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Infinitely many solutions for a differential inclusion problem in $\mathbb{R}^N$

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Abstract

In this paper we consider the differential inclusion problem

$$\begin{cases} -\triangle_p u + |u|^{p-2}u \in \alpha(x)\partial F(u(x)), & x \in \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), \end{cases} \tag{DI}$$

where $2 \leqslant N < p < +\infty$, $\alpha \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is radially symmetric, and ∂F stands for the generalized gradient of a locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$. Under suitable oscillatory assumptions on the potential F at zero or at infinity, we show the existence of infinitely many, radially symmetric solutions of (DI). No symmetry requirement on F is needed. Our approach is based on a non-smooth Ricceri-type variational principle, developed by Marano and Motreanu (J. Differential Equations 182 (2002) 108–120).

Keywords: p-Laplacian; Differential inclusion; Critical point; Locally Lipschitz function; Generalized gradient; Ricceri's variational principle

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1. Introduction and main results

Let Ω be an open domain in $\mathbb{R}^N$ and consider the problem,

$$\begin{cases} -\triangle_p u + |u|^{p-2}u = \alpha(x) f(u) & \text{in } \Omega, \\ u \in W^{1,p}(\Omega), \end{cases} \tag{P_Ω}$$

where $1 < p < +\infty$, $\triangle_p(\cdot) = \operatorname{div}(|\nabla(\cdot)|^{p-2}\nabla(\cdot))$ is the p-Laplacian, $\alpha \in L^1(\Omega)$, and $f : \mathbb{R} \to \mathbb{R}$ is a (not necessarily continuous) function. When Ω is *bounded*, (P_Ω) has been extensively studied; for a comprehensive treatment, as well as for updated list of references we refer the reader to the very recent monograph of Gasiński and Papageorgiou [8].

In the celebrated work [16], Ricceri elaborated a general variational principle (for Gâteaux differentiable functionals) which was successfully applied in the paper [17], in order to treat (P_Ω) subjected to the Neumann boundary condition whenever $p > N$. Marano and Motreanu [14] extended Ricceri's principle to a large class of non-differentiable functionals, applying their abstract result to a Neumann problem for an elliptic variational–hemivariational inequality which originates from (P_Ω). By means of [14], Candito [5] studied (P_Ω) (with Neumann boundary condition as well) when the nonlinearity f may possesses uncountable discontinuities. Through [16], Cammaroto et al. [4] treated a version of (P_Ω) subjected to Dirichlet boundary condition. The aforementioned papers [4,5,14,17] have the following common features: $p > N$; the domain Ω is bounded; and, without any symmetry requirement on the nonlinearity (f in (P_Ω)), infinitely many solutions are guaranteed for the studied problems. These results were achieved by providing the nonlinearity with a suitable *oscillatory behavior.* We point out that in their arguments, the compactness of embedding $W^{1,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ $(p > N)$ was indispensable.

A natural question arises: can we establish qualitatively similar result(s) studying (P_Ω), when Ω is allowed to be *unbounded*? The main objective of this paper, is to give an affirmative answer in the case when $\Omega = \mathbb{R}^N$, $N \geqslant 2$. Unlike to bounded domains, no compact embedding is available for $W^{1,p}(\mathbb{R}^N)$; although the embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ is continuous due to Morrey's theorem $(p > N)$, it is far to be compact. However, the subspace of radially symmetric functions of $W^{1,p}(\mathbb{R}^N)$, denoted further by $W_r^{1,p}(\mathbb{R}^N)$, can be embedded compactly into $L^\infty(\mathbb{R}^N)$ whenever $2 \leqslant N < p < +\infty$, (cf. Theorem 3.1); this result constitutes the key point of our investigations.

For $N > p = 2$ ($\triangle_p \equiv \triangle$), problem ($\mathrm{P}_{\mathbb{R}^N}$) has been extensively studied, see [1,2,18,19] (f continuous); and [9,11] (f is allowed to be discontinuous). In the case $N > p \neq 2$, important contributions to ($\mathrm{P}_{\mathbb{R}^N}$) can be found in [12].

We emphasize that in our approach, no continuity hypothesis will be required on the function f. So, ($\mathrm{P}_{\mathbb{R}^N}$) need not has a solution. To avoid this situation, we consider such functions f which are locally essentially bounded and we 'fill the discontinuity gaps' of f, replacing $f(\cdot)$ by an interval $[\underline{f}(\cdot), \overline{f}(\cdot)]$, where

$$\underline{f}(s) = \lim_{\delta \to 0^+} \operatorname{essinf}_{|t-s|<\delta} f(t) \quad \text{and} \quad \overline{f}(s) = \lim_{\delta \to 0^+} \operatorname{esssup}_{|t-s|<\delta} f(t). \tag{1}$$

In this way, instead of $(\mathrm{P}_{\mathbb{R}^N})$ we are dealing with a set-valued problem. On the other hand, it is well known that if $F(s) = \int_0^s f(t)\,dt$ with $f \in L^\infty_{\mathrm{loc}}(\mathbb{R})$, then F becomes locally Lipschitz and $\partial F(s) = [\underline{f}(s), \overline{f}(s)]$, for every $s \in \mathbb{R}$ (see [6], [15, Proposition 1.7]), where $\partial F(s)$ stands for the generalized gradient of F at $s \in \mathbb{R}$. This fact motivates the formulation of the *differential inclusion* problem:

$$\begin{cases} -\triangle_p u + |u|^{p-2}u \in \alpha(x)\partial F(u(x)), & x \in \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), \end{cases} \tag{DI}$$

where $F : \mathbb{R} \to \mathbb{R}$ is an arbitrary locally Lipschitz function. By a solution of (DI) it will be understood an element $u \in W^{1,p}(\mathbb{R}^N)$ for which there corresponds a mapping $\mathbb{R}^N \ni x \mapsto \zeta_x$ with $\zeta_x \in \partial F(u(x))$ for almost every $x \in \mathbb{R}^N$ having the property that for every $v \in W^{1,p}(\mathbb{R}^N)$, the function $x \mapsto \alpha(x)\zeta_x v(x)$ belongs to $L^1(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} \left(|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv\right) dx = \int_{\mathbb{R}^N} \alpha(x)\zeta_x v(x)\,dx. \tag{2}$$

In particular, if f is continuous and F is its primitive, as above, then $\partial F(s) = \{f(s)\}$ and the solutions of (DI) are exactly the *weak* solutions of $(\mathrm{P}_{\mathbb{R}^N})$, cf. (2); thus, the formulation of (DI) is completely justified.

For $l = 0$ or $l = +\infty$, set

$$F_l := \limsup_{|\rho| \to l} \frac{F(\rho)}{|\rho|^p}. \tag{3}$$

Problem (DI) will be studied in the following four cases:

- $0 < F_l < +\infty$, whenever $l = 0$ or $l = +\infty$ (see Theorem 2.1); and
- $F_l = +\infty$, whenever $l = 0$ or $l = +\infty$ (see Theorem 2.2).

In all the cases, under further additional assumptions, the compactness result (Theorem 3.1) makes possible the application of a particular form of [14, Theorem 1.1]. In this way, we obtain a sequence of critical points (in the sense of Chang [6]) of the energy functional associated to (DI), which is restricted to $W_r^{1,p}(\mathbb{R}^N)$. Applying then the non-smooth version of the principle of symmetric criticality (see [10]), these points will be as well critical points of the original functional, thus solutions of (DI), cf. Proposition 3.1. We emphasize that our results seem to be new even in the 'smooth' case, i.e., when we are finding solutions for $(\mathrm{P}_{\mathbb{R}^N})$ assuming the continuity of f.

