$$

is well defined and it is of class C^1. One readily has that for $\lambda > 0$ fixed, the critical points of $\mathcal{H}(\cdot,\cdot,\lambda)$ are exactly the weak solutions of (S_λ).

Taking into account the unboundedness of Ω (which causes, among other things, the non-compactness of the Sobolev embeddings $W_0^{1,\alpha}(\Omega) \hookrightarrow L^\beta(\Omega)$, $\beta \in [\alpha,\alpha^*]$, $\alpha \in \{p,q\}$), we construct a subspace of $W_0^{1,\alpha}(\Omega)$, $\alpha \in \{p,q\}$, which can be embedded compactly in $L^\beta(\Omega)$, $\beta \in (\alpha, \alpha^*)$. The compactness of this embedding will be useful in order to obtain critical points for $\mathcal{H}(\cdot,\cdot,\lambda)$. This construction can be described as follows.

The action of the compact group $\underline{G = \mathrm{id}^m \times \boldsymbol{O}(l)}$ on $W_0^{1,\alpha}(\Omega)$ is defined by Does 'id' denote the identity here?

$$gu(x,y) = u(x, g_0^{-1}y)$$

for every $(x,y) \in \omega \times \mathbb{R}^l$, $g = \mathrm{id}^m \times g_0 \in G$ and $u \in W_0^{1,\alpha}(\Omega)$, $\alpha \in \{p,q\}$. It is clear that the action G on $W_0^{1,\alpha}(\Omega)$ is isometric: that is,

$$\|gu\|_{1,\alpha} = \|u\|_{1,\alpha} \quad \text{for every } g \in G,\ u \in W_0^{1,\alpha}(\Omega),\ \alpha \in \{p,q\}. \tag{2.3}$$

The space $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ will be endowed with the norm

$$\|(u,v)\|_{1,p,q} = \|u\|_{1,p} + \|v\|_{1,q},$$

while the group G acts on it by

$$g(u,v) = (gu, gv) \quad \text{for every } g \in G,\ (u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega).$$

Let

$$W_{0,G}^{1,\alpha}(\Omega) \overset{\text{not}}{=} \mathrm{Fix}_G W_0^{1,\alpha}(\Omega) = \{u \in W_0^{1,\alpha}(\Omega) : gu = u \text{ for every } g \in G\}.$$

Use '$\neq$' instead of '$\overset{\text{not}}{=}$' here and below? OK that dot has been removed after 'not' if you want to keep this notation?

Since $l \geqslant 2$, the embedding $W^{1,\alpha}_{0,G}(\Omega) \hookrightarrow L^\beta(\Omega)$ with $\beta \in (\alpha, \alpha^*)$ is compact (see [**15**, Théorème III.2] or [**11**]). One clearly has that

$$\begin{aligned}\mathrm{Fix}_G(W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)) \\ &= \{(u,v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) : g(u,v) = (u,v) \text{ for every } g \in G\} \\ &= W^{1,p}_{0,G}(\Omega) \times W^{1,q}_{0,G}(\Omega). \end{aligned} \tag{2.4}$$

For abbreviation, we introduce further the following notation: $W^\alpha = W_0^{1,\alpha}(\Omega)$, $W_G^\alpha = W^{1,\alpha}_{0,G}(\Omega)$ ($\alpha \in \{p,q\}$), and $W^{p,q} = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, $W_G^{p,q} = W^{1,p}_{0,G}(\Omega) \times W^{1,q}_{0,G}(\Omega)$, respectively.

We say that a function $h : \Omega \to \mathbb{R}$ is *axially symmetric* if $h(x,y) = h(x,gy)$ for every $x \in \omega$, $y \in \mathbb{R}^l$ and $g \in \boldsymbol{O}(l)$. In particular, the elements of W_G^α are exactly the axially symmetric functions of W^α.

On the nonlinear term we will consider the following further hypotheses. Clarify sentence?

(F3) $\displaystyle\lim_{u,v\to 0} \frac{F_u(x,u,v)}{|u|^{p-1}} = \lim_{u,v\to 0} \frac{F_v(x,u,v)}{|v|^{q-1}} = 0$ uniformly for every $x \in \Omega$.

(F4) There exist $p_1 \in (0,p)$, $q_1 \in (0,q)$, $\mu \in [p,p^*]$, $\nu \in [q,q^*]$ and $a \in L^{\mu/(\mu-p_1)}(\Omega)$, $b \in L^{\nu/(\nu-q_1)}(\Omega)$, $c \in L^1(\Omega)$ such that

$$F(x,u,v) \leqslant a(x)|u|^{p_1} + b(x)|v|^{q_1} + c(x)$$

for every $x \in \Omega$ and $(u,v) \in \mathbb{R}^2$.

