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Existence of nonzero weak solutions for a class of elliptic variational inclusions systems in $\mathbb{R}^N$

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Abstract

We consider the following variational inclusions system of the form

$$\begin{aligned} -\triangle u + u &\in \partial_1 F(u, v) \qquad \text{in } \mathbb{R}^N, \\ -\triangle v + v &\in \partial_2 F(u, v) \qquad \text{in } \mathbb{R}^N, \end{aligned}$$

with $u, v \in H^1(\mathbb{R}^N)$, where $F : \mathbb{R}^2 \to \mathbb{R}$ is a locally Lipschitz function and $\partial_i F(u, v)$ $(i \in \{1, 2\})$ are the partial generalized gradients in the sense of Clarke. Under various growth conditions on the nonlinearity F we study the existence of nonzero weak solutions of the above system (in the sense of hemivariational inequalities), which are critical points of an appropriate locally Lipschitz function defined on $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. The main tool used in the paper is the principle of symmetric criticality for locally Lipschitz functions.

Keywords: Variational inclusions system; Hemivariational inequalities; Principle of symmetric criticality; Locally Lipschitz functions; Cerami condition; Palais–Smale condition

1. Introduction

This paper is devoted to the study of existence of nonzero *weak solution*s for a class of variational inclusions systems of the form

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$$
\begin{aligned}
&-\triangle u + u \in \partial_1 F(u, v) \qquad \text{in } \mathbb{R}^N,\\
&-\triangle v + v \in \partial_2 F(u, v) \qquad \text{in } \mathbb{R}^N,\\
&\quad u, v \in H^1(\mathbb{R}^N),
\end{aligned}
\tag{S}
$$

where $N \geq 2$ and $F : \mathbb{R}^2 \to \mathbb{R}$ is a locally Lipschitz function. We denote by $\partial_1 F(u, v)$ the partial generalized gradient of $F(\cdot, v)$ at the point u, and by $\partial_2 F(u, v)$ that of $F(u, \cdot)$ at v.

In recent years, nonlinear elliptic systems have been the objective of intensive investigations by many authors, motivated by their theoretical and practical importance (see for instance [1,3, 4,8–10,16,18], and references therein).

Costa [7] studied a class of semilinear elliptic system of the form

$$
\begin{aligned}
&-\triangle u + a(x)u = F_u(x, u, v) \qquad \text{in } \mathbb{R}^N,\\
&-\triangle v + b(x)v = F_v(x, u, v) \qquad \text{in } \mathbb{R}^N,
\end{aligned}
\tag{C}
$$

where $a, b : \mathbb{R}^N \to \mathbb{R}$ are continuous, coercive functions such that $a(x) \geq a_0 > 0$ and $b(x) \geq b_0 > 0$ for all $x \in \mathbb{R}^N$ and $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$. Due to the coercivity of functions a and b, the subspace $E_{a,b}$ of $H^1(\mathbb{R}^N, \mathbb{R}^2)$ defined by

$$
E_{a,b} = \left\{(u, v) \in H^1(\mathbb{R}^N, \mathbb{R}^2) : \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2 + a(x)u^2 + b(x)v^2)\mathrm{d}x < \infty\right\}
$$

is compactly embedded in $L^2(\mathbb{R}^N, \mathbb{R}^2)$. This makes possible the application of classical minimax theorems with suitable compactness condition (Palais–Smale or Cerami), depending on the behaviour of F "at infinity" (see [7]), obtaining in this manner weak solutions of (C) in the *usual* sense.

In our situation the picture is quite different. The difficulties in treating the system (S) arise from at least two facts. Firstly, we have no such compact embedding as above, i.e. the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$ is not compact (with $s \in [2, 2^*]$, where $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = \infty$ if $N = 2$). Secondly, the lack of differentiability of the nonlinear term F causes several technical obstructions; in concrete problems may appear such a non-differentiable term.

Concerning the lack of compactness, our strategy comes from the works of Bartsch and de Figueiredo [1] and Willem [19]. In [1], the following system is considered:

$$
\begin{aligned}
&-\triangle u + u = F_v(x, u, v) \qquad \text{in } \mathbb{R}^N,\\
&-\triangle v + v = F_u(x, u, v) \qquad \text{in } \mathbb{R}^N,
\end{aligned}
\tag{BdF}
$$

where $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$ verifies some growth and symmetry conditions. The Fountain theorem and a suitable compactness condition are applied, obtaining infinitely many (radial and non-radial) solutions of (BdF). In the radial case, Bartsch and de Figueiredo used the fact that the embedding $H_r^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ is compact for $p \in]2, 2^*[$ (see [17]), where $H_r^1(\mathbb{R}^N)$ is the subspace of the radially symmetric functions in $H^1(\mathbb{R}^N)$, i.e.

$$
H_r^1(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) : gu = u \text{ for all } g \in G = O(N)\}, \tag{1}
$$

where gu means $(gu)(x) = u(g^{-1}x)$ for all $g \in O(N), u \in H^1(\mathbb{R}^N)$ and $x \in \mathbb{R}^N$. In the non-radial case, they used the ingenious construction of Bartsch and Willem [2] (see also [19]); in dimensions $N = 4$ and $N \geq 6$, a subgroup of $O(N)$ (denoted by G_N) is constructed such that

$$
H_{G_N}^1(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) : gu = u \text{ for all } g \in G_N\} \tag{2}
$$

is compactly embedded in $L^p(\mathbb{R}^N)$, $p \in]2, 2^*[$ (see [2, p. 455–457]) and 0 is the only radial function of it. We will use these embeddings in our arguments.

Because of the non-differentiability of F, it is important to find an efficient method to treat the system (S). This method relies on the theory of hemivariational inequalities (see [13, 14]), where instead of an inclusion (which involves the generalized gradient of a given locally Lipschitz function) a *hemivariational inequality* is considered. We treat (S) in the same spirit; see Definition 1.1 below.

Throughout the paper, we will make the following assumption on F.

(F^1) There exist $c_1 > 0$ and $p \in]2, 2^*[$ such that

$$|w_1| + |w_2| \leq c_1(|u| + |v| + |u|^{p-1} + |v|^{p-1}), \tag{3}$$

for all $(u, v) \in \mathbb{R}^2$, and $w_i \in \partial_i F(u, v)$, $i \in \{1, 2\}$.

Definition 1.1. If $(u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ satisfies the following *hemivariational inequalities system* (HIS)

$$\int_{\mathbb{R}^N} (\nabla u \nabla w + uw)\mathrm{d}x + \int_{\mathbb{R}^N} F_1^0(u(x), v(x); -w(x))\mathrm{d}x \geq 0 \qquad \text{for all } w \in H^1(\mathbb{R}^N),$$
$$\int_{\mathbb{R}^N} (\nabla v \nabla y + vy)\mathrm{d}x + \int_{\mathbb{R}^N} F_2^0(u(x), v(x); -y(x))\mathrm{d}x \geq 0 \qquad \text{for all } y \in H^1(\mathbb{R}^N),$$

we say that (u, v) is a *weak solution* of (S).