The organization of the paper is as follows. In Section 2, we state the main results. In Section 3 we prove an embedding theorem, and a non-smooth Ricceri-type variational principle is recalled, specializing as well this abstract framework to our appropriate setting. In Sections 4 and 5 we prove Theorems 2.1 and 2.2, respectively, while in Section 6 we give some examples showing the applicability of our results.

Notations

- $L^p(\Omega)$ is the usual Lebesgue space, with norm $\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p\right)^{1/p}$, $1 \leqslant p < +\infty$, and $\|u\|_{L^\infty(\Omega)} = \operatorname{esssup}_{x\in\Omega}|u(x)|$. When $\Omega = \mathbb{R}^N$, we write $\|\cdot\|_{L^p}$.
- $W^{1,p}(\Omega)$ is the usual Sobolev space, endowed with the norm $\|u\|_{W^{1,p}(\Omega)} = (\|\nabla u\|^p_{L^p(\Omega)} + \|u\|^p_{L^p(\Omega)})^{1/p}$. If $\Omega = \mathbb{R}^N$, we write $\|u\|_{W^{1,p}}$.
- "$\rightharpoonup$" means weak convergence, "$\to$" strong convergence.
- $B_N(y,r)$ and $B_N[y,r]$ denote the open and closed N-dimensional balls with center $y \in \mathbb{R}^N$ and radius $r > 0$, respectively.

2. Main results

We say that a function $h : \mathbb{R}^N \to \mathbb{R}$ is *radially symmetric* if $h(gx) = h(x)$ for every $g \in O(N)$, and $x \in \mathbb{R}^N$. (Here, $O(N)$ denotes the orthogonal group of $\mathbb{R}^N$.) Throughout the paper, we assume that

(H) • $F : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz, $F(0) = 0$, and $F(s) \geqslant 0$, $\forall s \in \mathbb{R}$;
• $\alpha \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is radially symmetric, and $\alpha(x) \geqslant 0$, $\forall x \in \mathbb{R}^N$.

The main results of this paper are as follow.

Theorem 2.1 (*The case* $0 < F_l < +\infty$). *Let* $l = 0$ *or* $l = +\infty$, *and let* $2 \leqslant N < p < +\infty$. *Let* $F : \mathbb{R} \to \mathbb{R}$ *and* $\alpha : \mathbb{R}^N \to \mathbb{R}$ *be two functions which satisfy the hypotheses* (H) *and* $0 < F_l < +\infty$. *Assume that* $\|\alpha\|_{L^\infty} F_l > 2^N p^{-1}$ *and there exists a number* $\beta_l \in]2^N (pF_l)^{-1}, \|\alpha\|_{L^\infty}[$ *such that*

$$\frac{2}{(2^{-N} p \beta_l F_l - 1)^{1/p}} < \sup\{r : \operatorname{meas}(B_N(0,r) \setminus \alpha^{-1}(]\beta_l, +\infty[)) = 0\}. \tag{4}$$

Assume further that there are sequences $\{a_k\}$ *and* $\{b_k\}$ *in* $]0,+\infty[$ *with* $a_k < b_k$, $\lim_{k\to+\infty} b_k = l$, $\lim_{k\to+\infty} \frac{b_k}{a_k} = +\infty$ *such that*

$$\sup\{\operatorname{sign}(s)\xi : \xi \in \partial F(s), |s| \in]a_k, b_k[\} \leqslant 0. \tag{5}$$

Then, problem (DI) *possesses a sequence* $\{u_n\}$ *of solutions which are radially symmetric and*

$$\lim_{n\to+\infty} \|u_n\|_{W^{1,p}} = l.$$

In addition, if $F(s) = 0$ *for every* $s \in]-\infty, 0[$, *then the elements* u_n *are non-negative.*

Theorem 2.2 (*The case* $F_l = +\infty$). *Let* $l = 0$ *or* $l = +\infty$, *and let* $2 \leqslant N < p < +\infty$. *Let* $F : \mathbb{R} \to \mathbb{R}$ *and* $\alpha : \mathbb{R}^N \to \mathbb{R}$ *be two functions which satisfy* (H) *and* $F_l = +\infty$. *Assume that* $\|\alpha\|_{L^\infty} > 0$, *and there exist* $\mu > 0$ *and* $\beta_l \in]0, \|\alpha\|_{L^\infty}[$ *such that*

$$\operatorname{meas}(B_N(0,\mu) \setminus \alpha^{-1}(]\beta_l, +\infty[)) = 0 \tag{6}$$

and there are sequences $\{a_k\}$ *and* $\{b_k\}$ *in* $]0, +\infty[$ *with* $a_k < b_k$, $\lim_{k\to+\infty} b_k = l$, $\lim_{k\to+\infty} \frac{b_k}{a_k} = +\infty$ *such that*

$$\sup\{\operatorname{sign}(s)\xi : \xi \in \partial F(s), |s| \in]a_k, b_k[\} \leqslant 0$$

and

$$\limsup_{k\to+\infty} \frac{\max_{[-a_k,a_k]} F}{b_k^p} < (pc_\infty^p \|\alpha\|_{L^1})^{-1}, \tag{7}$$

where c_∞ *is the embedding constant of* $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$. *Then the conclusions of Theorem* 2.1 *hold.*

Remark 2.1. Relation (4), as well as (6), imply that α does not vanish in certain neighborhoods of the origin. By (5), one can deduce in particular that F is non-decreasing on $[-b_k, -a_k]$ and it is non-increasing on $[a_k, b_k]$. This fact, together with $F_l > 0$ gives rise to an oscillatory behavior of the potential F at zero or at infinity.

Remark 2.2. In Theorem 2.2 the embedding constant c_∞ of $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ appears explicitly. For applications, it is worth to point out an upper bound of it. After elementary estimations (see [3]), one certainly has $c_\infty \leqslant 2p(p-N)^{-1}$.

Remark 2.3. Every function $u \in W^{1,p}(\mathbb{R}^N)$ $(p > N)$ admits a continuous representation, see [3, p. 166]; in the sequel, we will replace u by this element.

3. Auxiliary results

3.1. A key embedding result

The action of the orthogonal group $O(N)$ on $W^{1,p}(\mathbb{R}^N)$ can be defined by $(gu)(x) = u(g^{-1}x)$, for every $g \in O(N)$, $u \in W^{1,p}(\mathbb{R}^N)$, $x \in \mathbb{R}^N$. It is clear that this group acts linearly and isometrically; in particular $\|gu\|_{W^{1,p}} = \|u\|_{W^{1,p}}$, for every $g \in O(N)$ and $u \in W^{1,p}(\mathbb{R}^N)$. Defining the subspace of radially symmetric functions of $W^{1,p}(\mathbb{R}^N)$ by

$$W_r^{1,p}(\mathbb{R}^N) = \{u \in W^{1,p}(\mathbb{R}^N) : gu = u \text{ for all } g \in O(N)\},$$

we can state the following crucial result.

Theorem 3.1. *The embedding* $W_r^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ *is compact whenever* $2 \leqslant N < p < +\infty$.