(F5) There exist $(u_0,v_0) \in W_G^{p,q}$ such that

$$\int_\Omega F(x,u_0(x),v_0(x))\,\mathrm{d}x > 0.$$

Remark 2.1. Let us denote by $\mathcal{H}_G(\cdot\,,\cdot\,,\lambda)$ the restriction of $\mathcal{H}(\cdot\,,\cdot\,,\lambda)$ to the space $W_G^{p,q}$. Then (F3) and (F4) imply that $\mathcal{H}_G(\cdot\,,\cdot\,,\lambda)$ is bounded from below and it satisfies the Palais–Smale condition for every $\lambda > 0$ (see § 3). Therefore, for every $\lambda > 0$ the functional $\mathcal{H}_G(\cdot\,,\cdot\,,\lambda)$ has a minimizer (u_λ, v_λ). Moreover, for large λ, (F5) forces that $\mathcal{H}_G(u_0,v_0,\lambda) < 0$, hence $\mathcal{H}_G(u_\lambda,v_\lambda,\lambda) < 0$. The element (u_λ,v_λ) will be a critical point not only of $\mathcal{H}_G(\cdot\,,\cdot\,,\lambda)$ but also of $\mathcal{H}(\cdot\,,\cdot\,,\lambda)$, due to the principle of symmetric criticality. On the other hand, (2.1) and (2.2) imply that $F_u(x,0,0) = F_v(x,0,0) = 0$. Therefore, $(0,0)$ is a solution of (S_λ) and $\mathcal{H}_G(0,0,\lambda) = 0$ for every $\lambda > 0$. This means, in particular, that $(u_\lambda,v_\lambda) \neq (0,0)$. But we are interested to obtain further information about the existence and behaviour of solutions of (S_λ), which requires a finer analysis. Actually, we can formulate the following theorem which constitutes the main result of this paper.

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Opening parenthesis added – OK?

'keen'?

Theorem 2.2. *Let $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ be a function which satisfies* (F1)–(F5). *If F is axially symmetric in the first variable and $ps = qr$, then there exist an open interval $\Lambda \subset (0,\infty)$ and $\sigma > 0$ such that for all $\lambda \in \Lambda$ the system (S_λ) has at least two distinct, non-trivial weak solutions (denote them by $(u^i_\lambda, v^i_\lambda)$, $i \in \{1,2\}$), the functions u^i_λ, v^i_λ are axially symmetric, and $\|u^i_\lambda\|_{1,p} < \sigma$, $\|v^i_\lambda\|_{1,q} < \sigma$, $i \in \{1,2\}$.*

Remark 2.3. Unlike to a bounded domain Ω where one clearly has $L^\gamma(\Omega) \subset L^\mu(\Omega)$ whenever $1 \leqslant \mu \leqslant \gamma \leqslant \infty$, in our case, i.e. $\Omega = \omega \times \mathbb{R}^l$, this inclusion is no longer valid, although it would be important in several estimations. The hypothesis $ps = qr$ (p, q, r, s from (F2)) is destined to compensate the unboundedness of the domain and it seems to be indispensable in our arguments (see Lemma 3.4 and relations (3.7), (3.8)). However, if in (S_λ) one has $p = q$, the above hypothesis disappears in the sense that, without loosing the generality, we may take $s = r$. Clarify sentence? Clarify sentence?

Example 2.4. Let $\Omega = \omega \times \mathbb{R}^2$, where ω is a bounded open interval in $\mathbb{R}$. Let $\gamma : \Omega \to \mathbb{R}$ be a continuous, non-negative, not identically zero, axially symmetric function with compact support in Ω. Then there exist an open interval $\Lambda \subset (0,\infty)$ and a number $\sigma > 0$ such that for every $\lambda \in \Lambda$, the system

$$\begin{aligned} -\Delta_{3/2} u &= \tfrac{5}{2}\lambda\gamma(x)|u|^{1/2}u\cos(|u|^{5/2}+|v|^3) && \text{in } \Omega,\\ -\Delta_{9/4} v &= 3\lambda\gamma(x)|v|v\cos(|u|^{5/2}+|v|^3) && \text{in } \Omega,\\ u &= v = 0 && \text{on } \partial\Omega \end{aligned}$$

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has at least two distinct, non-trivial weak solutions with the properties from Theorem 2.2.

