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An eigenvalue problem for hemivariational inequalities with combined nonlinearities on an infinite strip☆

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Abstract

In this paper a class of eigenvalue problems for hemivariational inequalities is studied which is defined on domains of the type $\omega \times \mathbb{R}$ (ω is a bounded open subset of $\mathbb{R}^m$, $m \geqslant 1$) and it involves concave–convex nonlinearities. Under suitable conditions on the nonlinearities, two nontrivial solutions are obtained which belong to a special closed convex cone of $H_0^1(\omega \times \mathbb{R})$ whenever the eigenvalues are of certain range. Our approach is variational, the main tool in our investigation is the critical point theory developed by Motreanu and Panagiotopoulos [Minimax Theorems and Qualitative Properties of the Solutions of Hemivariational Inequalities, Kluwer Academic Publishers, Dordrecht, 1999, Chapter 3].

Keywords: Hemivariational inequalities; Infinite strips; Critical points; Eigenvalue; Palais–Smale condition; Motreanu–Panagiotopoulos type functional

1. Introduction

Elliptic equations defined on domains of the form $\Omega = \omega \times \mathbb{R}^{N-m}$ (where ω is a bounded open set in $\mathbb{R}^m$ with smooth boundary, $m \geqslant 1$ and $N - m \geqslant 1$) have been studied by

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many authors, motivated by various problems from mathematical physics (as the Klein-Gordon or Schrödinger equations); without seeking completeness, we refer the readers to [5,7,9,10,18,20]. In these papers the approach is variational; an appropriate (smooth) energy functional on $H_0^1(\Omega)$ is defined whose critical points are (weak) solutions of the investigated problem. Since Ω is unbounded, the use of standard variational techniques become delicate, due to the lack of compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega),\ p \in [2, 2^*)$. As usually, 2^* is the Sobolev critical exponent, i.e. $2^* = 2N/(N-2)$ when $N \geqslant 3$, and $2^* = +\infty$ when $N = 2$.

First of all, we point out that there is a fundamental difference between problems where $N-m \geqslant 2$ and $N-m=1$, respectively. Indeed, the subspace of axially symmetric functions of $H_0^1(\Omega)$, denoted further by $H_{0,s}^1(\Omega)$, regains the compactness of the embedding into $L^p(\Omega)$ as soon as $N-m \geqslant 2$ and $p \in (2, 2^*)$, see [8, Theorem 1]. Since $H_{0,s}^1(\Omega)$ is exactly the fixed point set of $H_0^1(\Omega)$ under the group action $id^m \times O(N-m)$, it is enough to find critical points for the functional which is restricted to $H_{0,s}^1(\Omega)$. Due to the principle of symmetric criticality of Palais, these points will be critical points also for the initial functional, thus solutions for the studied problem. Such points can be obtained by various techniques (for example by minimization [7], minimax arguments [9,20]). We notice that one can also meet some concrete problems which involve discontinuous nonlinearities on this type of domains; in such cases a non-smooth approach is used, see [12,13].

On the other hand, when $N-m=1$, i.e. $\Omega = \omega \times \mathbb{R}$ (with ω as above), the situation changes radically. In spite of that one encounters important problems in this case (see for instance [2,4]), only a few existence results are known, see [3,7]. The difficulty lies in the fact that $H_{0,s}^1(\omega \times \mathbb{R})$ cannot be embedded compactly into $L^p(\omega \times \mathbb{R})$ for any $p \in [2, 2^*)$, thus the above described machinery does not work any more. However, Lions [14, Théorème III.2] (see also [8]) observed that defining the closed convex cone

$$\begin{aligned}\mathscr{K} = \{u \in H_0^1(\omega \times \mathbb{R}) : \ & u \text{ is nonnegative,} \\ & y \mapsto u(x, y) \text{ is nonincreasing for } x \in \omega,\ y \geqslant 0, \text{ and} \\ & y \mapsto u(x, y) \text{ is nondecreasing for } x \in \omega,\ y \leqslant 0\},\end{aligned} \tag{K}$$

the bounded subsets of $\mathscr{K}$ are relatively compact in $L^p(\omega \times \mathbb{R})$ whenever $p \in (2, 2^*)$. Burton [5] was the first who exploited in its entirety the above 'compactness'; namely, by means of a version of the Mountain Pass theorem (due to Hofer [11] for an order-preserving operator on Hilbert spaces), he was able to establish the existence of a nontrivial solution for an elliptic equation on domains of the type $\omega \times \mathbb{R}$. The main ingredient in his proof was the symmetric decreasing rearrangement of the suitable functions, proving that the cone $\mathscr{K}$ remains invariant under a carefully chosen nonlinear operator, which is an indispensable hypothesis in the Hofer's result.

The main goal of this paper is to give a new approach to treat elliptic (eigenvalue) problems on domains of the type $\Omega = \omega \times \mathbb{R}$. The genesis of our method can be found in [17, Chapter 3], where Motreanu and Panagiotopoulos developed a new critical point theory for locally Lipschitz functions which are perturbed by convex, proper and lower semicontinuous functionals. Since the indicator function of a closed convex subset of a vector space possesses exactly the latter properties, this approach arises in a natural manner as it was already forecasted in [12].