Proof. Let u_n be a bounded sequence in $W_r^{1,p}(\mathbb{R}^N)$. Up to a subsequence, $u_n \rightharpoonup u$ in $W_r^{1,p}(\mathbb{R}^N)$ for some $u \in W_r^{1,p}(\mathbb{R}^N)$. Let $\rho > 0$ be an arbitrarily fixed number. Due to the radially symmetric properties of u and u_n, we have

$$\|u_n - u\|_{W^{1,p}(B_N(g_1 y,\rho))} = \|u_n - u\|_{W^{1,p}(B_N(g_2 y,\rho))} \tag{8}$$

for every $g_1, g_2 \in O(N)$ and $y \in \mathbb{R}^N$. For a fixed $y \in \mathbb{R}^N$, we can define

$$m(y,\rho) = \sup\{n \in \mathbb{N} : \exists\, g_i \in O(N),\ i \in \{1,\ldots,n\} \text{ such that } B_N(g_i y,\rho) \cap B_N(g_j y,\rho) = \emptyset,\ \forall i \neq j\}.$$

By virtue of (8), for every $y \in \mathbb{R}^N$ and $n \in \mathbb{N}$, we have

$$\|u_n - u\|_{W^{1,p}(B_N(y,\rho))} \leqslant \frac{\|u_n - u\|_{W^{1,p}}}{m(y,\rho)} \leqslant \frac{\sup_{n\in\mathbb{N}} \|u_n\|_{W_{1,p}} + \|u\|_{W^{1,p}}}{m(y,\rho)}.$$

The right-hand side does not depend on n, and $m(y,\rho) \to +\infty$ whenever $|y| \to +\infty$ (ρ is kept fixed, and $N \geqslant 2$). Thus, for every $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that for every $y \in \mathbb{R}^N$ with $|y| \geqslant R_\varepsilon$ one has

$$\|u_n - u\|_{W^{1,p}(B_N(y,\rho))} < \varepsilon(2S_\rho)^{-1} \quad \text{for every } n \in \mathbb{N}, \tag{9}$$

where $S_\rho > 0$ is the embedding constant of $W^{1,p}(B_N(0,\rho)) \hookrightarrow C^0(B_N[0,\rho])$. Furthermore, we observe that the embedding constant for $W^{1,p}(B_N(y,\rho)) \hookrightarrow C^0(B_N[y,\rho])$ can be chosen S_ρ as well, *independent* of the position of the point $y \in \mathbb{R}^N$. This fact can be concluded either by a simple translation of the functions $u \in W^{1,p}(B_N(y,\rho))$ into $B_N(0,\rho)$, i.e. $\tilde{u}(\cdot) = u(\cdot - y) \in W^{1,p}(B_N(0,\rho))$ (thus $\|u\|_{W^{1,p}(B_N(y,\rho))} = \|\tilde{u}\|_{W^{1,p}(B_N(0,\rho))}$ and $\|u\|_{C^0(B_N[y,\rho])} = \|\tilde{u}\|_{C^0(B_N[0,\rho])}$); or, by the invariance with respect to rigid motions of the cone property of the balls $B_N(y,\rho)$ when ρ is kept fixed. Thus, in view of (9), one has that

$$\sup_{|y|\geqslant R_\varepsilon} \|u_n - u\|_{C^0(B_N[y,\rho])} \leqslant \varepsilon/2 \quad \text{for every } n \in \mathbb{N}. \tag{10}$$

On the other hand, since $u_n \rightharpoonup u$ in $W_r^{1,p}(\mathbb{R}^N)$, then in particular, by Rellich theorem it follows that $u_n \to u$ in $C^0(B_N[0,R_\varepsilon])$, i.e., there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\|u_n - u\|_{C^0(B_N[0,R_\varepsilon])} < \varepsilon \quad \text{for every } n \geqslant n_\varepsilon. \tag{11}$$

Combining (10) with (11), one concludes that $\|u_n - u\|_{L^\infty} < \varepsilon$ for every $n \geqslant n_\varepsilon$, i.e., $u_n \to u$ in $L^\infty(\mathbb{R}^N)$. This ends the proof. $\square$

An alternate proof of Theorem 3.1. Lions [13, Lemme II.1] provided us with a Strauss-type estimation (see [18]) for radially symmetric functions of $W^{1,p}(\mathbb{R}^N)$; namely, for every $u \in W_r^{1,p}(\mathbb{R}^N)$ we have

$$|u(x)| \leqslant p^{1/p}(\text{Area } S^{N-1})^{-1/p}\|u\|_{W^{1,p}}|x|^{(1-N)/p}, \quad x \neq 0, \tag{12}$$

where S^{N-1} is the N-dimensional unit sphere.

Now, let $\{u_n\}$ be a sequence in $W_r^{1,p}(\mathbb{R}^N)$ which converges weakly to some $u \in W_r^{1,p}(\mathbb{R}^N)$. By applying inequality (12) for $u_n - u$, and taking into account that $\|u_n - u\|_{W^{1,p}}$ is bounded, and $N \geqslant 2$, then for every $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that

$$\|u_n - u\|_{L^\infty(|x| \geqslant R_\varepsilon)} \leqslant C|R_\varepsilon|^{(1-N)/p} < \varepsilon \quad \forall n \in \mathbb{N},$$

where $C > 0$ does not depend on n. The rest is similar as above. $\square$

Remark 3.1. It is well known that the embedding $W_r^{1,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is compact whenever $2 \leqslant N \leqslant p < +\infty$ and $q \in]p, +\infty[$, see Lions [13, Théorème II. 1], but is no longer compact neither for $N = 1$ nor for $q \in \{p, +\infty\}$. However, there is no incompatibility with Theorem 3.1. Indeed, our result works only in the case when $p > N$, but it fails if $p = N$; the reason is that for bounded domains $\Omega \subset \mathbb{R}^N$ the embedding $W^{1,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$ is compact for $p > N$ (Rellich theorem), while for $p = N$ the space $W^{1,p}(\Omega)$ cannot even be embedded continuously into $L^\infty(\Omega)$.

3.2. A non-smooth variational principle of Ricceri

Let $(X, \|\cdot\|)$ be a real Banach space and X^* its topological dual. A function $h : X \to \mathbb{R}$ is called *locally Lipschitz* if each point $u \in X$ possesses a neighborhood $\mathcal{N}_u$ such that $|h(u_1) - h(u_2)| \leqslant L\|u_1 - u_2\|$ for all $u_1, u_2 \in \mathcal{N}_u$, for a constant $L > 0$ depending on $\mathcal{N}_u$. The *generalized directional derivative* of h at the point $u \in X$ in the direction $v \in X$ is

$$h^0(u; v) = \limsup_{\substack{w \to u \\ t \searrow 0}} \frac{1}{t}(h(w + tv) - h(w)).$$

The *generalized gradient* of h at $u \in X$ is defined by

$$\partial h(u) = \{x^* \in X^* : \langle x^*, z\rangle_X \leqslant h^0(u; z) \quad \text{for all} \quad z \in X\},$$

which is a non-empty, convex and w^*-compact subset of X^*, where $\langle\cdot,\cdot\rangle_X$ is the duality pairing between X^* and X. We say that $u \in X$ is a *critical point* of h, if $0 \in \partial h(u)$, see Chang [6].

We shall apply the following critical point theorem whose smooth version is due to Ricceri [16, Theorem 2.5].