Indeed, let us choose

$$F(x,u,v) = \gamma(x)\sin(|u|^{5/2}+|v|^3), \qquad r = \tfrac{11}{4}, \qquad s = \tfrac{33}{8}.$$

(F1)–(F3) hold immediately. For (F4) we choose $a = b = 0$, $c = \gamma$. Since γ is an axially symmetric function, $\operatorname{supp}\gamma$ will be an $\mathrm{id}\times\boldsymbol{O}(2)$-invariant set, i.e. if $(x,y) \in \operatorname{supp}\gamma$ then $(x,gy) \in \operatorname{supp}\gamma$ for every $g \in \boldsymbol{O}(2)$. Therefore, it is possible to fix an element $u_0 \in W^{1,3/2}_{0,\mathrm{id}\times\boldsymbol{O}(2)}(\Omega)$ such that $u_0(x) = (\pi/2)^{2/5}$ for every $x \in \operatorname{supp}\gamma$. Choosing $v_0 = 0$, one has that

$$\int_\Omega F(x,u_0(x),v_0(x))\,\mathrm{d}x = \int_{\operatorname{supp}\gamma}\gamma(x)\sin|u_0(x)|^{5/2}\,\mathrm{d}x = \int_{\operatorname{supp}\gamma}\gamma(x)\,\mathrm{d}x > 0.$$

The conclusion follows from Theorem 2.2.

3. Proof of Theorem 2.2

To prove Theorem 2.2, we will apply the following abstract critical-point result of Ricceri.

Theorem 3.1 (Theorem 3 in [18]). *Let $(X,\|\cdot\|)$ be a separable and reflexive real Banach space, $I \subseteq \mathbb{R}$ an interval, and $g : X \times I \to \mathbb{R}$ a continuous function satisfying the following conditions:*

(i) *for every $x \in X$, the function $g(x,\cdot)$ is concave;*

(ii) *for every $\lambda \in I$, the function $g(\cdot\,,\lambda)$ is sequentially weakly lower semicontinuous and continuously Gâteaux differentiable, satisfies the Palais–Smale condition and*

$$\lim_{\|x\|\to+\infty} g(x,\lambda) = +\infty;$$

(iii) *there exists a continuous concave function* $h : I \to \mathbb{R}$ *such that*

$$\sup_{\lambda \in I} \inf_{x \in X} (g(x,\lambda) + h(\lambda)) < \inf_{x \in X} \sup_{\lambda \in I} (g(x,\lambda) + h(\lambda)).$$

Then there is an open interval $\Lambda \subseteq I$ *and a number* $\sigma > 0$ *such that for each* $\lambda \in \Lambda$, *the function* $g(\cdot\,,\lambda)$ *has at least three critical points in* X *having norm less than* σ.

Remark 3.2. Theorem 3.1 is a very efficient tool in the investigation of elliptic eigenvalue problems. The reader can consult the recent papers of Averna and Salvati [**3**], Bonnano [**5**], Marano and Motreanu [**16**] and Ricceri [**19**] for various extensions and applications of the above result. However, to the best of my knowledge, Theorem 2.2 is the first application of Ricceri's result to *non-scalar* elliptic problems.

In the rest of this section, we suppose that all the assumptions of Theorem 2.2 are fulfilled.

Lemma 3.3. *For every* $\varepsilon > 0$ *there exists* $c(\varepsilon) > 0$ *such that*

(i) $|F_u(x,u,v)| \leqslant \varepsilon(|u|^{p-1} + |v|^{(p-1)q/p}) + c(\varepsilon)(|u|^{r-1} + |v|^{(r-1)q/p})$,

(ii) $|F_v(x,u,v)| \leqslant \varepsilon(|v|^{q-1} + |u|^{(q-1)p/q}) + c(\varepsilon)(|v|^{s-1} + |u|^{(s-1)p/q})$,

(iii)
$$\begin{aligned}|F(x,u,v)| &\leqslant \varepsilon(|u|^p + |v|^{(p-1)q/p}|u| + |v|^q + |u|^{(q-1)p/q}|v|)\\ &\quad + c(\varepsilon)(|u|^r + |v|^{(r-1)q/p}|u| + |v|^s + |u|^{(s-1)p/q}|v|)\end{aligned}$$

for every $x \in \Omega$ *and* $(u,v) \in \mathbb{R}^2$.