In order to formulate our problem, we shall consider a Carathéodory function $F : (\omega \times \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ which is locally Lipschitz in the second variable such that

(F1) $F(x, 0) = 0$, and there exist $c_1 > 0$ and $p \in (2, 2^*)$ such that

$$|\xi| \leqslant c_1(|s| + |s|^{p-1}), \quad \forall \xi \in \partial F(x, s),\ (x, s) \in (\omega \times \mathbb{R}) \times \mathbb{R}.$$

We denoted by $\partial F(x, s)$ the generalized gradient of $F(x, \cdot)$ at the point $s \in \mathbb{R}$. Let $a \in L^1(\omega \times \mathbb{R}) \cap L^\infty(\omega \times \mathbb{R})$ with $a \geqslant 0$, $a \not\equiv 0$, and $q \in (1, 2)$. Keeping in mind the notation $\mathscr{K}$ from (K), for $\lambda > 0$, denote by (P_λ) the following *variational-hemivaritional inequality problem*:

Find $u \in \mathscr{K}$ such that

$$\int_{\omega\times\mathbb{R}} \nabla u(x)\nabla(v(x) - u(x))\mathrm{d}x + \int_{\omega\times\mathbb{R}} F^0(x, u(x); -v(x) + u(x))\mathrm{d}x$$
$$\geqslant \lambda \int_{\omega\times\mathbb{R}} a(x)|u(x)|^{q-2}u(x)(v(x) - u(x))\mathrm{d}x, \quad \forall v \in \mathscr{K}.$$

The expression $F^0(x, s; t)$ stands for the generalized directional derivative of $F(x, \cdot)$ at the point $s \in \mathbb{R}$ in the direction $t \in \mathbb{R}$.

To investigate the existence of solutions of (P_λ) we shall construct a functional $\mathscr{I}_\lambda : H_0^1(\omega \times \mathbb{R}) \to \mathbb{R}$ associated to (P_λ) which is defined by

$$\mathscr{I}_\lambda(u) = \frac{1}{2}\int_{\omega\times\mathbb{R}} |\nabla u|^2 - \int_{\omega\times\mathbb{R}} F(x, u(x))\mathrm{d}x - \frac{\lambda}{q}\int_{\omega\times\mathbb{R}} a(x)|u|^q + \psi_{\mathscr{K}}(u),$$

where $\psi_{\mathscr{K}}$ is the indicator function of the set $\mathscr{K}$. We will see that

$$\mathscr{H}_\lambda : u \mapsto \frac{1}{2}\int_{\omega\times\mathbb{R}} |\nabla u|^2 - \int_{\omega\times\mathbb{R}} F(x, u(x))\mathrm{d}x - \frac{\lambda}{q}\int_{\omega\times\mathbb{R}} a(x)|u|^q$$

is a locally Lipschitz function. Therefore, $\mathscr{I}_\lambda = \mathscr{H}_\lambda + \psi_{\mathscr{K}}$ is a Motreanu–Panagiotopoulos type functional (see [17, Chapter 3]). If $u \mapsto \mathscr{H}_\lambda(u)$ is of class $\mathscr{C}^1$ (for further comments, see Remark 2.3) then the functional $\mathscr{I}_\lambda = \mathscr{H}_\lambda + \psi_{\mathscr{K}}$ is of Szulkin type, see [19]. Proposition 4.2 asserts that the critical points of $\mathscr{I}_\lambda$ in the sense of Motreanu–Panagiotopoulos are solutions of the (P_λ). Hence, it remains to study the critical points of $\mathscr{I}_\lambda$, where we shall apply the Mountain Pass theorem adopted to this kind of functional and a local minimization argument as well, in order to obtain two nontrivial solutions of (P_λ) for certain values of $\lambda \in \mathbb{R}$. It is not clear under which conditions we are able to guarantee infinitely many solutions for (P_λ), not even in the case when $\mathscr{H}_\lambda$ is an even functional; unfortunately, the symmetric version of the Mountain Pass theorem, see [17, Corollary 3.6], cannot be applied to $\mathscr{I}_\lambda$ since $\psi_{\mathscr{K}}$ is not even. We point out that although by means of the Motreanu–Panagiotopoulos type functional several important questions have been solved (see for instance [15–17]), our result seems to give a genuinely new applicability of this critical point theory.

The paper is organized as follows. In the next section we give further hypotheses on F and we will formulate our main result. In Section 3 we recall some basic notions about the Motreanu–Panagiotopoulos type functionals; in Section 4 some auxiliary results are collected; in Section 5 we verify the Palais–Smale condition; in Section 6 the first solution of (P_λ) is constructed by means of the Mountain Pass theorem, while in the last section we prove the existence of the second solution of (P_λ) by means of the Ekeland variational principle.

Notations.

- $\Omega = \omega \times \mathbb{R}$.
- The norm of $L^\alpha(\Omega)$ will be denoted by $\|\cdot\|_\alpha$, $\alpha \geqslant 1$.
- $H_0^1(\Omega)$ is the usual Sobolev space endowed with the inner product $\langle u, v\rangle_0 = \int_\Omega \nabla u \nabla v \mathrm{d}x$ and norm $\|\cdot\|_0 = \sqrt{\langle\cdot,\cdot\rangle_0}$. Since Ω has the cone property, we have the continuous embedding $H_0^1(\Omega) \hookrightarrow L^\alpha(\Omega)$, $\alpha \in [2, 2^*]$, that is, there exists $k_\alpha > 0$ such that $\|u\|_\alpha \leqslant k_\alpha \|u\|_0$ for all $u \in H_0^1(\Omega)$.

2. Main result

Besides (F1) we make the following assumptions on the nonlinearity F:

(F2)

$$\lim_{s\to 0} \frac{\max\{|\xi| : \xi \in \partial F(x,s)\}}{s} = 0 \quad \text{uniformly for every } x \in \Omega.$$

(F3) There exists $v > 2$ such that

$$vF(x,s) + F^0(x,s;-s) \leqslant 0, \qquad \forall (x,s) \in \Omega \times \mathbb{R}.$$

(F4) There exists $R > 0$ such that

$$\inf\{F(x,s) : (x,|s|) \in \omega \times [R,\infty)\} > 0.$$

Remark 2.1. One can deduce that $0 \in \mathscr{K}$ is a solution of (P_λ) for every $\lambda \in \mathbb{R}$. Indeed, take a sequence $\{s_n\}$ which tends to 0 and $\xi_n \in \partial F(x, s_n)$. In view of (F2) one has necessarily that

$$\lim_{s\to 0} \max\{|\xi| : \xi \in \partial F(x,s)\} = 0,$$

thus, in particular, $\xi_n \to 0$ as $n \to +\infty$. Due to the (weakly*-) closedness of the set-valued map $\partial F(x,\cdot)$, see [6, Proposition 2.1.5(b), p. 29], we have $0 \in \partial F(x, 0)$, i.e. $F^0(x, 0, w) \geqslant 0$ for every $(x, w) \in \Omega \times \mathbb{R}$.