Theorem 3.2 (*Marano and Motreanu [14, Theorem 1.1]*). *Let* $(X, \|\cdot\|)$ *be a reflexive real Banach space, and* $\tilde{X}$ *another real Banach spaces such that* X *is compactly embedded into* $\tilde{X}$. *Let* $\Phi : \tilde{X} \to \mathbb{R}$ *and* $\Psi : X \to \mathbb{R}$ *be two locally Lipschitz functions, such that* Ψ *is weakly sequentially lower semicontinuous and coercive. For every* $\rho > \inf_X \Psi$, *put*

$$\varphi(\rho) = \inf_{u \in \Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(u) - \inf_{v \in \overline{(\Psi^{-1}(]-\infty,\rho[))}_w} \Phi(v)}{\rho - \Psi(u)}, \tag{13}$$

where $\overline{(\Psi^{-1}(]-\infty,\rho[))}_w$ *is the closure of* $\Psi^{-1}(]-\infty,\rho[)$ *in the weak topology. Furthermore, set*

$$\gamma := \liminf_{\rho \to +\infty} \varphi(\rho), \qquad \delta := \liminf_{\rho \to (\inf_X \Psi)^+} \varphi(\rho). \tag{14}$$

Then, the following conclusions hold.

(A) *If* $\gamma < +\infty$ *then, for every* $\lambda > \gamma$, *either*
  (A1) $\Phi + \lambda\Psi$ *possesses a global minimum, or*
  (A2) *there is a sequence* $\{u_n\}$ *of critical points of* $\Phi + \lambda\Psi$ *such that* $\lim_{n\to+\infty} \Psi(u_n) = +\infty$.

(B) *If* $\delta < +\infty$ *then, for every* $\lambda > \delta$, *either*
  (B1) $\Phi + \lambda\Psi$ *possesses a local minimum, which is also a global minimum of* Ψ, *or*
  (B2) *there is a sequence* $\{u_n\}$ *of pairwise distinct critical points of* $\Phi + \lambda\Psi$, *with* $\lim_{n\to+\infty} \Psi(u_n) = \inf_X \Psi$, *weakly converging to a global minimum of* Ψ.

Remark 3.2. Note that Theorem 3.2 is a particular form of Marano and Motreanu [14, Theorem 1.1], where the authors put themselves within a very general framework, considering instead of Φ functions of the form $\Phi + \psi$, where $\psi : X \to]-\infty, +\infty]$ is convex, proper, and lower semicontinuous, i.e. functions of Motreanu–Panagiotopoulos type (see [15, Chapter 3]).

3.3. Our setting

Consider two functions, F and α, which fulfill (H). Let $\mathcal{F} : L^\infty(\mathbb{R}^N) \to \mathbb{R}$ be a function defined by

$$\mathcal{F}(u) = \int_{\mathbb{R}^N} \alpha(x) F(u(x))\, dx.$$

Since F is continuous and $\alpha \in L^1(\mathbb{R}^N)$, we easily seen that $\mathcal{F}$ is well-defined. Moreover, if we fix a $u \in L^\infty(\mathbb{R}^N)$ arbitrarily, there exists $k_u \in L^1(\mathbb{R}^N)$ such that for every $x \in \mathbb{R}^N$ and $v_i \in \mathbb{R}$ with $|v_i - u(x)| < 1$, $(i \in \{1, 2\})$ one has

$$|\alpha(x)F(v_1) - \alpha(x)F(v_2)| \leqslant k_u(x)|v_1 - v_2|.$$

Indeed, if we fix some small open intervals I_j $(j \in J)$, such that $F|_{I_j}$ is Lipschitz function (with Lipschitz constant $L_j > 0$) and $[-\|u\|_{L^\infty} - 1, \|u\|_{L^\infty} + 1] \subset \cup_{j\in J} I_j$, then we choose $k_u = \alpha \max_{j\in J} L_j$. (Here, without losing the generality, we supposed that $\operatorname{card} J < +\infty$.) Thus, we are in the position to apply Theorem 2.7.3 from [7, p. 80]; namely, $\mathcal{F}$ is a locally Lipschitz function on $L^\infty(\mathbb{R}^N)$ and for every closed subspace E of $L^\infty(\mathbb{R}^N)$ we have

$$\partial(\mathcal{F}|_E)(u) \subseteq \int_{\mathbb{R}^N} \alpha(x)\partial F(u(x))\,dx \quad \text{for every } u \in E, \tag{15}$$

where $\mathcal{F}|_E$ stands for the restriction of $\mathcal{F}$ to E. The interpretation of (15) is as follows (see also [7]): For every $\zeta \in \partial(\mathcal{F}|_E)(u)$ there corresponds a mapping $\mathbb{R}^N \ni x \mapsto \zeta_x$ such that $\zeta_x \in \partial F(u(x))$ for almost every $x \in \mathbb{R}^N$ having the property that for every $v \in E$ the function $x \mapsto \alpha(x)\zeta_x v(x)$ belongs to $L^1(\mathbb{R}^N)$ and

$$\langle \zeta, v\rangle_E = \int_{\mathbb{R}^N} \alpha(x)\zeta_x v(x)\,dx.$$

Now, let $\mathcal{E} : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ be the energy functional associated to our problem (DI), i.e., for every $u \in W^{1,p}(\mathbb{R}^N)$ set

$$\mathcal{E}(u) = \frac{1}{p}\,\|u\|^p_{W^{1,p}} - \mathcal{F}(u).$$

It is clear that $\mathcal{E}$ is locally Lipschitz on $W^{1,p}(\mathbb{R}^N)$ and we have

Proposition 3.1. *Any critical point $u \in W^{1,p}(\mathbb{R}^N)$ of $\mathcal{E}$ is a solution of* (DI).

Proof. Combining $0 \in \partial\mathcal{E}(u) = -\triangle_p u + |u|^{p-2}u - \partial(\mathcal{F}|_{W^{1,p}(\mathbb{R}^N)})(u)$ with the interpretation of (15), the desired requirement yields, see (2). □

Since α is radially symmetric, then $\mathcal{E}$ is $O(N)$-invariant, i.e. $\mathcal{E}(gu) = \mathcal{E}(u)$ for every $g \in O(N)$ and $u \in W^{1,p}(\mathbb{R}^N)$. Therefore, we are in the position to apply a non-smooth version of the *principle of symmetric criticality,* proved by Krawcewicz and Marzantowicz [10], whose form in our setting is as follows.

Proposition 3.2. *Any critical point of $\mathcal{E}_r = \mathcal{E}|_{W^{1,p}_r(\mathbb{R}^N)}$ will be also a critical point of $\mathcal{E}$.*

Remark 3.3. In view of Propositions 3.1 and 3.2, it is enough to find critical points of $\mathcal{E}_r$ in order to guarantee solutions for (DI). This fact will be carried out by means of Theorem 3.2, setting

$$X := W_r^{1,p}(\mathbb{R}^N),\ \ \tilde{X} := L^\infty(\mathbb{R}^N),\ \ \Phi := -\mathcal{F} \quad \text{and} \quad \Psi := \|\cdot\|_r^p, \tag{16}$$

where the notation $\|\cdot\|_r$ stands for the restriction of $\|\cdot\|_{W^{1,p}}$ into $W_r^{1,p}(\mathbb{R}^N)$. A few assumptions are already verified. Indeed, the embedding $X \hookrightarrow \tilde{X}$ is compact (cf. Theorem 3.1), $\Phi = -\mathcal{F}$ is locally Lipschitz, while $\Psi = \|\cdot\|_r^p$ is of class C^1 (thus, locally Lipschitz as well), coercive and weakly sequentially lower semicontinuous (see [3, Proposition III.5]). Moreover, $\mathcal{E}_r \equiv \Phi|_{W_r^{1,p}(\mathbb{R}^N)} + \frac{1}{p}\Psi$. According to (16), the function φ (defined in (13)) becomes

$$\varphi(\rho) = \inf_{\|u\|_r^p < \rho} \frac{\sup_{\|v\|_r^p \leqslant \rho} \mathcal{F}(v) - \mathcal{F}(u)}{\rho - \|u\|_r^p}, \quad \rho > 0. \tag{17}$$

The investigation of the numbers γ and δ (defined in (14)), as well as the cases (A) and (B) from Theorem 3.2 constitute the objective of the next two sections.