Proof. (i) Let $\varepsilon > 0$ be arbitrary. Let us prove the first inequality, the second one being similar. From the first limit of (F3) we have in particular that

$$\lim_{u,v \to 0} \frac{F_u(x,u,v)}{|u|^{p-1} + |v|^{(p-1)q/p}} = 0.$$

Therefore, there exists $\delta(\varepsilon) > 0$ such that if $|u|^{p-1} + |v|^{(p-1)q/p} < \delta(\varepsilon)$ then $|F_u(x,u,v)| \leqslant \varepsilon(|u|^{p-1} + |v|^{(p-1)q/p})$. If $|u|^{p-1} + |v|^{(p-1)q/p} \geqslant \delta(\varepsilon)$ then (2.1) implies that

$$\begin{aligned}|F_u(x,u,v)| &\leqslant c_1[(|u|^{p-1} + |v|^{(p-1)q/p})^{(r-1)/(p-1)}\delta(\varepsilon)^{(p-r)/(p-1)} + |u|^{r-1}]\\ &\leqslant c(\varepsilon)(|u|^{r-1} + |v|^{(r-1)q/p}).\end{aligned}$$

Combining the above inequalities, we obtain the desired relation. Part (iii) follows from the mean value theorem, (i), (ii) and $F(x,0,0) = 0$. □

We define the function $\mathcal{F} : W^{p,q} \to \mathbb{R}$ by

$$\mathcal{F}(u,v) = \int_{\Omega} F(x,u(x),v(x))\,\mathrm{d}x.$$

Using the Sobolev embeddings, (F1) and (F2), one can prove in a standard way that $\mathcal{F}$ is of class C^1, its differential being

$$\mathcal{F}'(u,v)(w,y) = \int_\Omega [F_u(x,u,v)w + F_v(x,u,v)y]\,\mathrm{d}x \tag{3.1}$$

for every $u, w \in W^p$ and $v, y \in W^q$.

Below, let us denote by $\|\cdot\|_{1,\alpha,G}$ the restriction of $\|\cdot\|_{1,\alpha}$ to W_G^α, $\alpha \in \{p,q\}$, and by $\mathcal{F}_G$, $\mathcal{H}_G(\cdot\,,\cdot\,,\lambda)$, $\|\cdot\|_{1,p,q,G}$ the restrictions of $\mathcal{F}$, $\mathcal{H}(\cdot\,,\cdot\,,\lambda)$, $\|\cdot\|_{1,p,q}$ to $W_G^{p,q}$, respectively. The norm of $L^\beta(\Omega)$ will be denoted by $\|\cdot\|_\beta$, as usual.

Lemma 3.4. *$\mathcal{F}_G$ is a sequentially weakly continuous function on $W_G^{p,q}$.*

Proof. Suppose the contrary, i.e. let $\{(u_n,v_n)\} \subset W_G^{p,q}$ be a sequence which converges weakly to $(u,v) \in W_G^{p,q}$ and $\mathcal{F}_G(u_n,v_n) \nrightarrow \mathcal{F}_G(u,v)$. Therefore, there exists $\varepsilon_0 > 0$ and a subsequence of $\{(u_n,v_n)\}$ (denoting again by $\{(u_n,v_n)\}$) such that Clarify sentence?

$$0 < \varepsilon_0 \leqslant |\mathcal{F}_G(u_n,v_n) - \mathcal{F}_G(u,v)| \quad \text{for every } n \in \mathbb{N}.$$

For some $0 < \theta_n < 1$ we have

$$0 < \varepsilon_0 \leqslant |\mathcal{F}_G'(u_n + \theta_n(u-u_n), v_n + \theta_n(v-v_n))(u_n-u, v_n-v)| \tag{3.2}$$

for every $n \in \mathbb{N}$. Let us denote by $w_n = u_n + \theta_n(u-u_n)$ and $y_n = v_n + \theta_n(v-v_n)$. Since the embeddings $W_G^p \hookrightarrow L^r(\Omega)$ and $W_G^q \hookrightarrow L^s(\Omega)$ are compact, up to a subsequence, $\{(u_n,v_n)\}$ converges strongly to (u,v) in $L^r(\Omega) \times L^s(\Omega)$. By (3.1), Lemma 3.3, Hölder's inequality and $ps = qr$ one has ‘define’?

$$\begin{aligned}
&|\mathcal{F}_G'(w_n,y_n)(u_n-u,v_n-v)| \\
&\quad\leqslant \int_\Omega [|F_u(x,w_n,y_n)||u_n-u| + |F_v(x,w_n,y_n)||v_n-v|]\,\mathrm{d}x \\
&\quad\leqslant \varepsilon \int_\Omega [(|w_n|^{p-1} + |y_n|^{(p-1)q/p})|u_n-u| + (|y_n|^{q-1} + |w_n|^{(q-1)p/q})|v_n-v|]\,\mathrm{d}x \\
&\qquad + c(\varepsilon)\int_\Omega [(|w_n|^{r-1} + |y_n|^{(r-1)q/p})|u_n-u| + (|y_n|^{s-1} + |w_n|^{(s-1)p/q})|v_n-v|]\,\mathrm{d}x \\
&\quad\leqslant \varepsilon[(\|w_n\|_p^{p-1} + \|y_n\|_q^{(p-1)q/p})\|u_n-u\|_p + (\|y_n\|_q^{q-1} + \|w_n\|_p^{(q-1)p/q})\|v_n-v\|_q] \\
&\qquad + c(\varepsilon)[(\|w_n\|_r^{r-1} + \|y_n\|_s^{(r-1)q/p})\|u_n-u\|_r + (\|y_n\|_s^{s-1} + \|w_n\|_r^{(s-1)p/q})\|v_n-v\|_s].
\end{aligned}$$