In order to obtain nontrivial solutions of (P_λ), we shall prove the following theorem which constitutes the main result of this paper.

Theorem 2.1. *Let $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function which satisfies* (F1)–(F4). *Then there exists $\lambda_0 > 0$ such that* (P_λ) *has at least two nontrivial, distinct solutions $u_\lambda^1, u_\lambda^2 \in \mathscr{K}$ whenever $\lambda \in (0, \lambda_0)$.*

Next we make some further remarks about the hypotheses we considered.

Remark 2.2. (P_λ) is a problem which involves concave–convex nonlinearities. Indeed, the number q is supposed to be in the interval $(1, 2)$ while in Lemma 4.2 we shall prove that $s \mapsto F(x, s)$ has a superquadratic growth at infinity, due to (F1)–(F4).

Remark 2.3. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a measurable (not necessarily continuous) function and suppose that there exists $c > 0$ such that for some $p \in (2, 2^*)$ one has $|f(x, s)| \leqslant c(|s| + |s|^{p-1})$ for every $(x, s) \in \mathbb{R} \times \Omega$. Define $F : \Omega \times \mathbb{R} \to \mathbb{R}$ by $F(x, s) = \int_0^s f(x, t)\mathrm{d}t$. Then F is a Carathéodory function which is locally Lipschitz in the second variable which satisfies the growth condition from (F1). Indeed, since $f(x, \cdot) \in L^\infty_{loc}(\mathbb{R})$, by [17, Proposition 1.7] we have $\partial F(x, s) = [\underline{f}(x, s), \overline{f}(x, s)]$ for every $(x, s) \in \Omega \times \mathbb{R}$ where

$$\underline{f}(x, s) = \lim_{\delta \to 0^+} \operatorname*{essinf}_{|t-s|<\delta} f(x, t) \quad \text{and} \quad \overline{f}(x, s) = \lim_{\delta \to 0^+} \operatorname*{esssup}_{|t-s|<\delta} f(x, t).$$

If f is continuous in the second variable, then $\partial F(x, s) = \{f(x, s)\}$ for every $(x, s) \in \Omega \times \mathbb{R}$. Thus, hypothesis (F1) is the non-smooth reformulation of the classical subcritical condition while (F2) reduces to $f(x, s) = o(s)$ as $s \to 0$, uniformly for every $x \in \Omega$. Moreover, (F3) becomes the well-known Ambrosetti–Rabinowitz type inequality $\nu F(x, s) \leqslant sf(x, s)$, see [1]. In this continuous case, our problem (P_λ) can be handled by means of the Szulkin type functional, see [19].

Now, we give some examples where the hypotheses of Theorem 2.1 hold true.

Example 2.1. $F(x, s) = F(s) = |s|^p$, $p \in (2, 2^*)$. In this case, $F^0(x, s; -s) = -p|s|^p$. One can choose $\nu = p$.

Example 2.2. $F(x, s) = F(s) = -s^3/3$ if $s \leqslant 0$, and $F(x, s) = F(s) = s^3 \ln(2 + s)$ if $s \geqslant 0$. One can choose arbitrary $p \in (2, 2^*)$ and $\nu \in (2, 3]$ in (F1) and (F3), respectively.

Example 2.3. Let $a_0 = 0$, and $a_k = k^{-2}$ for $k \geqslant 1$. Let us consider $F(x, s) = F(s) = |s|^3/3 + \sum_{k=0}^n a_k(u^2 - k^2)/2$ when $|s| \in [n, n + 1)$, $n \in \mathbb{N}$. This function is locally Lipschitz; the hypotheses of Theorem 2.1 are verified as soon as $p = 3$, $\nu = 5/2$ and $N \in \{2, 3, 4, 5\}$. Notice that it is enough to verify (F3) only for positive numbers, since F is even and thus $F^0(s; -s) = F^0(-s; s)$ for every $s \in \mathbb{R}$.

3. Motreanu–Panagiotopoulos type functionals

Let $(X, \|\cdot\|)$ be a real Banach space and X^* its topological dual. A function $h : X \to \mathbb{R}$ is called *locally Lipschitz* if each point $u \in X$ possesses a neighborhood $\mathcal{N}_u$ such that $|h(u_1) - h(u_2)| \leqslant L\|u_1 - u_2\|$ for all $u_1, u_2 \in \mathcal{N}_u$, for a constant $L > 0$ depending on $\mathcal{N}_u$. The *generalized directional derivative* of h at the point $u \in X$ in the direction $z \in X$ is

$$h^0(u; z) = \limsup_{w \to u, t \to 0^+} \frac{h(w + tz) - h(w)}{t}.$$

The *generalized gradient* of h at $u \in X$ is defined by

$$\partial h(u) = \{x^* \in X^* : \langle x^*, z\rangle_X \leqslant h^0(u; z),\ \ \forall z \in X\},$$

which is a nonempty, convex and w^*-compact subset of X^*, where $\langle \cdot, \cdot \rangle_X$ is the duality pairing between X^* and X.

Let $\mathscr{I} = h + \psi$, with $h : X \to \mathbb{R}$ locally Lipschitz and $\psi : X \to (-\infty, +\infty]$ convex, proper (i.e., $\psi \not\equiv +\infty$), and lower semicontinuous. $\mathscr{I}$ is called a *Motreanu–Panagioutopoulos type* functional, see [17, Chapter 3].