Here $F_1^0(u, v; w)$ denotes the partial generalized directional derivative of $F(\cdot, v)$ at the point $u \in \mathbb{R}$ in the direction $w \in \mathbb{R}$ (see Section 2). $F_2^0(u, v; w)$ is defined in a similar way.

Remark 1.1. If $F \in C^1(\mathbb{R}^2, \mathbb{R})$ then $(u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ solves (HIS) if and only if it is a weak solution (in the usual sense) of

$$-\triangle u + u = F_u(u, v) \qquad \text{in } \mathbb{R}^N,$$
$$-\triangle v + v = F_v(u, v) \qquad \text{in } \mathbb{R}^N.$$

Therefore, the notion introduced in Definition 1.1 is natural. We mention that several papers deal with the *scalar* case of the above problem; see e.g. Strauss [17] and Bartsch and Willem [2], the later one treating the non-autonomous case.

Our goal is to find weak solutions of (S) in the spirit of Definition 1.1. To do this, let us introduce a further set of assumptions.

(F^2) F is regular on $\mathbb{R}^2$ (in the sense of Clarke [6]).

(F^3)

$$\lim_{|u|+|v|\to 0} \frac{\max |\partial_i F(u, v)|}{|u| + |v|} = 0, \quad i \in \{1, 2\}.$$

(F_α^4) There exists $\alpha > 2$ such that

$$\alpha F(u, v) + F_1^0(u, v; -u) + F_2^0(u, v; -v) \leq 0$$

for all $(u, v) \in \mathbb{R}^2$.

(F_ν^4) There exist $c_2, \nu > 0$ such that

$$2F(u, v) + F_1^0(u, v; -u) + F_2^0(u, v; -v) \leq -c_2(|u|^\nu + |v|^\nu)$$

for all $(u, v) \in \mathbb{R}^2$.

(F^5) $F \geq 0$ and $F(u, v) > 0$ for all $(u, v) \neq (0, 0)$.

To present our main results, we consider the Sobolev space $H^1(\mathbb{R}^N)$ with the standard inner product $(u, v)_1 = \int_{\mathbb{R}^N} (\nabla u \nabla v + uv)\mathrm{d}x$ and the corresponding norm $\|u\|_1 = \sqrt{(u, u)_1}$. We define the function $\mathcal{J} : H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \to \mathbb{R}$ by

$$\mathcal{J}(u, v) = \frac{1}{2}\|u\|_1^2 + \frac{1}{2}\|v\|_1^2 - \int_{\mathbb{R}^N} F(u, v)\mathrm{d}x.$$

We will prove in Section 2 that (due to (F^1) and (F^2)) $\mathcal{J}$ is a locally Lipschitz function and its critical points (in the sense of Chang [5]) are weak solutions of (S). In order to obtain critical points of $\mathcal{J}$, we first consider the space $H_r^1(\mathbb{R}^N)$. Restricting $\mathcal{J}$ to $H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$, we will verify the Palais–Smale condition (see Chang [5]), respectively the Cerami condition (see Kourogenis and Papageorgiou [11]) in the cases (F_α^4), respectively (F_ν^4). Applying the Mountain Pass Theorem proved by Kourogenis and Papageorgiou [11] (involving the Cerami condition for locally Lipschitz functions), we are able to guarantee critical points of the restricted function $\mathcal{J}$. Using then the Principle of Symmetric Criticality of Palais [15] for locally Lipschitz functions, proved by Krawcewicz and Marzantowicz [12], the above points will be critical points on the whole space $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. With a minimal adjustment, using the "non-radial" construction from (2), we can obtain further critical points of $\mathcal{J}$ in certain dimensions. More precisely, our main results can be formulated as follows.

Theorem 1.1. *If $F : \mathbb{R}^2 \to \mathbb{R}$ is a locally Lipschitz function satisfying (F^1)–(F^3), (F_α^4) and (F^5), then system (S) possesses at least one nonzero weak solution. If, in addition, $N = 4$ or $N \geq 6$ and F is even, then (S) has at least two nonzero weak solutions.*

Theorem 1.2. *Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a non-negative locally Lipschitz function satisfying (F^1)–(F^3). If (F_ν^4) holds for some $\nu \in\,]\max\{2, \frac{N}{2}(p-2)\}, 2^*[$, then system (S) possesses at least one nonzero weak solution. If, in addition, $N = 4$ or $N \geq 6$ and F is even, then (S) has at least two nonzero weak solutions.*

Next we make some remarks about the hypotheses we considered.

Remark 1.2. The regularity of F in the sense of Clarke (see Section 2) is not very restrictive. Indeed, the class of the regular functions is large, containing for example the continuously differentiable, respectively the convex and locally Lipschitz functions (see [6, Proposition 2.3.6]).

Remark 1.3. It would be interesting to investigate under which conditions it can be possible to obtain infinitely many weak solutions of (S) like in [1]. Due to the lack of differentiability of $\mathcal{J}$ this problem is more delicate. Therefore, at the moment, our investigations will be restricted only to Theorems 1.1 and 1.2; a possible attempt will be considered in future.

Remark 1.4. The hypotheses (F_α^4) and (F_ν^4) are the non-smooth versions of the "superquadraticity at infinity", respectively the "nonquadraticity at infinity" of F (see [7]).

Finally, we give some examples.

Example 1.1. Let $F_1(u, v) = |u|^3 + |v|^3 + |uv|^{\frac{3}{2}}$. Then $F_1 \in C^1(\mathbb{R}^2, \mathbb{R})$ satisfies each hypothesis listed above for $N \in \{2, 3, 4, 5\}$, choosing $p = \alpha = \nu = 3$. Thus, Theorems 1.1 and 1.2 can be applied.

Example 1.2. Let $F_2(u, v) = \max\{|u|^{\frac{1}{2}}, |v|^{\frac{1}{2}}\}(|u|^{\frac{5}{2}} + |v|^{\frac{5}{2}})$. This function is a locally Lipschitz, non-differentiable, convex function which satisfies (F^1)–(F^3), (F^4_α) and (F^5) in dimensions $N \in \{2, 3, 4, 5\}$, if we choose $p = 3$ and $\alpha = \frac{5}{2}$. Thus, Theorem 1.1 can be applied. A simple calculation shows that F_2 does not verify (F^4_ν) for any $\nu > 0$. We mention that in the differentiable case, Costa [7, Ex. 5)] constructed a function (of class C^2) which is nonquadratic and it is not superquadratic at infinity, in the sense of (F^4_α), respectively (F^4_ν).

Example 1.3. If we perturb F_2 from Example 1.2 by $(u, v) \mapsto |u|^3 + |v|^3$, i.e. we define

$$F_3(u, v) = F_2(u, v) + |u|^3 + |v|^3,$$

then F_3 verifies (F^1)–(F^3) and (F^4_ν) in dimensions $N \in \{2, 3, 4, 5\}$, choosing $p = \nu = 3$ and $c_2 = 1$. Therefore, Theorem 1.2 can be applied.