Since $\{w_n\}$ and $\{y_n\}$ are bounded in $W_G^p \hookrightarrow L^p(\Omega) \cap L^r(\Omega)$ and $W_G^q \hookrightarrow L^q(\Omega) \cap L^s(\Omega)$, respectively, while $u_n \to u$ and $v_n \to v$ strongly in $L^r(\Omega)$ and $L^s(\Omega)$, respectively, choosing $\varepsilon > 0$ small arbitrary, we obtain that $\mathcal{F}_G'(w_n,y_n)(u_n-u,v_n-v) \to 0$, as $n \to \infty$. But this contradicts (3.2). □ Clarify sentence?

It is clear that

$$\mathcal{H}_G(u,v,\lambda) = \frac{1}{p}\|u\|_{1,p,G}^p + \frac{1}{q}\|v\|_{1,q,G}^q - \lambda\mathcal{F}_G(u,v)$$

for $(u,v) \in W_G^{p,q}$. For a fixed $\lambda \geqslant 0$ we denote by $\mathcal{H}_G'(u,v,\lambda)$ the differential of $\mathcal{H}_G(\cdot\,,\cdot\,,\lambda)$ at $(u,v) \in W_G^{p,q}$.

Lemma 3.5. *Let $\lambda \geqslant 0$ be fixed and let $\{(u_n, v_n)\}$ be a bounded sequence in $W_G^{p,q}$ such that*

$$\|\mathcal{H}_G'(u_n, v_n, \lambda)\|_{(W_G^{p,q})^*} \to 0$$

as $n \to \infty$. Then $\{(u_n, v_n)\}$ contains a strongly convergent subsequence in $W_G^{p,q}$.

Proof. Up to a subsequence, we can assume that

$$(u_n, v_n) \to (u, v) \text{ weakly in } W_G^{p,q}, \tag{3.3}$$

$$(u_n, v_n) \to (u, v) \text{ strongly in } L^r(\Omega) \times L^s(\Omega). \tag{3.4}$$

On the other hand, we have

$$\begin{aligned}
&\mathcal{H}_G'(u_n, v_n, \lambda)(u - u_n, v - v_n) \\
&\quad = \int_\Omega |\nabla u_n|^{p-2} \nabla u_n (\nabla u - \nabla u_n) \\
&\quad\quad + \int_\Omega |\nabla v_n|^{q-2} \nabla v_n (\nabla v - \nabla v_n) - \lambda \mathcal{F}_G'(u_n, v_n)(u - u_n, v - v_n)
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{H}_G'(u, v, \lambda)(u_n - u, v_n - v) \\
&\quad = \int_\Omega |\nabla u|^{p-2} \nabla u (\nabla u_n - \nabla u) \\
&\quad\quad + \int_\Omega |\nabla v|^{q-2} \nabla v (\nabla v_n - \nabla v) - \lambda \mathcal{F}_G'(u, v)(u_n - u, v_n - v).
\end{aligned}$$

Adding these two relations, one has

$$\begin{aligned}
a_n &\overset{\text{not}}{=} \int_\Omega (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u|^{p-2} \nabla u)(\nabla u_n - \nabla u) \\
&\quad + \int_\Omega (|\nabla v_n|^{q-2} \nabla v_n - |\nabla v|^{q-2} \nabla v)(\nabla v_n - \nabla v) \\
&= -\mathcal{H}_G'(u_n, v_n, \lambda)(u - u_n, v - v_n) - \mathcal{H}_G'(u, v, \lambda)(u_n - u, v_n - v) \\
&\quad - \lambda \mathcal{F}_G'(u_n, v_n)(u - u_n, v - v_n) - \lambda \mathcal{F}_G'(u, v)(u_n - u, v_n - v).
\end{aligned}$$