4. Proof of Theorem 2.1 (The case $0 < F_l < +\infty$)

First, we deal with the case when $l = +\infty$. For simplicity, we write F_∞ and β_∞ instead of $F_{+\infty}$ and $\beta_{+\infty}$, respectively.

4.1. The case $0 < F_\infty < +\infty$.

Since $\lim_{k\to+\infty} b_k = +\infty$, instead of the sequence $\{b_k\}$, we may consider a non-decreasing subsequence of it, denoted again by $\{b_k\}$. Fix an $s \in \mathbb{R}$ such that $|s| \in]a_k, b_k]$. By using Lebourg's mean value theorem (see [7, Theorem 2.3.7]), there exists $\theta \in]0,1[$ and $\xi_\theta \in \partial F(\theta s + (1-\theta)\mathrm{sign}(s)a_k)$ such that

$$\begin{aligned} F(s) - F(\mathrm{sign}(s)a_k) &= \xi_\theta(s - \mathrm{sign}(s)a_k) = \mathrm{sign}(s)\xi_\theta(|s| - a_k) \\ &= \mathrm{sign}(\theta s + (1-\theta)\mathrm{sign}(s)a_k)\xi_\theta(|s| - a_k). \end{aligned}$$

According now to (5), we obtain that $F(s) \leqslant F(\mathrm{sign}(s)a_k)$ for every $s \in \mathbb{R}$ complying with $|s| \in]a_k, b_k]$. In particular, we are led to $\max_{[-a_k,a_k]} F = \max_{[-b_k,b_k]} F$ for every $k \in \mathbb{N}$. Therefore, one can fix a $\overline{\rho}_k \in [-a_k, a_k]$ such that

$$F(\overline{\rho}_k) = \max_{[-a_k,a_k]} F = \max_{[-b_k,b_k]} F. \tag{18}$$

Moreover, since $\{b_k\}$ is non-decreasing, the sequence $\{|\overline{\rho}_k|\}$ can be chosen non-decreasingly as well.

In view of (4) we can choose a number μ such that

$$\frac{2}{(2^{-N}p\beta_\infty F_\infty - 1)^{1/p}} < \mu < \sup\{r : \text{meas}(B_N(0,r) \setminus \alpha^{-1}(]\beta_\infty, +\infty[)) = 0\}. \tag{19}$$

In particular, one has

$$\alpha(x) > \beta_\infty \quad \text{for a.e. } x \in B_N(0,\mu). \tag{20}$$

Inspired by Cammaroto et al. [4], for every $k \in \mathbb{N}$ we define

$$u_k(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R}^N \setminus B_N(0,\mu), \\ \overline{\rho}_k & \text{if } x \in B_N\left(0, \frac{\mu}{2}\right), \\ \frac{2\overline{\rho}_k}{\mu}(\mu - |x|) & \text{if } x \in B_N(0,\mu) \setminus B_N\left(0, \frac{\mu}{2}\right). \end{cases} \tag{21}$$

It is easy to see that u_k belongs to $W^{1,p}(\mathbb{R}^N)$ and it is radially symmetric. Thus, $u_k \in W_r^{1,p}(\mathbb{R}^N)$. Let $\rho_k = (\frac{b_k}{c_\infty})^p$, where c_∞ is the embedding constant of $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$.

Claim 4.1. *There exists a $k_0 \in \mathbb{N}$ such that $\|u_k\|_r^p < \rho_k$, for every $k > k_0$.*

Since $\lim_{k\to+\infty} \frac{b_k}{a_k} = +\infty$, there exists a $k_0 \in \mathbb{N}$ such that

$$\frac{b_k}{a_k} > c_\infty(\mu^N \omega_N K(p,N,\mu))^{1/p} \quad \text{for every } k > k_0, \tag{22}$$

where ω_N denotes the volume of the N-dimensional unit ball and

$$K(p,N,\mu) := \frac{2^p}{\mu^p}\left(1 - \frac{1}{2^N}\right) + 1. \tag{23}$$

Thus, for every $k > k_0$ one has

$$\begin{aligned} \|u_k\|_r^p &= \int_{\mathbb{R}^N} |\nabla u_k|^p\,dx + \int_{\mathbb{R}^N} |u_k|^p\,dx \\ &\leqslant \left(\frac{2|\overline{\rho}_k|}{\mu}\right)^p \left(\text{vol}\,B_N(0,\mu) - \text{vol}\,B_N\left(0,\frac{\mu}{2}\right)\right) + |\overline{\rho}_k|^p \text{vol}\,B_N(0,\mu) \end{aligned}$$

$$= |\overline{\rho}_k|^p \mu^N \omega_N K(p,N,\mu) \leqslant a_k^p \mu^N \omega_N K(p,N,\mu) \qquad \text{(cf. (22))}$$
$$< \left(\frac{b_k}{c_\infty}\right)^p = \rho_k,$$

which proves Claim 4.1.

Now, let φ from (17) and $\gamma = \liminf_{\rho\to+\infty} \varphi(\rho)$ defined in (14).

Claim 4.2. $\gamma = 0$.

By definition, $\gamma \geqslant 0$. Suppose that $\gamma > 0$. Since $\lim_{k\to+\infty} \frac{\rho_k}{|\overline{\rho}_k|^p} = +\infty$, there is a number $k_1 \in \mathbb{N}$ such that for every $k > k_1$ we have

$$\frac{\rho_k}{|\overline{\rho}_k|^p} > \frac{2}{\gamma}(F_\infty + 1)(\|\alpha\|_{L^1} - \beta_\infty \overline{\mu}^N \omega_N) + \mu^N \omega_N K(p,N,\mu), \tag{24}$$

where $\overline{\mu}$ is an arbitrary fixed number complying with

$$0 < \overline{\mu} < \min\left\{\left(\frac{\|\alpha\|_{L^1}}{\beta_\infty \omega_N}\right)^{1/N}, \frac{\mu}{2}\right\}. \tag{25}$$

Moreover, since $|\overline{\rho}_k| \to +\infty$ as $k \to +\infty$ (otherwise we would have $F_\infty = 0$), by the definition of F_∞, see (3), there exists a $k_2 \in \mathbb{N}$ such that

$$\frac{F(\overline{\rho}_k)}{|\overline{\rho}_k|^p} < F_\infty + 1 \quad \text{for every } k > k_2. \tag{26}$$