The paper is organized as follows. In Section 2 the basic notions and preliminary results are collected; in Section 3 we investigate the Palais–Smale, Cerami and the geometric conditions from the Mountain Pass Theorem (see [11]) for an appropriate function, while in Section 4 we prove Theorems 1.1 and 1.2.

2. Auxiliary results

Let $(X, \|\cdot\|)$ be a real Banach space and X^* its dual. A function $h : X \to \mathbb{R}$ is called locally Lipschitz if each point $u \in X$ possesses a neighborhood U_u such that

$$|h(u_1) - h(u_2)| \leq L\|u_1 - u_2\|, \qquad \text{for all } u_1, u_2 \in U_u,$$

for a constant $L > 0$ depending on U_u. The generalized gradient of h at $u \in X$ is defined as being the subset of X^*

$$\partial h(u) = \{x^* \in X^* : \langle x^*, z\rangle_X \leq h^0(u; z) \text{ for all } z \in X\},$$

which is nonempty, convex and w^*-compact, where $\langle\cdot,\cdot\rangle_X$ is the duality pairing between X^* and X, $h^0(u; z)$ is the generalized directional derivative of h at the point $u \in X$ along the direction $z \in X$, namely

$$h^0(u; z) = \limsup_{w \to u, t \to 0^+} \frac{h(w + tz) - h(w)}{t}$$

(see [6]). Now, we list some fundamental properties of the generalized gradient and directional derivative which will be used through the paper.

Proposition 2.1 (*[6]*).

(i) $(-h)^0(u; z) = h^0(u; -z)$ *for all* $u, z \in X$.
(ii) $h^0(u; z) = \max\{\langle x^*, z\rangle_X : x^* \in \partial h(u)\}$ *for all* $u, z \in X$.
(iii) *Let* $j : X \to \mathbb{R}$ *be a continuously differentiable function. Then* $\partial j(u) = \{j'(u)\}$, $j^0(u; z)$ *coincides with* $\langle j'(u), z\rangle_X$ *and* $(h + j)^0(u; z) = h^0(u; z) + \langle j'(u), z\rangle_X$ *for all* $u, z \in X$.
(iv) *(Lebourg's mean value theorem) Let* u *and* v *be two points in* X. *Then there exists a point* w *in the open segment between* u *and* v, *and* $x^*_w \in \partial h(w)$ *such that*

$$h(u) - h(v) = \langle x^*_w, u - v\rangle_X.$$

(v) *(Second Chain Rule) Let Y be a Banach space and $j : Y \to X$ a continuously differentiable function. Then $h \circ j$ is locally Lipschitz and*

$$\partial(h \circ j)(y) \subseteq \partial h(j(y)) \circ j'(y) \qquad \text{for all } y \in Y.$$

A point $u \in X$ is a *critical point of* h if $0 \in \partial h(u)$, i.e. $h^0(u, w) \geq 0$ for all $w \in X$. In this case, $h(u)$ is a *critical value of* h. We define $\lambda_h(u) = \inf\{\|x^*\|_X : x^* \in \partial h(u)\}$. (We will use the notation $\|x^*\|_X$ instead of $\|x^*\|_{X^*}$.) Of course, this infimum is attained, since $\partial h(u)$ is w^*-compact.

The function h *satisfies the Palais–Smale condition at level* $c \in \mathbb{R}$ (denoted $(PS)_c$), if every sequence $\{x_n\} \subset X$ such that $h(x_n) \to c$ and $\lambda_h(x_n) \to 0$ contains a convergent subsequence in X (see [5]).

The function h *satisfies the Cerami condition at level* $c \in \mathbb{R}$ (denoted $(C)_c$), if every sequence $\{x_n\} \subset X$ such that $h(x_n) \to c$ and $(1 + \|x_n\|)\lambda_h(x_n) \to 0$ contains a convergent subsequence in X (see [11]).

It is clear that $(PS)_c$ implies $(C)_c$.

We say that h is *regular* at $u \in X$ (in the sense of Clarke [6]) if for all $z \in X$ the usual one-sided directional derivative

$$h'(u; z) = \lim_{t \to 0^+} \frac{h(u + tz) - h(u)}{t}$$

exists and $h'(u; z) = h^0(u; z)$. h is *regular on* X, if it is regular in every point $u \in X$.

Proposition 2.2. *Let $h : X \times X \to \mathbb{R}$ be a locally Lipschitz function which is regular at $(u, v) \in X \times X$. Then*

(i) $\partial h(u, v) \subseteq \partial_1 h(u, v) \times \partial_2 h(u, v)$, *where $\partial_1 h(u, v)$ denotes the partial generalized gradient of $h(\cdot, v)$ at the point u, and $\partial_2 h(u, v)$ that of $h(u, \cdot)$ at v.*
(ii) $h^0(u, v; w, z) \leq h_1^0(u, v; w) + h_2^0(u, v; z)$ *for all $w, z \in X$.*

Proof. For (i), see [6, Proposition 2.3.15]. Now, let us fix $w, z \in X$. From Proposition 2.1(ii) it follows that there exists $x^* \in \partial h(u, v)$ such that

$$h^0(u, v; w, z) = \langle x^*, (w, z)\rangle_{X \times X}.$$

By (i) we have $x^* = (x_1^*, x_2^*)$, where $x_i^* \in \partial_i h(u, v)$, and using the definition of the generalized gradient, we obtain $h^0(u, v; w, z) = \langle x_1^*, w\rangle_X + \langle x_2^*, z\rangle_X \leq h_1^0(u, v; w) + h_2^0(u, v; z)$. □

Let E be a closed subspace of $H^1(\mathbb{R}^N)$. The inner product and the norm on E will be denoted by $(\cdot, \cdot)_E$, respectively $\|\cdot\|_E$. The Cartesian product $E \times E$ will be also a Hilbert space which is endowed with the inner product

$$((u, v), (w, y))_{E \times E} = (u, w)_E + (v, y)_E$$

for every $u, v, w, y \in E$. The norm is $\|(u, v)\|_{E \times E} = \sqrt{((u, v), (u, v))_{E \times E}}$. The norm on $L^p(\mathbb{R}^N)$ is $\|u\|_p = (\int_{\mathbb{R}} |u|^p \mathrm{d}x)^{1/p}$. Now we prove a crucial result.

Lemma 2.1. *Suppose that $F : \mathbb{R}^2 \to \mathbb{R}$ is a locally Lipschitz function which satisfies (F^1) and (F^2). Then $\mathcal{F} : H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by*

$$\mathcal{F}(u, v) = \int_{\mathbb{R}^N} F(u, v)\mathrm{d}x$$

is locally Lipschitz. Moreover, if E is a closed subspace of $H^1(\mathbb{R}^N)$ and $\mathcal{F}_E$ denotes the restriction of $\mathcal{F}$ to $E \times E$ then

$$\mathcal{F}_E^0(u, v; w, y) \le \int_{\mathbb{R}^N} F^0(u(x), v(x); w(x), y(x))\mathrm{d}x$$

for all $u, v, w, y \in E$.