Using (3.3) and (3.4), similar estimations as in Lemma 3.4 show that the last two terms tends to 0 as $n \to \infty$. Due to (3.3), the second terms tends to 0, while the inequality

$$|\mathcal{H}_G'(u_n, v_n, \lambda)(u - u_n, v - v_n)| \leqslant \|\mathcal{H}_G'(u_n, v_n, \lambda)\|_{(W_G^{p,q})^*} \|(u - u_n, v - v_n)\|_{1,p,q,G}$$

and the assumption implies that the first term tends to 0 too. Thus,

$$\lim_{n \to \infty} a_n = 0. \tag{3.5}$$

From the well-known inequality

$$|t - s|^\alpha \leqslant \begin{cases} (|t|^{\alpha-2} t - |s|^{\alpha-2} s)(t - s), & \text{if } \alpha \geqslant 2, \\ ((|t|^{\alpha-2} t - |s|^{\alpha-2} s)(t - s))^{\alpha/2} (|t|^\alpha + |s|^\alpha)^{(2-\alpha)/2}, & \text{if } 1 < \alpha < 2, \end{cases}$$

for all $t, s \in \mathbb{R}^N$, and (3.5), we conclude that

$$\lim_{n\to\infty} \int_\Omega (|\nabla u_n - \nabla u|^p + |\nabla v_n - \nabla v|^q) = 0,$$

hence, the sequence $\{(u_n, v_n)\}$ converges strongly to (u, v) in $W_G^{p,q}$. $\square$

Proof of Theorem 2.2 completed. We will show that the assumptions of Theorem 3.1 are fulfilled with the following choice: $X = W_G^{p,q}$, $I = [0, \infty)$ and $g = \mathcal{H}_G$.

Since the function $\lambda \mapsto \mathcal{H}_G(u, v, \lambda)$ is affine, (i) is true.

Now, we fix $\lambda \geqslant 0$. It is clear that

$$W_G^{p,q} \ni (u, v) \mapsto \frac{1}{p}\|u\|^p_{1,p,G} + \frac{1}{q}\|v\|^q_{1,q,G}$$

is sequentially weakly lower semicontinuous (see [**6**, Proposition III.5]). Thus, from Lemma 3.4 it follows that $\mathcal{H}_G(\cdot\,,\cdot\,,\lambda)$ is also sequentially weakly lower semicontinuous. Word added – OK?

We first prove that

$$\lim_{\|(u,v)\|_{1,p,q,G}\to\infty} \mathcal{H}_G(u, v, \lambda) = +\infty. \tag{3.6}$$

Indeed, from (F4) and Hölder's inequalities, one has

$$\begin{aligned}\mathcal{H}_G(u, v, \lambda) &\geqslant \frac{1}{p}\|u\|^p_{1,p,G} + \frac{1}{q}\|v\|^q_{1,q,G} - \lambda \int_\Omega [a(x)|u|^{p_1} + b(x)|v|^{q_1} + c(x)]\,\mathrm{d}x \\ &\geqslant \frac{1}{p}\|u\|^p_{1,p,G} + \frac{1}{q}\|v\|^q_{1,q,G} - \lambda[\|a\|_{\mu/(\mu-p_1)}\|u\|^{p_1}_\mu + \|b\|_{\nu/(\nu-q_1)}\|v\|^{q_1}_\nu + \|c\|_1].\end{aligned}$$

Since $W_G^p \hookrightarrow L^\mu(\Omega)$ and $W_G^q \hookrightarrow L^\nu(\Omega)$ are continuous, while $p_1 < p$ and $q_1 < q$, relation (3.6) yields immediately. To conclude (ii) completely from Theorem 3.1, we prove that $\mathcal{H}_G(\cdot\,,\cdot\,,\lambda)$ satisfies the Palais–Smale condition. To this end, let $\{(u_n, v_n)\}$ be a sequence in $W_G^{p,q}$ such that $\sup_{n\to\infty}|\mathcal{H}_G(u_n, v_n, \lambda)| < +\infty$ and $\lim_{n\to\infty}\|\mathcal{H}'_G(u_n, v_n, \lambda)\|_{(W_G^{p,q})^*} = 0$. According to (3.6), $\{(u_n, v_n)\}$ must be bounded in $W_G^{p,q}$. The conclusion follows now by Lemma 3.5. Clarify sentence?