Definition 3.1 (*Motreanu and Panagiotopoulos [17, Definition 3.1]*). An element $u \in X$ is said to be a *critical point of* $\mathscr{I} = h + \psi$, if

$$h^0(u; v - u) + \psi(v) - \psi(u) \geqslant 0, \forall v \in X.$$

In this case, $\mathscr{I}(u)$ is a *critical value of* $\mathscr{I}$.

Definition 3.2 (*Motreanu and Panagiotopoulos [17, Definition 3.2]*). The functional $\mathscr{I} = h + \psi$ is said to *satisfy the Palais–Smale condition at level* $c \in \mathbb{R}$ (shortly, $(PS)_c$) if every sequence $\{u_n\}$ in X satisfying $\mathscr{I}(u_n) \to c$ and

$$h^0(u_n; v - u_n) + \psi(v) - \psi(u_n) \geqslant -\varepsilon_n \|v - u_n\|, \quad \forall v \in X,$$

for a sequence $\{\varepsilon_n\}$ in $[0, \infty)$ with $\varepsilon_n \to 0$, contains a convergent subsequence. If $(PS)_c$ is verified for all $c \in \mathbb{R}$, $\mathscr{I}$ is said to *satisfy the Palais–Smale condition* (shortly, (PS)).

The following version of the Mountain Pass theorem will be used in Section 6.

Proposition 3.1 (*Motreanu and Panagiotopoulos [17, Corollary 3.2]*). *Assume that the functional $\mathscr{I} = h + \psi$ on the Banach space X is of Motreanu–Panagiotopoulos type, to satisfies the (PS), $\mathscr{I}(0) = 0$ and*

(i) *there exist constants $\alpha > 0$ and $\rho > 0$ such that $\mathscr{I}(u) \geqslant \alpha$ for all $\|u\| = \rho$,*
(ii) *there exists $e \in X$ with $\|e\| > \rho$ and $\mathscr{I}(e) \leqslant 0$.*

Then the number $c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} \mathscr{I}(\gamma(t))$, where $\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = e\}$, is a critical value of $\mathscr{I}$ with $c \geqslant \alpha$.

Below, we collected those basic properties of the generalized directional derivative and gradient which will be used through the whole paper.

Proposition 3.2 (*see Clarke [6]*). *Let $h : X \to \mathbb{R}$ be a locally Lipschitz function. Then we have*:

(i) $(-h)^0(u; z) = h^0(u; -z),\ \ \forall u, z \in X.$
(ii) $h^0(u; z) = \max\{\langle x^*, z\rangle_X : x^* \in \partial h(u)\},\ \ \forall u, z \in X.$
(iii) *Let $j : X \to \mathbb{R}$ be a continuously differentiable function. Then $\partial j(u) = \{j'(u)\}$, $j^0(u; z)$ coincides with $\langle j'(u), z\rangle_X$ and $(h + j)^0(u; z) = h^0(u; z) + \langle j'(u), z\rangle_X$ for all*

$u, z \in X$. *Moreover,* $\partial(h+j)(u) = \partial h(u) + j'(u)$, $\partial(hj)(u) \subseteq j(u)\partial h(u) + h(u)j'(u)$ *and* $\partial(\lambda h)(u) = \lambda \partial h(u)$ *for all* $u \in X$ *and* $\lambda \in \mathbb{R}$.

(iv) (Lebourg's mean value theorem) *Let* u *and* v *two points in* X. *Then there exists a point* w *in the open segment between* u *and* v, *and* $x_w^* \in \partial h(w)$ *such that*

$$h(u) - h(v) = \langle x_w^*, u - v\rangle_X.$$

(v) (Second Chain Rule) *Let* Y *be a Banach space and* $j : Y \to X$ *a continuously differentiable function. Then* $h \circ j$ *is locally Lipschitz and*

$$\partial(h \circ j)(y) \subseteq \partial h(j(y)) \circ j'(y), \quad \forall y \in Y.$$

(vi) *The function* $(u, z) \mapsto h^0(u; z)$ *is upper semicontinuous.*

4. Auxiliary results

Define $\mathscr{F} : H_0^1(\Omega) \to \mathbb{R}$ by

$$\mathscr{F}(u) = \int_\Omega F(x, u(x))\mathrm{d}x. \tag{1}$$

Proposition 4.1. *Let* $F : \Omega \times \mathbb{R} \to \mathbb{R}$ *be a locally Lipschitz function which satisfies* (F1). *Then* $\mathscr{F}$ (*from* (1)) *is well-defined and it is locally Lipschitz. Moreover,*

$$\mathscr{F}^0(u; w) \leqslant \int_\Omega F^0(x, u(x); w(x))\mathrm{d}x, \quad \forall u, w \in H_0^1(\Omega). \tag{2}$$

Since the proof of the above lemma is similar to that of [12, Lemma 4.2], we shall omit it.

By standard arguments we have that the functionals $A_1, A_2 : H_0^1(\Omega) \to \mathbb{R}$, defined by $A_1(u) = \|u\|_0^2$ and $A_2(u) = \int_\Omega a(x)|u|^q \mathrm{d}x$ are of class $\mathscr{C}^1$ with derivatives $\langle A_1'(u), v\rangle_{H_0^1(\Omega)} = 2\langle u, v\rangle_0$ and $\langle A_2'(u), v\rangle_{H_0^1(\Omega)} = q\int_\Omega a(x)|u|^{q-2}uv\mathrm{d}x$, respectively. Therefore, due to Proposition 4.1 the functional

$$\mathscr{H}_\lambda(u) = \frac{1}{2}\|u\|_0^2 - \frac{\lambda}{q}\int_\Omega a(x)|u|^q - \mathscr{F}(u)$$

on $H_0^1(\Omega)$ is locally Lipschitz. On the other hand, the indicator function of the set $\mathscr{K}$, i.e.

$$\psi_{\mathscr{K}}(u) = \begin{cases} 0 & \text{if } u \in \mathscr{K}, \\ +\infty & \text{if } u \notin \mathscr{K}, \end{cases}$$

is convex, proper, and lower semicontinuous. In conclusion, $\mathscr{I}_\lambda = \mathscr{H}_\lambda + \psi_{\mathscr{K}}$ is a Motreanu–Panagiotopoulos type functional. Moreover, one easily obtain the following:

Proposition 4.2. *Fix* $\lambda > 0$ *arbitrary. Every critical point* $u \in H_0^1(\Omega)$ *of* $\mathscr{I}_\lambda = \mathscr{H}_\lambda + \psi_{\mathscr{K}}$ *is a solution of* (P_λ).