Proof. First, let $(u_1, u_2), (u_3, u_4) \in \mathbb{R}^2$ be fixed elements. Applying Lebourg's mean value theorem, we obtain a $w \in \partial F(\xi, \vartheta)$ such that

$$F(u_1, u_2) - F(u_3, u_4) = \langle w, (u_1 - u_3, u_2 - u_4)\rangle_{\mathbb{R}^2},$$

where (ξ, ϑ) is in the open line segment between (u_1, u_2) and (u_3, u_4). Using the regularity of F at (ξ, ϑ) and Proposition 2.2(i), there exist $w_i \in \partial_i F(\xi, \vartheta)$ $(i \in \{1, 2\})$, such that

$$F(u_1, u_2) - F(u_3, u_4) = w_1(u_1 - u_3) + w_2(u_2 - u_4). \tag{4}$$

From relations (3) and (4), we obtain after a straightforward computation that

$$|F(u_1, u_2) - F(u_3, u_4)| \le c_3 \sum_{i=1}^{4} (|u_i| + |u_i|^{p-1})(|u_1 - u_3| + |u_2 - u_4|), \tag{5}$$

where $c_3 > 0$ does not depend on the above points. Now, we fix $u_i \in H^1(\mathbb{R}^N)$, $(i \in \{1, \ldots, 4\})$. Using (5), Holder's inequality and the fact that the embedding $H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ is continuous (the embedding constant being $c_{1,p} > 0$), we have

$$|\mathcal{F}(u_1, u_2) - \mathcal{F}(u_3, u_4)| \le c_3 \sum_{i=1}^{4} (\|u_i\|_1 + c_{1,p}^p \|u_i\|_1^{p-1})(\|u_1 - u_3\|_1 + \|u_2 - u_4\|_1).$$

From this relation it follows that $\mathcal{F}$ is locally Lipschitz on $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.

Now, we fix $u, v, w, y \in E$. Since F is continuous, $F^0(u(x), v(x); w(x), y(x))$ can be expressed as the upper limit of

$$\frac{F(z^1 + tw(x), z^2 + ty(x)) - F(z^1, z^2)}{t},$$

where $t \to 0^+$ taking rational values and $(z^1, z^2) \to (u(x), v(x))$ taking values in a countable dense subset of $\mathbb{R}^2$. Being the upper limit of measurable functions of $x \in \mathbb{R}^N$, the function $x \mapsto F^0(u(x), v(x); w(x), y(x))$ is also measurable and it is from $L^1(\mathbb{R}^N)$ (due to (F^1)).

E being a closed subspace of a separable Hilbert space, there exist functions $z_n^1, z_n^2 \in E$ and numbers $t_n \to 0^+$ such that $(z_n^1, z_n^2) \to (u, v)$ in $E \times E$ and

$$\mathcal{F}_E^0(u, v; w, y) = \lim_{n\to\infty} \frac{\mathcal{F}_E(z_n^1 + t_n w, z_n^2 + t_n y) - \mathcal{F}_E(z_n^1, z_n^2)}{t_n}$$

and without loss of generality, we may assume $z_n^1(x) \to u(x)$ and $z_n^2(x) \to v(x)$ a.e. $x \in \mathbb{R}^N$, as $n \to \infty$.

We define $g_n : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ by

$$g_n(x) = -\frac{F(z_n^1(x) + t_n w(x), z_n^2(x) + t_n y(x)) - F(z_n^1(x), z_n^2(x))}{t_n} + c_3(|w(x)| + |y(x)|)$$

$$\times \Big[|z_n^1(x)| + |z_n^2(x)| + |z_n^1(x) + t_n w(x)| + |z_n^2(x) + t_n y(x)| + |z_n^1(x)|^{p-1} + |z_n^2(x)|^{p-1} + |z_n^1(x) + t_n w(x)|^{p-1} + |z_n^2(x) + t_n y(x)|^{p-1}\Big].$$

The function g_n is measurable, and due to (5) it is non-negative. From Fatou's lemma we have

$$A = \int_{\mathbb{R}^N} \limsup_{n\to\infty} [-g_n(x)]\mathrm{d}x \geq \limsup_{n\to\infty} \int_{\mathbb{R}^N} [-g_n(x)]\mathrm{d}x = B.$$

Let $g_n = -C_n + D_n$, where

$$C_n(x) = \frac{F(z_n^1(x) + t_n w(x), z_n^2(x) + t_n y(x)) - F(z_n^1(x), z_n^2(x))}{t_n}$$

and

$$D_n(x) = c_3(|w(x)| + |y(x)|)\Big[|z_n^1(x)| + |z_n^2(x)| + |z_n^1(x) + t_n w(x)| + |z_n^2(x) + t_n y(x)| + |z_n^1(x)|^{p-1} + |z_n^2(x)|^{p-1} + |z_n^1(x) + t_n w(x)|^{p-1} + |z_n^2(x) + t_n y(x)|^{p-1}\Big].$$

Let $d_n = \int_{\mathbb{R}_N} D_n \mathrm{d}x$. Then

$$B = \limsup_{n\to\infty} \left(\int_{\mathbb{R}^N} C_n \mathrm{d}x - d_n \right). \tag{6}$$

First, we obtain the following estimation:

$$\begin{aligned}
&\left| d_n - 2c_3 \int_{\mathbb{R}^N} (|w| + |y|) \left(|u| + |v| + |u|^{p-1} + |v|^{p-1} \right) \mathrm{d}x \right| \\
&\quad \leq c_3 \Big\{ (\|w\|_E + \|y\|_E)(2\|z_n^1 - u\|_E + t_n\|w\|_E + 2\|z_n^2 - v\|_E + t_n\|y\|_E) \\
&\qquad + (p-1)2^{p-2}(\|w\|_p + \|y\|_p) \Big[\|z_n^1 - u\|_p (\|z_n^1\|_p^{p-2} + \|u\|_p^{p-2}) + (\|z_n^1 - u\|_p \\
&\qquad + t_n\|w\|_p)((\|z_n^1\|_p + t_n\|w\|_p)^{p-2} + \|u\|_p^{p-2}) + \|z_n^2 - v\|_p (\|z_n^2\|_p^{p-2} + \|v\|_p^{p-2}) \\
&\qquad + (\|z_n^2 - v\|_p + t_n\|y\|_p) \left((\|z_n^2\|_p + t_n\|y\|_p)^{p-2} + \|v\|_p^{p-2} \right) \Big] \Big\}.
\end{aligned}$$

Since the embedding $E \subseteq H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ is continuous and $\|z_n^1 - u\|_E \to 0$, $\|z_n^2 - v\|_E \to 0$ and $t_n \to 0^+$, we obtain that the sequence $\{d_n\}$ is convergent, its limit being

$$\lim_{n\to\infty} d_n = 2c_3 \int_{\mathbb{R}^N} (|w| + |y|) \left(|u| + |v| + |u|^{p-1} + |v|^{p-1} \right) \mathrm{d}x.$$