Now, let $v \in W_r^{1,p}(\mathbb{R}^N)$ arbitrarily fixed with $\|v\|_r^p \leqslant \rho_k$. Due to the continuous embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$, we have $\|v\|_{L^\infty}^p \leqslant c_\infty^p \rho_k = b_k^p$. Therefore, one has

$$\sup_{x\in\mathbb{R}^N} |v(x)| \leqslant b_k.$$

In view of (18), we obtain

$$F(v(x)) \leqslant \max_{[-b_k,b_k]} F = F(\overline{\rho}_k) \quad \text{for every } x \in \mathbb{R}^N. \tag{27}$$

Hence, for every $k > \max\{k_0, k_1, k_2\}$, one has

$$\sup_{\|v\|_r^p \leqslant \rho_k} \mathcal{F}(v) - \mathcal{F}(u_k)$$
$$= \sup_{\|v\|_r^p \leqslant \rho_k} \int_{\mathbb{R}^N} \alpha(x) F(v(x))\,dx - \int_{\mathbb{R}^N} \alpha(x) F(u_k(x))\,dx \qquad \text{(cf. (27), (H))}$$

$$\leqslant F(\overline{\rho}_k)\|\alpha\|_{L^1} - \int_{B_N(0,\overline{\mu})} \alpha(x)F(u_k(x))\,dx \qquad \text{(cf. (20), (21), (25))}$$

$$\leqslant F(\overline{\rho}_k)(\|\alpha\|_{L^1} - \beta_\infty \overline{\mu}^N \omega_N) \qquad \text{(cf. (25), (26))}$$

$$\leqslant (F_\infty + 1)|\overline{\rho}_k|^p(\|\alpha\|_{L^1} - \beta_\infty \overline{\mu}^N \omega_N) \qquad \text{(cf. (24))}$$

$$\leqslant \frac{\gamma}{2}(\rho_k - |\overline{\rho}_k|^p \mu^N \omega_N K(p,N,\mu))$$

$$\leqslant \frac{\gamma}{2}(\rho_k - \|u_k\|_r^p).$$

Since $\|u_k\|_r^p < \rho_k$ (cf. Claim 4.1), and $\rho_k \to +\infty$ as $k \to +\infty$, we obtain

$$\gamma = \liminf_{\rho\to+\infty} \varphi(\rho) \leqslant \liminf_{k\to+\infty} \varphi(\rho_k) \leqslant \liminf_{k\to+\infty} \frac{\sup_{\|v\|_r^p \leqslant \rho_k} \mathcal{F}(v) - \mathcal{F}(u_k)}{\rho_k - \|u_k\|_r^p} \leqslant \frac{\gamma}{2},$$

contradiction. This proves Claim 4.2.

Claim 4.3. *$\mathcal{E}_r$ is not bounded below on $W_r^{1,p}(\mathbb{R}^N)$.*

By (19), we find a number ε_∞ such that

$$0 < \varepsilon_\infty < F_\infty - \frac{2^N}{p\beta_\infty}\left(\left(\frac{2}{\mu}\right)^p + 1\right). \tag{28}$$

In particular, for every $k \in \mathbb{N}$, $\sup_{|\rho|\geqslant k} \frac{F(\rho)}{|\rho|^p} > F_\infty - \varepsilon_\infty$. Therefore, we can fix $\tilde{\rho}_k$ with $|\tilde{\rho}_k| \geqslant k$ such that

$$\frac{F(\tilde{\rho}_k)}{|\tilde{\rho}_k|^p} > F_\infty - \varepsilon_\infty. \tag{29}$$

Now, define $w_k \in W_r^{1,p}(\mathbb{R}^N)$ in the same way as u_k, see (21), replacing $\overline{\rho}_k$ by $\tilde{\rho}_k$. We obtain

$$\begin{aligned}
\mathcal{E}_r(w_k) &= \frac{1}{p}\|w_k\|_r^p - \mathcal{F}(w_k) \\
&\leqslant \frac{1}{p}|\tilde{\rho}_k|^p \mu^N \omega_N K(p,N,\mu) - \int_{B_N(0,\frac{\mu}{2})} \alpha(x)F(w_k(x))\,dx \qquad \text{(cf. (20), (29))} \\
&\leqslant \frac{1}{p}|\tilde{\rho}_k|^p \mu^N \omega_N K(p,N,\mu) - (F_\infty - \varepsilon_\infty)|\tilde{\rho}_k|^p \beta_\infty \omega_N \left(\frac{\mu}{2}\right)^N
\end{aligned}$$

$$= |\tilde{\rho}_k|^p \mu^N \omega_N \left(\frac{1}{p} K(p, N, \mu) - \frac{1}{2^N} (F_\infty - \varepsilon_\infty) \beta_\infty \right) \qquad \text{(cf. (23), (28))}$$

$$< -\frac{1}{p} |\tilde{\rho}_k|^p \omega_N \left(\frac{2}{\mu} \right)^{p-N}.$$

Since $|\tilde{\rho}_k| \to +\infty$ as $k \to +\infty$, we obtain $\lim_{k\to+\infty} \mathcal{E}_r(w_k) = -\infty$, which ends the proof of Claim 4.3.

Proof concluded ($0 < F_\infty < +\infty$). It is enough to apply Remark 3.3. Indeed, since $\gamma = 0$ (cf. Claim 4.2) and the function $\mathcal{E}_r \equiv -\mathcal{F}|_{W_r^{1,p}(\mathbb{R}^N)} + \frac{1}{p}\|\cdot\|_r^p$ is not bounded below (cf. Claim 4.3), the alternative (A1) from Theorem 3.2, applied to $\lambda = \frac{1}{p}$, is excluded. Thus, there exists a sequence $\{u_n\} \subset W_r^{1,p}(\mathbb{R}^N)$ of critical points of $\mathcal{E}_r$ with $\lim_{n\to+\infty} \|u_n\|_r = +\infty$.

Now, let us suppose that $F(s) = 0$ for every $s \in]-\infty, 0[$, and let u be a solution of (DI). Denote $S = \{x \in \mathbb{R}^N : u(x) < 0\}$, and assume that $S \neq \emptyset$. In virtue of Remark 2.3, the set S is open. Define $u_S : \mathbb{R}^N \to \mathbb{R}$ by $u_S = \min\{u, 0\}$. Applying (2) for $v := u_S \in W^{1,p}(\mathbb{R}^N)$ and taking into account that $\zeta_x \in \partial F(u(x)) = \{0\}$ for every $x \in S$, one has

$$0 = \int_{\mathbb{R}^N} (|\nabla u|^{p-2} \nabla u \nabla u_S + |u|^{p-2} u u_S)\, dx = \int_S (|\nabla u|^p + |u|^p)\, dx = \|u\|^p_{W^{1,p}(S)},$$

which contradicts the choice of the set S. This ends the proof. □

Remark 4.1. A closer inspection of the proof allows us to replace hypothesis (4) by a weaker, but a more technical condition. More specifically, it is enough to require that $p\|\alpha\|_{L^\infty} F_l > 1$, and instead of (4), put

$$\sup_M \left\{ N_{\beta_l} - \frac{1}{(1-\sigma)(p\beta_l F_l \sigma^N - 1)^{1/p}} \right\} > 0, \tag{30}$$

where

$$M = \{(\sigma, \beta_l) : \sigma \in](p\|\alpha\|_{L^\infty} F_l)^{-1/N}, 1[,\ \ \beta_l \in](pF_l\sigma^N)^{-1}, \|\alpha\|_{L^\infty}[\}$$

and

$$N_{\beta_l} = \sup\{r : \text{meas}(B_N(0,r) \setminus \alpha^{-1}(]\beta_l, +\infty[)) = 0\}.$$

Now, in the construction of the functions w_k we replace the radius $\frac{\mu}{2}$ of the ball by $\sigma\mu$, where σ is chosen according to (30).