Proof. Since $u \in H_0^1(\Omega)$ is a critical point of $\mathscr{J}_\lambda = \mathscr{H}_\lambda + \psi_{\mathscr{K}}$, one has

$$\mathscr{H}_\lambda^0(u; v-u) + \psi_{\mathscr{K}}(v) - \psi_{\mathscr{K}}(u) \geqslant 0, \quad \forall v \in H_0^1(\Omega).$$

We have immediately that u belongs to $\mathscr{K}$. Otherwise, we would have $\psi_{\mathscr{K}}(u) = +\infty$ which led us to a contradiction, letting for instance $v = 0 \in \mathscr{K}$ in the above inequality. Now, we fix $v \in \mathscr{K}$ arbitrary. By using relation (2) and the properties (i) and (iii) from Proposition 3.2, we obtain the desired inequality. □

Lemma 4.1 (Kristály [12, Lemma 4.1]). *If $F : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies* (F1) *and* (F2), *for every $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that*

(i) $|\xi| \leqslant \varepsilon |s| + c(\varepsilon)|s|^{p-1}$, $\forall \xi \in \partial F(x,s)$, $(x,s) \in \Omega \times \mathbb{R}$.
(ii) $|F(x,s)| \leqslant \varepsilon s^2 + c(\varepsilon)|s|^p$, $\forall (x,s) \in \Omega \times \mathbb{R}$.

Lemma 4.2. *If $F : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies* (F1), (F3) *and* (F4) *then there exist $c_2, c_3 > 0$ such that*

$$F(x,s) \geqslant c_2|s|^\nu - c_3 s^2, \quad \forall (x,s) \in \Omega \times \mathbb{R}.$$

Proof. First, for arbitrary fixed $(x,u) \in \Omega \times \mathbb{R}$ we consider the function $g : (0, +\infty) \to \mathbb{R}$ defined by

$$g(t) = t^{-\nu} F(x, tu).$$

Clearly, g is a locally Lipschitz function and by the properties (iii) and (v) of Proposition 3.2 we have

$$\partial g(t) \subseteq -\nu t^{-\nu-1} F(x,tu) + t^{-\nu} u \partial F(x,tu), \ \ t > 0.$$

Let $t > 1$. By Lebourg's mean value theorem, there exist $\tau = \tau(x,u) \in (1,t)$ and $w_\tau = w_\tau(x,u) \in \partial g(\tau)$ such that $g(t) - g(1) = w_\tau(t-1)$. Therefore, there exists $\xi_\tau = \xi_\tau(x,u) \in \partial F(x, \tau u)$ such that $w_\tau = -\nu\tau^{-\nu-1} F(x,\tau u) + \tau^{-\nu} u \xi_\tau$ and by Proposition 3.2(ii)

$$g(t) - g(1) \geqslant -\tau^{-\nu-1}[\nu F(x,\tau u) + F^0(x, \tau u; -\tau u)](t-1).$$

By (F3) one has $g(t) \geqslant g(1)$, i.e. $F(x,tu) \geqslant t^\nu F(x,u)$, for every $t \geqslant 1$. Let $c_R = \inf\{F(x,s) : (x, |s|) \in \omega \times [R, \infty)\}$, which is a strictly positive number, due to (F4). Combining the above facts we derive

$$F(x,s) \geqslant \frac{c_R}{R^\nu}|s|^\nu, \quad \forall (x,s) \in \Omega \times \mathbb{R} \text{ with } |s| \geqslant R. \tag{3}$$

On the other hand, by (F1) we have $|F(x,s)| \leqslant c_1(s^2 + |s|^p)$ for every $(x,s) \in \Omega \times \mathbb{R}$. In particular, we have

$$-F(x,s) \leqslant c_1(s^2 + |s|^p) \leqslant c_1(1 + R^{p-2} + R^{\nu-2})s^2 - c_1|s|^\nu$$

for every $(x,s) \in \Omega \times \mathbb{R}$ with $|s| \leqslant R$. Combining the above inequality with (3), the desired inequality yields if one chooses $c_2 = \min\{c_1, c_R/R^\nu\}$ and $c_3 = c_1(1 + R^{p-2} + R^{\nu-2})$. □

Remark 4.1. In particular, from Lemma 4.2 we observe that $2 < \nu < p$.

5. The Palais–Smale condition

Proposition 5.1. *If* $F : \Omega \times \mathbb{R} \to \mathbb{R}$ *verifies* (F1)–(F3) *then* $\mathscr{I}_\lambda = \mathscr{H}_\lambda + \psi_{\mathscr{K}}$ *satisfies* (PS) *for every* $\lambda > 0$.