From (6), we obtain

$$\begin{aligned}
B &= \limsup_{n\to\infty} \frac{\mathcal{F}_E(z_n^1 + t_n w, z_n^2 + t_n y) - \mathcal{F}_E(z_n^1, z_n^2)}{t_n} - \lim_{n\to\infty} d_n \\
&= \mathcal{F}_E^0(u, v; w, y) - 2c_3 \int_{\mathbb{R}^N} (|w| + |y|) \left(|u| + |v| + |u|^{p-1} + |v|^{p-1} \right) \mathrm{d}x.
\end{aligned}$$

On the other hand, $A \leq A_1 - A_2$, where

$$A_1 = \int_{\mathbb{R}^N} \limsup_{n\to\infty} C_n(x)\mathrm{d}x \quad \text{and} \quad A_2 = \int_{\mathbb{R}^N} \liminf_{n\to\infty} D_n(x)\mathrm{d}x.$$

Since $z_n^1(x) \to u(x), z_n^2(x) \to v(x)$ a.e. $x \in \mathbb{R}^N$ and $t_n \to 0^+$ we have

$$A_2 = 2c_3 \int_{\mathbb{R}^N} (|w| + |y|) \left(|u| + |v| + |u|^{p-1} + |v|^{p-1}\right) \mathrm{d}x$$

and

$$\begin{aligned} A_1 &= \int_{R^N} \limsup_{n\to\infty} \frac{F(z_n^1(x) + t_n w(x), z_n^2(x) + t_n y(x)) - F(z_n^1(x), z_n^2(x))}{t_n} \mathrm{d}x \\ &\leq \int_{R^N} \limsup_{\substack{(z^1,z^2)\to(u(x),v(x)) \\ t\to 0^+}} \frac{F(z^1 + tw(x), z^2 + ty(x)) - F(z^1, z^2)}{t} \mathrm{d}x \\ &= \int_{\mathbb{R}^N} F^0(u(x), v(x); w(x), y(x)) \mathrm{d}x. \end{aligned}$$

This completes the proof. □

Remark 2.1. If we suppose in addition that (F^3) holds in Lemma 2.1, then for all $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$|F(u_1, u_2) - F(u_3, u_4)| \leq \left(\varepsilon \sum_{i=1}^{4} |u_i| + c_\varepsilon \sum_{i=1}^{4} |u_i|^{p-1}\right) (|u_1 - u_3| + |u_2 - u_4|) \tag{7}$$

for all $u_i \in \mathbb{R}$ $(i \in \{1, 2, 3, 4\})$.

Indeed, (F^3) implies that for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$|w_1| + |w_2| \leq \varepsilon(|u| + |v|) \tag{8}$$

for all $w_i \in \partial_i F(u, v)$ $(i \in \{1, 2\})$, with $|u| + |v| < \delta$. On the other hand, if $|u| + |v| \geq \delta$, from (3) we have

$$\begin{aligned} |w_1| + |w_2| &\leq c_1((|u| + |v|)^{p-1}\delta^{2-p} + |u|^{p-1} + |v|^{p-1}) \\ &\leq c_1(2^{p-1}\delta^{2-p} + 1)(|u|^{p-1} + |v|^{p-1}). \end{aligned}$$

Combining the above estimation with (8) we find that:

for all $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$|w_1| + |w_2| \leq \varepsilon(|u| + |v|) + c_\varepsilon(|u|^{p-1} + |v|^{p-1}) \tag{9}$$

for all $(u, v) \in \mathbb{R}^2$ and $w_i \in \partial_i F(u, v)$ $(i \in \{1, 2\})$. Now, if in (4) we use the estimation (9) instead of (3), we obtain (7) which will be useful in Section 3.

Since $H^1(\mathbb{R}^N) \ni u \mapsto \frac{1}{2}\|u\|_1^2$ is of class C^1, the function $\mathcal{J}$ is locally Lipschitz on $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Now, we are in the position to establish the following.

Proposition 2.3. *If the function F satisfies (F^1) and (F^2) then every critical point $(u, v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ of $\mathcal{J}$ is a weak solution of* (S).

Proof. We will apply Lemma 2.1 for $E = H^1(\mathbb{R}^N)$. Since (u, v) is a critical point of $\mathcal{J}$, for every $w, y \in H^1(\mathbb{R}^N)$ we have

$$\begin{aligned} 0 \leq \mathcal{J}^0(u, v; w, y) &= (u, w)_1 + (v, y)_1 + (-\mathcal{F})^0(u, v; w, y) \\ &= (u, w)_1 + (v, y)_1 + \mathcal{F}^0(u, v; -w, -y) \\ &\leq (u, w)_1 + (v, y)_1 + \int_{\mathbb{R}^N} F^0(u(x), v(x); -w(x), -y(x)) \mathrm{d}x. \end{aligned}$$

Using Proposition 2.2(ii), we obtain

$$0 \le (u, w)_1 + (v, y)_1 + \int_{\mathbb{R}^N} F_1^0(u(x), v(x); -w(x))\mathrm{d}x + \int_{\mathbb{R}^N} F_2^0(u(x), v(x); -y(x))\mathrm{d}x.$$

Taking $y = 0$, respectively $w = 0$, in the above inequality, we are led to the required inequalities from (HIS), i.e. (u, v) is a weak solution of (S). $\square$

Now we recall some notions which will be used in Section 4. Let G be a compact Lie group which acts linear isometrically on the real Banach space $(X, \|\cdot\|)$, i.e. the action $G \times X \to X : [g, u] \mapsto gu$ is continuous, $1 \cdot u = u$, $(g_1 g_2)u = g_1(g_2 u)$ for every $g_1, g_2 \in G$, and the map $u \mapsto gu$ is linear such that $\|gu\| = \|u\|$ for every $g \in G$ and $u \in X$.

A function $h : X \to \mathbb{R}$ is *G-invariant* if $h(gu) = h(u)$ for all $g \in G,\ u \in X$. The action on X induces an action of the same type on the dual space X^*, defined by $(gx^*)(u) = x^*(gu)$ for all $g \in G,\ u \in X$ and $x^* \in X^*$. We have $\|gx^*\| = \|x^*\|$ for all $g \in G, x^* \in X^*$. Supposing that $h : X \to \mathbb{R}$ is a G-invariant, locally Lipschitz functional, then $g\partial h(u) = \partial h(gu) = \partial h(u)$ for all $g \in G, u \in X$. Therefore, the function $u \mapsto \lambda_h(u)$ is G-invariant (see [12]). Let

$$X^G = \{u \in X : gu = u \text{ for all } g \in G\}.$$

We recall the *Principle of Symmetric Criticality* of Krawcewicz and Marzantowicz [12, p. 1045], which will be crucial in the proof of our theorems.