Now we deal with (iii). Let us define the function $f : (0, \infty) \to \mathbb{R}$ by

$$f(t) = \sup\left\{\mathcal{F}_G(u, v) : \frac{1}{p}\|u\|^p_{1,p,G} + \frac{1}{q}\|v\|^q_{1,q,G} \leqslant t\right\}.$$

After an integration in Lemma 3.3 (iii), using the Young inequality, Sobolev embeddings and the relation $ps = qr$, for an arbitrary $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that

$$\mathcal{F}_G(u, v) \leqslant \varepsilon(\|u\|^p_{1,p,G} + \|v\|^q_{1,q,G}) + c(\varepsilon)(\|u\|^r_{1,p,G} + \|v\|^s_{1,q,G}) \tag{3.7}$$

for every $(u, v) \in W_G^{p,q}$. Since the function $x \mapsto (a^x + b^x)^{1/x}$, $x > 0$ is non-increasing $(a, b \geqslant 0)$, using again $ps = qr$, one has that

$$\|u\|^r_{1,p,G} + \|v\|^s_{1,q,G} \leqslant [\|u\|^p_{1,p,G} + \|v\|^q_{1,q,G}]^{r/p}. \tag{3.8}$$

Therefore,

$$f(t) \leqslant \varepsilon \max\{p,q\}t + c(\varepsilon)(\max\{p,q\}t)^{r/p}, \quad t>0.$$

On the other hand, clearly $f(t) \geqslant 0$, $t>0$. Taking into account the arbitrariness of $\varepsilon > 0$ and the fact that $r>p$, we conclude that

$$\lim_{t\to 0^+} \frac{f(t)}{t} = 0. \tag{3.9}$$

By (F5) it is clear that $(u_0, v_0) \neq (0,0)$ (note that $\mathcal{F}_G(0,0)=0$). Therefore, it is possible to choose a number η such that

$$0 < \eta < \mathcal{F}_G(u_0,v_0)\left[\frac{1}{p}\|u_0\|^p_{1,p,G} + \frac{1}{q}\|v_0\|^q_{1,q,G}\right]^{-1}.$$

Due to (3.9), there exists

$$t_0 \in \left(0, \frac{1}{p}\|u_0\|^p_{1,p,G} + \frac{1}{q}\|v_0\|^q_{1,q,G}\right)$$

such that $f(t_0) < \eta t_0$. Thus,

$$f(t_0) < \mathcal{F}_G(u_0,v_0)t_0\left[\frac{1}{p}\|u_0\|^p_{1,p,G} + \frac{1}{q}\|v_0\|^q_{1,q,G}\right]^{-1}.$$

Let $\rho_0 > 0$ such that

$$f(t_0) < \rho_0 < \mathcal{F}_G(u_0,v_0)t_0\left[\frac{1}{p}\|u_0\|^p_{1,p,G} + \frac{1}{q}\|v_0\|^q_{1,q,G}\right]^{-1}. \tag{3.10}$$

Define $h : I = [0,\infty) \to \mathbb{R}$ by $h(\lambda) = \rho_0\lambda$. We prove that h fulfils the inequality (iii) from Theorem 3.1.

Due to the choice of t_0 and (3.10), one has

$$\rho_0 < \mathcal{F}_G(u_0,v_0). \tag{3.11}$$

The function

$$I \ni \lambda \mapsto \inf_{(u,v)\in W_G^{p,q}}\left[\frac{1}{p}\|u\|^p_{1,p,G} + \frac{1}{q}\|v\|^q_{1,q,G} + \lambda(\rho_0 - \mathcal{F}_G(u,v))\right]$$

is clearly upper semicontinuous on I. Thanks to (3.11), we have

$$\begin{aligned}&\lim_{\lambda\to\infty}\inf_{(u,v)\in W_G^{p,q}}(\mathcal{H}_G(u,v,\lambda)+\rho_0\lambda)\\ &\qquad\leqslant \lim_{\lambda\to\infty}\left[\frac{1}{p}\|u_0\|^p_{1,p,G} + \frac{1}{q}\|v_0\|^q_{1,q,G} + \lambda(\rho_0 - \mathcal{F}_G(u_0,v_0))\right] = -\infty.\end{aligned}$$

Thus we find an element $\bar{\lambda} \in I$ such that

$$\sup_{\lambda \in I} \inf_{(u,v)\in W_G^{p,q}} (\mathcal{H}_G(u,v,\lambda) + \rho_0\lambda)$$
$$= \inf_{(u,v)\in W_G^{p,q}} \left[\frac{1}{p}\|u\|_{1,p,G}^p + \frac{1}{q}\|v\|_{1,q,G}^q + \bar{\lambda}(\rho_0 - \mathcal{F}_G(u,v))\right]. \quad (3.12)$$

Since $f(t_0) < \rho_0$, for all $(u,v) \in W_G^{p,q}$ such that

$$\frac{1}{p}\|u\|_{1,p,G}^p + \frac{1}{q}\|v\|_{1,q,G}^q \leqslant t_0,$$

we have $\mathcal{F}_G(u,v) < \rho_0$. Thus,

$$t_0 \leqslant \inf\left\{\frac{1}{p}\|u\|_{1,p,G}^p + \frac{1}{q}\|v\|_{1,q,G}^q : \mathcal{F}_G(u,v) \geqslant \rho_0\right\}. \quad (3.13)$$