Proof. Let $\lambda > 0$ and $c \in \mathbb{R}$ be some fixed numbers and let $\{u_n\}$ be a sequence from $H_0^1(\Omega)$ such that

$$\mathscr{I}_\lambda(u_n) = \mathscr{H}_\lambda(u_n) + \psi_{\mathscr{K}}(u_n) \to c, \tag{4}$$

$$\mathscr{H}_\lambda^0(u_n; v - u_n) + \psi_{\mathscr{K}}(v) - \psi_{\mathscr{K}}(u_n) \geqslant -\varepsilon_n \|v - u_n\|_0, \quad \forall v \in H_0^1(\Omega), \tag{5}$$

for a sequence $\{\varepsilon_n\}$ in $[0, \infty)$ with $\varepsilon_n \to 0$. By (4) one concludes that the sequence $\{u_n\}$ belongs entirely to $\mathscr{K}$. Setting $v = 2u_n$ in (5), we obtain

$$\mathscr{H}_\lambda^0(u_n; u_n) \geqslant -\varepsilon_n \|u_n\|_0.$$

Due to Proposition 4.1, from the above inequality we derive

$$\|u_n\|_0^2 - \lambda \int_\Omega a(x)|u_n|^q + \int_\Omega F^0(x, u_n(x); -u_n(x))\mathrm{d}x \geqslant -\varepsilon_n \|u_n\|_0. \tag{6}$$

By (4) one has for large $n \in \mathbb{N}$ that

$$c + 1 \geqslant \frac{1}{2}\|u_n\|_0^2 - \frac{\lambda}{q}\int_\Omega a(x)|u_n|^q - \int_\Omega F(x, u_n(x))\mathrm{d}x. \tag{7}$$

Multiplying (6) by v^{-1} and adding this one to (7), by Hölder's inequality we have for large $n \in \mathbb{N}$

$$\begin{aligned} c + 1 + \frac{1}{v}\|u_n\|_0 &\geqslant \left(\frac{1}{2} - \frac{1}{v}\right)\|u_n\|_0^2 - \lambda\left(\frac{1}{q} - \frac{1}{v}\right)\int_\Omega a(x)|u_n|^q \\ &\quad - \frac{1}{v}\int_\Omega [F^0(x, u_n(x); -u_n(x)) + vF(x, u_n(x))]\mathrm{d}x \\ &\overset{\text{(F3)}}{\geqslant} \left(\frac{1}{2} - \frac{1}{v}\right)\|u_n\|_0^2 - \lambda\left(\frac{1}{q} - \frac{1}{v}\right)\|a\|_{v/(v-q)}\|u_n\|_v^q \\ &\geqslant \left(\frac{1}{2} - \frac{1}{v}\right)\|u_n\|_0^2 - \lambda\left(\frac{1}{q} - \frac{1}{v}\right)\|a\|_{v/(v-q)}k_v^q\|u_n\|_0^q. \end{aligned}$$

In the above inequalities we used Remark 4.1 and the hypothesis $a \in L^1(\Omega) \cap L^\infty(\Omega)$ thus, in particular, $a \in L^{v/(v-q)}(\Omega)$. Since $q < 2 < v$, from the above estimation we derive that the sequence $\{u_n\}$ is bounded in $\mathscr{K}$. Therefore, $\{u_n\}$ is relatively compact in $L^p(\Omega)$, $p \in (2, 2^*)$. Up to a subsequence, we can suppose that

$$u_n \to u \quad \text{weakly in } H_0^1(\Omega), \tag{8}$$

$$u_n \to u \quad \text{strongly in } L^\mu(\Omega),\ \mu \in (2, 2^*). \tag{9}$$

Since $\mathscr{K}$ is (weakly) closed then $u \in \mathscr{K}$. Setting $v = u$ in (5), we have

$$\langle u_n, u - u_n \rangle_0 + \int_\Omega F^0(x, u_n(x); u_n(x) - u(x))\mathrm{d}x$$
$$-\lambda \int_\Omega a(x)|u_n|^{q-2} u_n(u - u_n) \geqslant -\varepsilon_n \|u - u_n\|_0.$$

Therefore, in view of Proposition 3.2(ii) and Lemma 4.1(i) we derive

$$\begin{aligned}\|u - u_n\|_0^2 &\leqslant \langle u, u - u_n \rangle_0 + \int_\Omega F^0(x, u_n(x); u_n(x) - u(x))\mathrm{d}x \\ &\quad - \lambda \int_\Omega a(x)|u_n|^{q-2} u_n(u - u_n) + \varepsilon_n \|u - u_n\|_0 \\ &\leqslant \langle u, u - u_n \rangle_0 + \lambda \|a\|_{v/(v-q)} \|u_n\|_v^{q-1} \|u - u_n\|_v + \varepsilon_n \|u - u_n\|_0 \\ &\quad + \int_\Omega \max\{\xi_n(x)(u_n(x) - u(x)) : \xi_n(x) \in \partial F(x, u_n(x))\}\mathrm{d}x \\ &\leqslant \langle u, u - u_n \rangle_0 + \lambda \|a\|_{v/(v-q)} \|u_n\|_v^{q-1} \|u - u_n\|_v + \varepsilon_n \|u - u_n\|_0 \\ &\quad + \varepsilon \|u_n\|_0 \|u_n - u\|_0 + c(\varepsilon) \|u_n\|_p^{p-1} \|u_n - u\|_p,\end{aligned}$$

where $\varepsilon > 0$ is arbitrary small. Taking into account relations (8) and (9), the facts that $v, p \in (2, 2^*)$, the arbitrariness of $\varepsilon > 0$ and $\varepsilon_n \to 0^+$, one has that $\{u_n\}$ converges strongly to u in $H_0^1(\Omega)$. This completes the proof. $\square$

6. Mountain Pass geometry of $\mathscr{I}_\lambda = \mathscr{H}_\lambda + \psi_{\mathscr{K}}$; the first solution of (P_λ)

Proposition 6.1. *If $F : \Omega \times \mathbb{R} \to \mathbb{R}$ verifies* (F1)–(F4) *then there exists a $\lambda_0 > 0$ such that for every $\lambda \in (0, \lambda_0)$ the following assertions are true*:

(i) *there exist constants $\alpha_\lambda > 0$ and $\rho_\lambda > 0$ such that $\mathscr{I}_\lambda(u) \geqslant \alpha_\lambda$ for all $\|u\|_0 = \rho_\lambda$,*
(ii) *there exists $e_\lambda \in H_0^1(\Omega)$ with $\|e_\lambda\|_0 > \rho_\lambda$ and $\mathscr{I}_\lambda(e_\lambda) \leqslant 0$.*

Proof. (i) Due to Lemma 4.1(ii), for every $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that $\mathscr{F}(u) \leqslant \varepsilon \|u\|_0^2 + c(\varepsilon)\|u\|_p^p$ for every $u \in H_0^1(\Omega)$. It suffices to restrict our attention to elements u which belong to $\mathscr{K}$; otherwise $\mathscr{I}_\lambda(u)$ will be $+\infty$, i.e. (i) holds trivially. Fix $\varepsilon_0 \in (0, \frac{1}{2})$. One has

$$\begin{aligned}\mathscr{I}_\lambda(u) &\geqslant \left(\frac{1}{2} - \varepsilon_0\right) \|u\|_0^2 - k_p^p c(\varepsilon_0) \|u\|_0^p - \frac{\lambda k_p^q}{q} \|a\|_{p/(p-q)} \|u\|_0^q \\ &= (A - B\|u\|_0^{p-2} - \lambda C \|u\|_0^{q-2}) \|u\|_0^2,\end{aligned} \tag{10}$$

where $A = (\frac{1}{2} - \varepsilon_0) > 0$, $B = k_p^p c(\varepsilon_0) > 0$ and $C = k_p^q \|a\|_{p/(p-q)}/q > 0$.

For every $\lambda > 0$, let us define a function $g_\lambda : (0, \infty) \to \mathbb{R}$ by

$$g_\lambda(t) = A - Bt^{p-2} - \lambda C t^{q-2}.$$

Clearly, $g'_\lambda(t_\lambda)=0$ if and only if $t_\lambda=(\lambda\frac{2-q}{p-2}\frac{C}{B})^{1/(p-q)}$. Moreover, $g_\lambda(t_\lambda)=A-D\lambda^{(p-2)/(p-q)}$, where $D=D(p,q,B,C)>0$. Choosing $\lambda_0>0$ such that $g_{\lambda_0}(t_{\lambda_0})>0$, one clearly has for every $\lambda\in(0,\lambda_0)$ that $g_\lambda(t_\lambda)>0$. Therefore, for every $\lambda\in(0,\lambda_0)$, setting $\rho_\lambda=t_\lambda$ and $\alpha_\lambda=g_\lambda(t_\lambda)t_\lambda^2$, the assertion from (i) holds true.

(ii) By Lemma 4.2 we have $\mathscr{F}(u)\geqslant c_2\|u\|_\nu^\nu-c_3\|u\|_2^2$ for every $u\in H_0^1(\Omega)$. Let us fix $u\in\mathscr{K}$. Then we have

$$\mathscr{I}_\lambda(u)\leqslant\left(\frac{1}{2}+c_3k_2^2\right)\|u\|_0^2-c_2\|u\|_\nu^\nu+\frac{\lambda}{q}\|a\|_{\nu/(\nu-q)}k_\nu^q\|u\|_0^q. \tag{11}$$

Fix arbitrary $u_0\in\mathscr{K}\setminus\{0\}$. Letting $u=su_0$ $(s>0)$ in (11), we have that $\mathscr{I}_\lambda(su_0)\to-\infty$ as $s\to+\infty$, since $\nu>2>q$. Thus, for every $\lambda\in(0,\lambda_0)$, it is possible to set $s=s_\lambda$ so large that for $e_\lambda=s_\lambda u_0$, we have $\|e_\lambda\|_0>\rho_\lambda$ and $\mathscr{I}_\lambda(e_\lambda)\leqslant0$. This ends the proof of the proposition. □

We now turn to establish the existence of the first nontrivial solution of (P_λ). By Proposition 5.1, the functional $\mathscr{I}_\lambda$ satisfies (*PS*) and clearly $\mathscr{I}_\lambda(0)=0$ for every $\lambda>0$. Let us fix $\lambda\in(0,\lambda_0)$. By Proposition 6.1 it follows that there are constants $\alpha_\lambda,\rho_\lambda>0$ and $e_\lambda\in H_0^1(\Omega)$ such that $\mathscr{I}_\lambda$ fulfills the properties (i) and (ii) from Proposition 3.1. Therefore, the number $c_\lambda^1=\inf_{\gamma\in\Gamma}\sup_{t\in[0,1]}\mathscr{I}_\lambda(\gamma(t))$, where $\Gamma=\{\gamma\in C([0,1],H_0^1(\Omega)):\gamma(0)=0,\gamma(1)=e_\lambda\}$, is a critical value of $\mathscr{I}_\lambda$ with $c_\lambda^1\geqslant\alpha_\lambda>0$. It is clear that the critical point $u_\lambda^1\in H_0^1(\Omega)$ which corresponds to c_λ^1 cannot be trivial since $\mathscr{I}_\lambda(u_\lambda^1)=c_\lambda^1>0=\mathscr{I}_\lambda(0)$. It remains to apply Proposition 4.2 which concludes that u_λ^1 is actually an element of $\mathscr{K}$ and it is a solution of (P_λ).