4.2. The case $0 < F_0 < +\infty$.

The proof works similarly as in §4.1; we will show only the differences. The sequence $\{\rho_k\}$ defined as above, converges now to 0, while the same holds for $\{\overline{\rho}_k\}$. Instead of Claim 4.2 we can prove that $\delta = \liminf_{\rho\to 0^+} \varphi(\rho) = 0$. Since 0 is the unique global minimum of $\Psi = \|\cdot\|_r^p$, it would be enough to show that 0 is not a local minimum of $\mathcal{E}_r \equiv -\mathcal{F}|_{W_r^{1,p}(\mathbb{R}^N)} + \frac{1}{p}\|\cdot\|_r^p$, in order to exclude alternative (B1) from Theorem 3.2. To this end, we fix $\tilde{\rho}_k$ with $|\tilde{\rho}_k| \leqslant \frac{1}{k}$ such that

$$\frac{F(\tilde{\rho}_k)}{|\tilde{\rho}_k|^p} > F_0 - \varepsilon_0,$$

where ε_0 is fixed in a similar manner as in (28), replacing β_∞, F_∞ by β_0, F_0, respectively. If we take w_k as in §4.1, then it is clear that $\{w_k\}$ strongly converges now to 0 in $W_r^{1,p}(\mathbb{R}^N)$, while $\mathcal{E}_r(w_k) < -\frac{1}{p}|\tilde{\rho}_k|^p \omega_N (2/\mu)^{p-N} < 0 = \mathcal{E}_r(0)$. Thus, 0 is not a local minimum of $\mathcal{E}_r$. So, there exists a sequence $\{u_n\} \subset W_r^{1,p}(\mathbb{R}^N)$ of critical points of $\mathcal{E}_r$ such that $\lim_{n\to+\infty} \|u_n\|_r = 0 = \inf_{W_r^{1,p}(\mathbb{R}^N)} \Psi$. This concludes completely the proof of Theorem 2.1.

5. Proof of Theorem 2.2 (The case $F_l = +\infty$)

5.1. The case $F_\infty = +\infty$

Due to (6),

$$\alpha(x) > \beta_\infty \quad \text{for a.e. } x \in B_N(0, \mu). \tag{31}$$

Let $\overline{\rho}_k$ and ρ_k as in §4.1, as well as u_k, defined this time by means of $\mu > 0$ from (31).

Claim 5.1. *There exists a* $k_0 \in \mathbb{N}$ *such that* $\|u_k\|_r^p < \rho_k$, *for every* $k > k_0$.

The proof is similarly as in §4.1.

Claim 5.2. $\gamma < \frac{1}{p}$.

Note that $F(\overline{\rho}_k) = \max_{[-a_k, a_k]} F$, cf. (18). Since $|\overline{\rho}_k| \leqslant a_k$, then $\lim_{k\to+\infty} \frac{|\overline{\rho}_k|}{b_k} = 0$. Combining this fact with (7), and choosing $\varepsilon > 0$ sufficiently small, one has

$$\limsup_{k\to+\infty} \frac{F(\overline{\rho}_k) + |\overline{\rho}_k|^p \mu^N \omega_N p^{-1} \|\alpha\|_{L^1}^{-1} K(p, N, \mu)}{b_k^p} < ((p+\varepsilon) c_\infty^p \|\alpha\|_{L^1})^{-1},$$

where $K(p, N, \mu)$ is from (23). According to the above inequality, there exists $k_3 \in \mathbb{N}$ such that for every $k > k_3$ we readily have

$$\begin{aligned} F(\overline{\rho}_k)\|\alpha\|_{L^1} &\leqslant (p+\varepsilon)^{-1} c_\infty^{-p} b_k^p - p^{-1}|\overline{\rho}_k|^p \mu^N \omega_N K(p, N, \mu) \\ &\leqslant \frac{1}{p+\varepsilon}\left(\rho_k - \frac{p+\varepsilon}{p}\|u_k\|_r^p\right) < \frac{1}{p+\varepsilon}\left(\rho_k - \|u_k\|_r^p\right). \end{aligned}$$

Thus, for every $k > k_3$, one has

$$\sup_{\|v\|_r^p \leqslant \rho_k} \mathcal{F}(v) - \mathcal{F}(u_k) < F(\overline{\rho}_k)\|\alpha\|_{L^1} < \frac{1}{p+\varepsilon}(\rho_k - \|u_k\|_r^p).$$

Hence $\gamma \leqslant \frac{1}{p+\varepsilon} < \frac{1}{p}$, which concludes the proof of Claim 5.2.

Claim 5.3. *$\mathcal{E}_r$ is not bounded below on $W_r^{1,p}(\mathbb{R}^N)$.*

Since $F_\infty = +\infty$, for an arbitrarily large number $M > 0$, we can fix $\tilde{\rho}_k$ with $|\tilde{\rho}_k| \geqslant k$ such that

$$\frac{F(\tilde{\rho}_k)}{|\tilde{\rho}_k|^p} > M. \tag{32}$$

Define $w_k \in W_r^{1,p}(\mathbb{R}^N)$ as in (21), putting $\tilde{\rho}_k$ instead of $\overline{\rho}_k$. We obtain

$$\begin{aligned} \mathcal{E}_r(w_k) &= \frac{1}{p}\|w_k\|_r^p - \mathcal{F}(w_k) \\ &\leqslant \frac{1}{p}\mu^N \omega_N |\tilde{\rho}_k|^p K(p, N, \mu) - \int_{B_N(0,\frac{\mu}{2})} \alpha(x) F(w_k(x))\,dx \quad \text{(cf. (31), (32))} \\ &\leqslant |\tilde{\rho}_k|^p \mu^N \omega_N \left(\frac{1}{p} K(p, N, \mu) - \frac{1}{2^N} M\beta_\infty\right). \end{aligned}$$

Since $|\tilde{\rho}_k| \to +\infty$ as $k \to +\infty$, and M is large enough we obtain that $\lim_{k\to+\infty} \mathcal{E}_r(w_k) = -\infty$. The proof of Claim 5.3 is concluded.

Proof concluded. Since $\gamma < \frac{1}{p}$ (cf. Claim 5.2), we can apply Theorem 3.2 (A) for $\lambda = \frac{1}{p}$. The rest is the same as in §4.1. □

5.2. The case $F_0 = +\infty$

We follow the line of §5.1. The sequences $\{\rho_k\}$, $\{\overline{\rho}_k\}$ are defined as above; they converge to 0. Let $\mu > 0$ be as in (31), replacing β_∞ by β_0. Instead of Claim 5.2, we may prove that $\delta = \liminf_{\rho\to 0^+} \varphi(\rho) < \frac{1}{p}$. Now, we are in the position to apply Theorem 3.2 (B) with $\lambda = \frac{1}{p}$. Since $F_0 = +\infty$, for an arbitrarily large number $M > 0$, we may choose $\tilde{\rho}_k$ with $|\tilde{\rho}_k| \leqslant \frac{1}{k}$ such that

$$\frac{F(\tilde{\rho}_k)}{|\tilde{\rho}_k|^p} > M.$$

Define $w_k \in W_r^{1,p}(\mathbb{R}^N)$ by means of $\tilde{\rho}_k$ as above. It is clear that $\{w_k\}$ strongly converges to 0 in $W_r^{1,p}(\mathbb{R}^N)$ while

$$\mathcal{E}_r(w_k) \leqslant |\tilde{\rho}_k|^p \mu^N \omega_N \left(\frac{1}{p}K(p, N, \mu) - \frac{1}{2^N} M\beta_0\right) < 0 = \mathcal{E}_r(0).$$

Consequently, in spite of the fact that 0 is the unique global minimum of $\Psi = \|\cdot\|_r^p$, it is not a local minimum of $\mathcal{E}_r$; thus, (B1) can be excluded. The rest is the same as in §4.2. This completes the proof of Theorem 2.2. □

6. Examples

Throughout this section we suppose that $2 \leqslant N < p < +\infty$.