Proposition 2.4. *Assume that a compact Lie group G acts linear isometrically on a Banach space X. If $h : X \to \mathbb{R}$ is a G-invariant, locally Lipschitz functional and if $u \in X^G$ is a critical point of h restricted to X^G, then u is a critical point of h.*

If we endow the Cartesian product $X \times X$ with the norm $\|(u, v)\| = \sqrt{\|u\|^2 + \|v\|^2}$, $u, v \in X$, then G acts linear isometrically on $X \times X$, where the action $G \times (X \times X) \mapsto X \times X$ is defined by

$$g(u, v) = (gu, gv)$$

for all $g \in G$ and $u, v \in X$. Moreover,

$$(X \times X)^G = \{(u, v) \in X \times X : g(u, v) = (u, v) \text{ for all } g \in G\} = X^G \times X^G. \tag{10}$$

Finally, we recall the following “strong” form of the *Mountain Pass Theorem* which involves the Cerami condition for locally Lipschitz functions, proved by Kourogenis and Papageorgiou [11, Theorem 6].

Theorem 2.1. *Let X be a Banach space, and $h : X \to \mathbb{R}$ be a locally Lipschitz function with $h(0) = 0$. Suppose that there exist a point $e \in X$ and constants $\rho, \eta > 0$ such that*

(i) *$h(u) \ge \eta$ for all $u \in X$ with $\|u\| = \rho$,*
(ii) *$\|e\| > \rho$ and $h(e) \le 0$,*
(iii) *h satisfies $(C)_c$, with*

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} h(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = e\}.$$

Then $c \ge \eta$ and $c \in \mathbb{R}$ is a critical value of h.

3. Palais–Smale and geometric conditions

In this section we study the Palais–Smale and Cerami conditions for $\mathcal{J}$, which will be restricted to a certain subspace of $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, and we investigate the geometric conditions of the Mountain Pass Theorem.

Proposition 3.1. *Suppose that $F : \mathbb{R}^2 \to \mathbb{R}$ is a locally Lipschitz function which satisfies (F^1)–(F^3). Let E be a closed subspace of $H^1(\mathbb{R}^N)$ which is compactly embedded in $L^p(\mathbb{R}^N)$ and denote by $\mathcal{J}_E$ the restriction of $\mathcal{J}$ to $E \times E$. Then:*

(i) *$\mathcal{J}_E$ satisfies $(PS)_c$ for all $c > 0$ when (F^4_α) holds.*
(ii) *$\mathcal{J}_E$ satisfies $(C)_c$ for all $c > 0$ when F is non-negative and (F^4_ν) holds for some $\nu \in]\max\{2, \frac{N}{2}(p-2)\}, 2^*[$.*

Proof. (i) Let $\{(u_n, v_n)\}$ be a sequence from $E \times E$ such that

$$\mathcal{J}_E(u_n, v_n) \to c > 0, \tag{11}$$

$$\lambda_{\mathcal{J}_E}(u_n, v_n) \to 0, \tag{12}$$

as $n \to \infty$. We prove that $\{(u_n, v_n)\}$ is bounded in $E \times E$. For every $n \in \mathbb{N}$ there exists $z_n^* \in \partial\mathcal{J}_E(u_n, v_n)$ such that $\|z_n^*\|_{E\times E} = \lambda_{\mathcal{J}_E}(u_n, v_n)$. Clearly, (12) implies that

$$\begin{aligned}\mathcal{J}_E^0(u_n, v_n; u_n, v_n) &\geq \langle z_n^*, (u_n, v_n)\rangle_{E\times E} \\ &\geq -\|z_n^*\|_{E\times E}\|(u_n, v_n)\|_{E\times E} \geq -\alpha\|(u_n, v_n)\|_{E\times E}\end{aligned}$$

for n large enough. Using Lemma 2.1, the above estimation, (11) and (F^4_α), we get that for n large enough

$$\begin{aligned}c + 1 + \|(u_n, v_n)\|_{E\times E} &\geq \mathcal{J}_E(u_n, v_n) - \frac{1}{\alpha}\mathcal{J}_E^0(u_n, v_n; u_n, v_n) \\ &\geq \frac{1}{2}(\|u_n\|_E^2 + \|v_n\|_E^2) - \mathcal{F}_E(u_n, v_n) - \frac{1}{\alpha}[(u_n, u_n)_E + (v_n, v_n)_E \\ &\quad + (-\mathcal{F}_E)^0(u_n, v_n; u_n, v_n)] \\ &= \left(\frac{1}{2} - \frac{1}{\alpha}\right)\|(u_n, v_n)\|^2_{E\times E} - \mathcal{F}_E(u_n, v_n) - \frac{1}{\alpha}\mathcal{F}_E^0(u_n, v_n; -u_n, -v_n) \\ &\geq \left(\frac{1}{2} - \frac{1}{\alpha}\right)\|(u_n, v_n)\|^2_{E\times E} - \int_{\mathbb{R}^N}\Big[F(u_n, v_n) \\ &\quad + \frac{1}{\alpha}F^0(u_n(x), v_n(x); -u_n(x), -v_n(x))\Big]\mathrm{d}x \geq \left(\frac{1}{2} - \frac{1}{\alpha}\right)\|(u_n, v_n)\|^2_{E\times E} \\ &\quad - \int_{\mathbb{R}^N}\Big[F(u_n, v_n) + \frac{1}{\alpha}\Big(F_1^0(u_n(x), v_n(x); -u_n(x)) \\ &\quad + F_2^0(u_n(x), v_n(x); -v_n(x))\Big)\Big]\mathrm{d}x \geq \left(\frac{1}{2} - \frac{1}{\alpha}\right)\|(u_n, v_n)\|^2_{E\times E}.\end{aligned}$$

This shows that $\{(u_n, v_n)\}$ is bounded in $E \times E$. Since the embedding $E \hookrightarrow L^p(\mathbb{R}^N)$ is compact, passing to a subsequence if necessary, we may suppose that

$$(u_n, v_n) \rightharpoonup (u, v) \qquad \text{weakly in } E \times E, \tag{13}$$

$$u_n \to u \qquad \text{strongly in } L^p(\mathbb{R}^N), \tag{14}$$

$$v_n \to v \qquad \text{strongly in } L^p(\mathbb{R}^N). \tag{15}$$

On the other hand, we have

$$\begin{aligned}\mathcal{J}_E^0(u_n, v_n; u-u_n, v-v_n) &= (u_n, u-u_n)_E + (v_n, v-v_n)_E \\ &\quad + \mathcal{F}_E^0(u_n, v_n; u_n-u, v_n-v),\end{aligned}$$

$$\mathcal{J}_E^0(u, v; u_n-u, v_n-v) = (u, u_n-u)_E + (v, v_n-v)_E + \mathcal{F}_E^0(u, v; u-u_n, v-v_n).$$