7. Local minimization; the second solution of (P_λ)

Let us fix $\lambda\in(0,\lambda_0)$ arbitrary, λ_0 being from the previous section. By Proposition 6.1, there exists $\rho_\lambda>0$ such that

$$\inf_{\|u\|_0=\rho_\lambda}\mathscr{I}_\lambda(u)>0. \tag{12}$$

On the other hand, since $a\geqslant0$, $a\not\equiv0$, there exists $u_0\in\mathscr{K}$ such that $\int_\Omega a(x)|u_0(x)|^q\,\mathrm{d}x>0$. Thus, for $t>0$ small one has

$$\mathscr{I}_\lambda(tu_0)\leqslant t^2\left(\frac{1}{2}+c_3k_2^2\right)\|u_0\|_0^2-c_2t^\nu\|u_0\|_\nu^\nu-\frac{\lambda}{q}t^q\int_\Omega a(x)|u_0(x)|^q\,\mathrm{d}x<0.$$

For $r>0$, let us denote by $B_r=\{u\in H_0^1(\Omega):\|u\|_0\leqslant r\}$ and $S_r=\{u\in H_0^1(\Omega):\|u\|_0=r\}$. With these notations, relation (12) and the above inequality can be summarized as

$$c_\lambda^2=\inf_{u\in B_{\rho_\lambda}}\mathscr{I}_\lambda(u)<0<\inf_{u\in S_{\rho_\lambda}}\mathscr{I}_\lambda(u). \tag{13}$$

We point out that c_λ^2 is finite, due to (10). Moreover, we will show that c_λ^2 is another critical point of $\mathscr{I}_\lambda$. To this end, let $n \in \mathbb{N}\setminus\{0\}$ such that

$$\frac{1}{n} < \inf_{u\in S_{\rho_\lambda}} \mathscr{I}_\lambda(u) - \inf_{u\in B_{\rho_\lambda}} \mathscr{I}_\lambda(u). \tag{14}$$

By Ekeland's variational principle, applied to the lower semicontinuous functional $\mathscr{I}_{\lambda|_{B_{\rho_\lambda}}}$, which is bounded below (see (13)), there is $u_{\lambda,n} \in B_{\rho_\lambda}$ such that

$$\mathscr{I}_\lambda(u_{\lambda,n}) \leqslant \inf_{u\in B_{\rho_\lambda}} \mathscr{I}_\lambda(u) + \frac{1}{n}, \tag{15}$$

$$\mathscr{I}_\lambda(w) \geqslant \mathscr{I}_\lambda(u_{\lambda,n}) - \frac{1}{n}\|w - u_{\lambda,n}\|_0, \quad \forall w \in B_{\rho_\lambda}. \tag{16}$$

By (14) and (15) we have that $\mathscr{I}_\lambda(u_{\lambda,n}) < \inf_{u\in S_{\rho_\lambda}} \mathscr{I}_\lambda(u)$; therefore $\|u_{\lambda,n}\|_0 < \rho_\lambda$.

Fix an element $v \in H_0^1(\Omega)$. It is possible to choose $t > 0$ so small that $w = u_{\lambda,n} + t(v - u_{\lambda,n}) \in B_{\rho_\lambda}$. Putting this element into (16), using the convexity of $\psi_{\mathscr{K}}$ and dividing by $t > 0$, one concludes

$$\frac{\mathscr{H}_\lambda(u_{\lambda,n} + t(v - u_{\lambda,n})) - \mathscr{H}_\lambda(u_{\lambda,n})}{t} + \psi_{\mathscr{K}}(v) - \psi_{\mathscr{K}}(u_{\lambda,n}) \geqslant -\frac{1}{n}\|v - u_{\lambda,n}\|_0.$$

Letting $t \to 0^+$, by the definition of the generalized directional derivative, we derive

$$\mathscr{H}_\lambda^0(u_{\lambda,n}; v - u_{\lambda,n}) + \psi_{\mathscr{K}}(v) - \psi_{\mathscr{K}}(u_{\lambda,n}) \geqslant -\frac{1}{n}\|v - u_{\lambda,n}\|_0. \tag{17}$$

By (13) and (15) we obtain that

$$\mathscr{I}_\lambda(u_{\lambda,n}) = \mathscr{H}_\lambda(u_{\lambda,n}) + \psi_{\mathscr{K}}(u_{\lambda,n}) \to c_\lambda^2 \tag{18}$$

as $n \to \infty$. Since v was arbitrary fixed in (17), the sequence $\{u_{\lambda,n}\}$ fulfills (4) and (5), respectively. Hence, it is possible to prove in a similar manner as in Proposition 5.1 that $\{u_{\lambda,n}\}$ contains a convergent subsequence; denote it again by $\{u_{\lambda,n}\}$ and its limit point by u_λ^2. It is clear that u_λ^2 belongs to B_{ρ_λ}. By the lower semicontinuity of $\psi_{\mathscr{K}}$ we have $\psi_{\mathscr{K}}(u_\lambda^2) \leqslant \liminf_{n\to\infty} \psi_{\mathscr{K}}(u_{\lambda,n})$ while from Proposition 3.2(vi) one has $\limsup_{n\to\infty} \mathscr{H}_\lambda^0(u_{\lambda,n}; v - u_{\lambda,n}) \leqslant \mathscr{H}_\lambda^0(u_\lambda^2; v - u_\lambda^2)$. Combining these inequalities with (17) we have

$$\mathscr{H}_\lambda^0(u_\lambda^2; v - u_\lambda^2) + \psi_{\mathscr{K}}(v) - \psi_{\mathscr{K}}(u_\lambda^2) \geqslant 0, \quad \forall v \in H_0^1(\Omega),$$

i.e. u_λ^2 is a critical point of $\mathscr{I}_\lambda$. Moreover,

$$c_\lambda^2 \overset{(13)}{=} \inf_{u\in B_{\rho_\lambda}} \mathscr{I}_\lambda(u) \leqslant \mathscr{I}_\lambda(u_\lambda^2) \leqslant \liminf_{n\to\infty} \mathscr{I}_\lambda(u_{\lambda,n}) \overset{(18)}{=} c_\lambda^2,$$

i.e. $\mathscr{I}_\lambda(u_\lambda^2) = c_\lambda^2$. Since $c_\lambda^2 < 0$, it follows that u_λ^2 is not trivial. We apply again Proposition 4.2, concluding that u_λ^2 is a solution of (P_λ) which differs from u_λ^1. This completes the proof of Theorem 2.1.

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