Example 6.1. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(s) = \frac{2^{N+p+3}}{p}|s|^p \max\{0, \sin \ln(\ln(|s| + 1) + 1)\},$$

and $\alpha : \mathbb{R}^N \to \mathbb{R}$ by

$$\alpha(x) = \frac{1}{(1 + |x|^N)^2}. \tag{33}$$

Then (DI) has an unbounded sequence of radially symmetric solutions.

Proof. The functions F and α clearly fulfill (H). Moreover, $F_\infty = \frac{2^{N+p+3}}{p}$. Since $\|\alpha\|_{L^\infty} = 1$, we may fix $\beta_\infty = \frac{1}{4}$ which verifies (4). For every $k \in \mathbb{N}$ let

$$a_k = e^{e^{(2k-1)\pi}-1} - 1 \quad \text{and} \quad b_k = e^{e^{2k\pi}-1} - 1.$$

If $a_k \leqslant |s| \leqslant b_k$, then $(2k-1)\pi \leqslant \ln(\ln(|s|+1)+1) \leqslant 2k\pi$, thus $F(s) = 0$ for every $s \in \mathbb{R}$ complying with $a_k \leqslant |s| \leqslant b_k$. So, $\partial F(s) = \{0\}$ for every $|s| \in]a_k, b_k[$ and (5) is verified. Thus, all the assumptions of Theorem 2.1 are satisfied. □

Example 6.2. Fix $\sigma \in \mathbb{R}$. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(s) = \begin{cases} \dfrac{8^{N+1}}{p} s^{p-\sigma} \max\left\{0, \sin \ln \ln \dfrac{1}{s}\right\}, & s \in]0, e^{-1}[, \\ 0, & s \notin]0, e^{-1}[, \end{cases}$$

and let $\alpha : \mathbb{R}^N \to \mathbb{R}$ be as in (33). Then, for every $\sigma \in [0, \min\{p-1, p(1-e^{-\pi})\}[$, (DI) admits a sequence of non-negative, radially symmetric solutions which strongly converges to 0 in $W^{1,p}(\mathbb{R}^N)$.

Proof. Since $\sigma < p-1$, (H) is verified. We distinguish two cases: $\sigma = 0$, and $\sigma \in]0, \min\{p-1, p(1-e^{-\pi})\}[$.

Case 1: $\sigma = 0$. We have $F_0 = \frac{8^{N+1}}{p}$. If we choose $\beta_0 = (1+2^N)^{-2}$, this clearly verifies (4). For every $k \in \mathbb{N}$ set

$$a_k = e^{-e^{2k\pi}} \quad \text{and} \quad b_k = e^{-e^{(2k-1)\pi}}. \tag{34}$$

For every $s \in [a_k, b_k]$, one has $(2k-1)\pi \leqslant \ln \ln \frac{1}{s} \leqslant 2k\pi$; thus $F(s) = 0$. So, $\partial F(s) = \{0\}$ for every $s \in]a_k, b_k[$ and (5) is verified. Now, we apply Theorem 2.1.

Case 2: $\sigma \in]0, \min\{p-1, p(1-e^{-\pi})\}[$. We have $F_0 = +\infty$. In order to verify (6), we fix for instance $\beta_0 = (1+2^N)^{-2}$ and $\mu = 2$. Take $\{a_k\}$ and $\{b_k\}$ in the same way as in (34). The inequality in (7) becomes obvious since

$$\limsup_{k\to+\infty} \frac{\max_{[-a_k,a_k]} F}{b_k^p} \leqslant \frac{8^{N+1}}{p} \limsup_{k\to+\infty} \frac{a_k^{p-\sigma}}{b_k^p} = \frac{8^{N+1}}{p} \lim_{k\to+\infty} e^{[p-e^{\pi}(p-\sigma)]e^{(2k-1)\pi}} = 0.$$

Therefore, we may apply Theorem 2.2. $\square$

Example 6.3. Let $\{a_k\}$ and $\{b_k\}$ be two sequences such that $a_1 = 1$, $b_1 = 2$ and $a_k = k^k$, $b_k = k^{k+1}$ for every $k \geqslant 2$. Define, for every $s \in \mathbb{R}$ the function

$$f(s) = \begin{cases} \dfrac{b_{k+1}^p - b_k^p}{a_{k+1} - b_k} & \text{if } s \in [b_k, a_{k+1}[, \\ 0 & \text{otherwise.} \end{cases}$$

Then the problem

$$\begin{cases} -\triangle_p u + |u|^{p-2}u \in \dfrac{\sigma}{(1+|x|^N)^2} [\underline{f}(u(x)), \overline{f}(u(x))], & x \in \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), \end{cases}$$

possesses an unbounded sequence of non-negative, radially symmetric solutions whenever $0 < \sigma < \frac{N}{p}\left(\frac{p-N}{2p}\right)^p (\text{Area}\, S^{N-1})^{-1}$. (The notations $\underline{f}$ and $\overline{f}$ come from (1).)

Proof. Let $F(s) = \int_0^s f(t)\,dt$. Since the function f is locally (essentially) bounded, F is locally Lipschitz. A more explicit expression of F is

$$F(s) = \begin{cases} b_k^p - b_1^p + \dfrac{b_{k+1}^p - b_k^p}{a_{k+1} - b_k}(s - b_k) & \text{if } s \in [b_k, a_{k+1}[, \\ b_k^p - b_1^p & \text{if } s \in [a_k, b_k[, \\ 0 & \text{otherwise.} \end{cases}$$

An easy calculation shows, as we expect, that $\partial F(s) = [\underline{f}(s), \overline{f}(s)]$ for every $s \in \mathbb{R}$. Taking $\alpha(x) = \frac{\sigma}{(1+|x|^N)^2}$, (H) is verified, and $\|\alpha\|_{L^1} = \frac{\sigma}{N}$ Area S^{N-1}. Moreover,

$$F_\infty = \limsup_{|s|\to+\infty} \frac{F(s)}{|s|^p} \geqslant \lim_{k\to+\infty} \frac{F(a_k)}{a_k^p} = \lim_{k\to+\infty} \frac{b_k^p - b_1^p}{a_k^p} = +\infty.$$

Choosing $\mu = 1$ and $\beta_\infty = \sigma/4$, (6) is verified, while (5) becomes trivial. Since $\max_{[-a_k,a_k]} F = F(a_k) = b_k^p - b_1^p$, relation (7) reduces to $pc_\infty^p\|\alpha\|_{L^1} < 1$ which is fulfilled due to the choice of σ and to Remark 2.2. It remains to apply Theorem 2.2. □

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