Adding these relations, this yields

$$\|(u_n-u, v_n-v)\|_{E\times E}^2 = \|u_n-u\|_E^2 + \|v_n-v\|_E^2 = I_n^1 - I_n^2 - I_n^3, \tag{16}$$

where

$$\begin{aligned}&I_n^1 = \mathcal{F}_E^0(u_n, v_n; u_n-u, v_n-v) + \mathcal{F}_E^0(u, v; u-u_n, v-v_n),\\ &I_n^2 = \mathcal{J}_E^0(u_n, v_n; u-u_n, v-v_n) \qquad \text{and} \qquad I_n^3 = \mathcal{J}_E^0(u, v; u_n-u, v_n-v).\end{aligned}$$

Now, we will estimate I_n^i ($i \in \{1, 2, 3\}$). Using Lemma 2.1, Proposition 2.1(ii) and (9), we obtain

$$\begin{aligned}I_n^1 &\le \int_{\mathbb{R}^N} [F^0(u_n(x), v_n(x); u_n(x)-u(x), v_n(x)-v(x)) \\ &\quad + F^0(u(x), v(x); u(x)-u_n(x), v(x)-v_n(x))]\mathrm{d}x \\ &\le \int_{\mathbb{R}^N} [|F_1^0(u_n(x), v_n(x); u_n(x)-u(x))| + |F_2^0(u_n(x), v_n(x); v_n(x)-v(x))| \\ &\quad + |F_1^0(u(x), v(x); u(x)-u_n(x))| + |F_2^0(u(x), v(x); v(x)-v_n(x))|]\mathrm{d}x \\ &= \int_{\mathbb{R}^N} |\max\{w_n^1(u_n(x)-u(x)) : w_n^1 \in \partial_1 F(u_n(x), v_n(x))\}|\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} |\max\{w_n^2(v_n(x)-v(x)) : w_n^2 \in \partial_2 F(u_n(x), v_n(x))\}|\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} |\max\{w^1(u(x)-u_n(x)) : w^1 \in \partial_1 F(u(x), v(x))\}|\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} |\max\{w^2(v(x)-v_n(x)) : w^2 \in \partial_2 F(u(x), v(x))\}|\mathrm{d}x \\ &\le \int_{\mathbb{R}^N} [\varepsilon(|u_n| + |v_n|) + c_\varepsilon(|u_n|^{p-1} + |v_n|^{p-1})](|u_n-u| + |v_n-v|)\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} [\varepsilon(|u| + |v|) + c_\varepsilon(|u|^{p-1} + |v|^{p-1})](|u-u_n| + |v-v_n|)\mathrm{d}x \\ &\le 4\varepsilon(\|u_n\|_E^2 + \|v_n\|_E^2 + \|u\|_E^2 + \|v\|_E^2) + c_\varepsilon(\|u_n\|_p^{p-1} + \|v_n\|_p^{p-1} \\ &\quad + \|u\|_p^{p-1} + \|v\|_p^{p-1})(\|u-u_n\|_p + \|v-v_n\|_p).\end{aligned}$$

Since the sequences $\{u_n\}$ and $\{v_n\}$ are bounded in E and keeping in mind the relations (14) and (15), from the arbitrariness of $\varepsilon > 0$ we obtain

$$\limsup_{n\to\infty} I_n^1 \le 0. \tag{17}$$

Since

$$\begin{aligned}I_n^2 &= \mathcal{J}_E^0(u_n, v_n; u-u_n, v-v_n) \ge \langle z_n^*, (u-u_n, v-v_n)\rangle_{E\times E} \\ &\ge -\|z_n^*\|_{E\times E}\|(u-u_n, v-v_n)\|_{E\times E},\end{aligned}$$

then due to (12), we have

$$\liminf_{n\to\infty} I_n^2 \geq 0. \tag{18}$$

Finally, let us fix an element $z^* \in \partial\mathcal{J}_E(u, v)$. Thus

$$I_n^3 \geq \langle z^*, (u_n - u, v_n - v)\rangle_{E\times E}.$$

From (13) we have

$$\liminf_{n\to\infty} I_n^3 \geq 0. \tag{19}$$

Thus the relations (17)–(19) and (16) imply $\|(u_n - u, v_n - v)\|_{E\times E}^2 \to 0$ as $n \to \infty$, i.e. the sequence $\{(u_n, v_n)\}$ converges strongly to (u, v) in $E \times E$.

(ii) Now, we consider a sequence $\{(u_n, v_n)\}$ from $E \times E$ which satisfies (11) and

$$(1 + \|(u_n, v_n)\|_{E\times E})\lambda_{\mathcal{J}_E}(u_n, v_n) \to 0, \tag{20}$$

as $n \to \infty$. As above, we will prove that $\{(u_n, v_n)\}$ is bounded in $E \times E$. Again, let $z_n^* \in \partial\mathcal{J}_E(u_n, v_n)$ such that $\|z_n^*\|_{E\times E} = \lambda_{\mathcal{J}_E}(u_n, v_n)$. Thus, (20) implies that

$$\mathcal{J}_E^0(u_n, v_n; u_n, v_n) \geq -(1 + \|(u_n, v_n)\|_{E\times E})\|z_n^*\|_{E\times E}$$

for n large enough. Therefore, Lemma 2.1, the above inequality, (11), (20) and (F_ν^4) imply that for n large enough

$$\begin{aligned} 2c + 1 &\geq 2\mathcal{J}_E(u_n, v_n) - \mathcal{J}_E^0(u_n, v_n; u_n, v_n) \\ &= -2\mathcal{F}_E(u_n, v_n) - (-\mathcal{F}_E)^0(u_n, v_n; u_n, v_n) \\ &\geq -\int_{\mathbb{R}^N} [2F(u_n, v_n) + F^0(u_n(x), v_n(x); -u_n(x), -v_n(x))]\mathrm{d}x \\ &\geq -\int_{\mathbb{R}^N} [2F(u_n, v_n) + F_1^0(u_n(x), v_n(x); -u_n(x)) \\ &\quad + F_2^0(u_n(x), v_n(x); -v_n(x))]\mathrm{d}x \geq c_2(\|u_n\|_\nu^\nu + \|v_n\|_\nu^\nu). \end{aligned}$$

Hence, the sequences

$$\{u_n\}, \{v_n\} \text{ are bounded in } L^\nu(\mathbb{R}^N). \tag{21}$$

Since $F(0, 0) = 0$, from (7) we have that there exists $c_4 > 0$ such that

$$F(u, v) \leq \left[\frac{1}{8}(|u| + |v|) + c_4(|u|^{p-1} + |v|^{p-1})\right](|u| + |v|) \tag{22}$$

for all $(u, v) \in \mathbb{R}^2$. Therefore, from (22)

$$\begin{aligned} \frac{1}{2}\|(u_n, v_n)\|_{E\times E}^2 - \mathcal{J}_E(u_n, v_n) &= \int_{\mathbb{R}^N} F(u_n, v_n)\mathrm{d}x \\ &\leq \frac{1}{4}\|(u_n, v_n)\|_{E\times E}^2 + 2c_4(\|u_n\|_p^p + \|v_n\|_p^p). \end{aligned}$$

In conclusion, for n large enough

$$\|(u_n, v_n)\|_{E\times E}^2 \leq 4(c + 1) + 8c_4(\|u_n\|_p^p + \|v_n\|_p^p). \tag{23}$$

Now, we distinguish three cases.