On the other hand,

$$\inf_{(u,v)\in W_G^{p,q}} \sup_{\lambda \in I} (\mathcal{H}_G(u,v,\lambda) + \rho_0\lambda)$$
$$= \inf_{(u,v)\in W_G^{p,q}} \left[\frac{1}{p}\|u\|_{1,p,G}^p + \frac{1}{q}\|v\|_{1,q,G}^q + \sup_{\lambda\in I}(\lambda(\rho_0 - \mathcal{F}_G(u,v)))\right]$$
$$= \inf\left\{\frac{1}{p}\|u\|_{1,p,G}^p + \frac{1}{q}\|v\|_{1,q,G}^q : \mathcal{F}_G(u,v) \geqslant \rho_0\right\}.$$

Thus, (3.13) is equivalent to Change OK?

$$t_0 \leqslant \inf_{(u,v)\in W_G^{p,q}} \sup_{\lambda\in I} (\mathcal{H}_G(u,v,\lambda) + \rho_0\lambda). \quad (3.14)$$

There are two distinct cases.

(I) If $0 \leqslant \bar{\lambda} < t_0/\rho_0$, we have

$$\inf_{(u,v)\in W_G^{p,q}} \left[\frac{1}{p}\|u\|_{1,p,G}^p + \frac{1}{q}\|v\|_{1,q,G}^q + \bar{\lambda}(\rho_0 - \mathcal{F}_G(u,v))\right] \leqslant \mathcal{H}_G(0,0,\bar{\lambda}) + \rho_0\bar{\lambda} = \bar{\lambda}\rho_0 < t_0.$$

Combining the above inequality with (3.12) and (3.14), the desired relation from Theorem 3.1 (iii) is obtained immediately. Change OK?

(II) If $t_0/\rho_0 \leqslant \bar{\lambda}$, from (3.11) and (3.10) we obtain

$$\inf_{(u,v)\in W_G^{p,q}} \left[\frac{1}{p}\|u\|_{1,p,G}^p + \frac{1}{q}\|v\|_{1,q,G}^q + \bar{\lambda}(\rho_0 - \mathcal{F}_G(u,v))\right]$$
$$\leqslant \frac{1}{p}\|u_0\|_{1,p,G}^p + \frac{1}{q}\|v_0\|_{1,q,G}^q + \bar{\lambda}(\rho_0 - \mathcal{F}_G(u_0,v_0))$$
$$\leqslant \frac{1}{p}\|u_0\|_{1,p,G}^p + \frac{1}{q}\|v_0\|_{1,q,G}^q + \frac{t_0}{\rho_0}(\rho_0 - \mathcal{F}_G(u_0,v_0)) < t_0.$$

The conclusion holds similarly as in the first case. Clarify sentence?

Author: does item (II) end here?

Thus, the hypotheses of Theorem 3.1 are fulfilled. This implies the existence of an open interval $\Lambda \subset [0,\infty)$ and $\sigma > 0$ such that for all $\lambda \in \Lambda$ the function $\mathcal{H}_G(\cdot,\cdot,\lambda)$ has at least three distinct critical points in $W_G^{p,q}$ (denote them by $(u_\lambda^i, v_\lambda^i)$, $i \in \{1,2,3\}$) and $\|(u_\lambda^i, v_\lambda^i)\|_{1,p,q,G} < \sigma$. In particular, the functions u_λ^i, v_λ^i are axially symmetric, and $\|u_\lambda^i\|_{1,p} < \sigma$, $\|v_\lambda^i\|_{1,q} < \sigma$, $i \in \{1,2,3\}$.

Since F is axially symmetric in the first variable, thanks to (2.3), the function $\mathcal{H}(\cdot,\cdot,\lambda)$ is G-invariant, i.e.

$$\mathcal{H}(g(u,v),\lambda) = \mathcal{H}(gu, gv, \lambda) = \mathcal{H}(u,v,\lambda)$$

for every $g \in G$, $(u,v) \in W^{p,q}$. Taking into account (2.4), i.e. $\mathrm{Fix}_G W^{p,q} = W_G^{p,q}$, we can apply the principle of symmetric criticality of Palais [**17**, Theorem 5.4], obtaining that $(u_\lambda^i, v_\lambda^i)$, $i \in \{1,2,3\}$, are also critical points of $\mathcal{H}(\cdot,\cdot,\lambda)$, hence, weak solutions of (S_λ). Since one of them may be the trivial one, as we pointed out in Remark 2.1, we will have at least two distinct, non-trivial solutions of (S_λ). This completely concludes the proof. $\square$

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