(I) *When* $\nu \in]p, 2^*[$. We have the interpolation inequality

$$\|u\|_p \leq \|u\|_\nu^{1-t}\|u\|_2^t \qquad \text{for all } u \in L^\nu(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \tag{24}$$

with $1/p = (1-t)/\nu + t/2$. Using the Sobolev embedding $E \subset H^1(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$, $(s \in [2, 2^*[)$, from (21), (23) and (24) we obtain

$$\begin{aligned}\|(u_n, v_n)\|_{E\times E}^2 &\leq 4(c+1) + c_5(\|u_n\|_E^{tp} + \|v_n\|_E^{tp}) \\ &\leq 4(c+1) + 2^{1-\frac{tp}{2}} c_5 \|(u_n, v_n)\|_{E\times E}^{tp},\end{aligned}$$

where $c_5 > 0$. Since $tp < 2$, then $\{(u_n, v_n)\}$ is bounded in $E \times E$.

(II) *When* $\nu \in]2, p[$. If $N \geq 3$, we have similarly

$$\|u\|_p \leq \|u\|_\nu^{1-t}\|u\|_{2^*}^t \qquad \text{for all } u \in L^\nu(\mathbb{R}^N) \cap L^{2^*}(\mathbb{R}^N)$$

with $1/p = (1-t)/\nu + t/2^*$. From the fact that $\nu > \frac{N}{2}(p-2)$ we have again $tp < 2$. Since $E \subset H^1(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$, $(s \in [2, 2^*])$ is continuous, a similar calculation as above shows that $\{(u_n, v_n)\}$ is bounded in $E \times E$. If $N = 2$, then $\nu < p < \nu + 2$. Therefore

$$\|u\|_p \leq \|u\|_\nu^{1-t}\|u\|_{\nu+2}^t \qquad \text{for all } u \in L^\nu(\mathbb{R}^2) \cap L^{\nu+2}(\mathbb{R}^2)$$

with $1/p = (1-t)/\nu + t/(\nu+2)$. The rest is similar to (I), using the continuous embedding $E \subset H^1(\mathbb{R}^2) \hookrightarrow L^s(\mathbb{R}^2)$, $(s \in [2, \infty[)$.

(III) *When* $\nu = p$. From (23) and (21) it follows that $\{(u_n, v_n)\}$ is bounded in $E \times E$.

In conclusion, the sequence $\{(u_n, v_n)\}$ is bounded in $E \times E$ in each case. Now, we can follow the line of the proof of (i); the only minor modification is in the estimation of I_n^2, where we use (20) instead of (12). This completes the proof. □

Proposition 3.2. *Let $E \neq \{0\}$ be a closed subspace of $H^1(\mathbb{R}^N)$ and suppose that a locally Lipschitz function $F : \mathbb{R}^2 \to \mathbb{R}$ satisfies (F^1)–(F^3). If either*

(i) (F_α^4) *and* (F^5) *hold or*

(ii) F *is non-negative and* (F_ν^4) *holds for some* $\nu \in]2, 2^*[$, *then there exist* $\eta > 0$, $\rho > 0$ *and* $e \in E$ *such that*

$$\mathcal{J}_E(u, v) \geq \eta \qquad \textit{for all } \|(u, v)\|_{E\times E} = \rho \tag{25}$$

and

$$\|(e, e)\|_{E\times E} > \rho \qquad \textit{and} \qquad \mathcal{J}_E(e, e) \leq 0. \tag{26}$$

Proof. In both cases, we have $F(0, 0) = 0$. To prove (25), we use (22) and the fact that the function $t \mapsto (a^t + b^t)^{\frac{1}{t}}$, $t > 0$ is non-increasing $(a, b \geq 0)$. We have

$$\begin{aligned}\mathcal{J}_E(u, v) &= \frac{1}{2}\|(u, v)\|_{E\times E}^2 - \int_{\mathbb{R}^N} F(u, v)\mathrm{d}x \\ &\geq \frac{1}{2}\|(u, v)\|_{E\times E}^2 - \int_{\mathbb{R}^N} \left[\frac{1}{8}(|u| + |v|) + c_4(|u|^{p-1} + |v|^{p-1})\right](|u| + |v|)\mathrm{d}x \\ &\geq \frac{1}{2}\|(u, v)\|_{E\times E}^2 - \frac{1}{4}\|(u, v)\|_{E\times E}^2 - 2c_4(\|u\|_p^p + \|v\|_p^p) \\ &\geq \frac{1}{4}\|(u, v)\|_{E\times E}^2 - 2c_4 c_{1,p}^p(\|u\|_E^p + \|v\|_E^p)\end{aligned}$$

$$\geq \frac{1}{4}\|(u,v)\|^2_{E\times E} - 2c_4c^p_{1,p}(\|u\|^2_E + \|v\|^2_E)^{\frac{p}{2}}$$
$$= \left(\frac{1}{4} - 2c_4c^p_{1,p}\|(u,v)\|^{p-2}_{E\times E}\right)\|(u,v)\|^2_{E\times E}.$$

Choosing $\|(u,v)\|_{E\times E} = \rho > 0$ small enough, the number

$$\eta = \left(\frac{1}{4} - 2c_4c^p_{1,p}\rho^{p-2}\right)\rho^2$$

will be strictly positive, due to the fact that $p > 2$. Thus (25) holds. To prove (26), we distinguish the two cases.

(I) *When* (F^4_α) *and* (F^5) *hold.*

We first show that

$$t^\alpha F(u,v) \leq F(tu,tv) \qquad \text{for all } t > 1 \qquad \text{and} \qquad (u,v)\in\mathbb{R}^2. \tag{27}$$

To this end, we fix arbitrarily $(u,v)\in\mathbb{R}^2$. From the Second Chain Rule and Proposition 2.2 (i) we have

$$\partial_t F(tu,tv) \subseteq \partial F(tu,tv)\circ(u,v) \subseteq \partial_1 F(tu,tv)u + \partial_2 F(tu,tv)v$$

for all $t > 0$, where ∂_t stands for the generalized gradient with respect to $t\in\mathbb{R}$. Since $t\mapsto t^{-\alpha}F(tu,tv)$, $t > 0$ is locally Lipschitz then for all $t > 0$

$$\partial_t(t^{-\alpha}F(tu,tv)) = -\alpha t^{-\alpha-1}F(tu,tv) + t^{-\alpha}\partial_t F(tu,tv).$$

Therefore

$$\partial_t(t^{-\alpha}F(tu,tv)) \subseteq t^{-\alpha-1}[-\alpha F(tu,tv) + tu\partial_1 F(tu,tv) + tv\partial_2 F(tu,tv)] \tag{28}$$

for all $t > 0$. Now, we fix $t > 1$. Due to the Lebourg's mean value theorem and (28), there exists $\tau\in]1,t[$ such that

$$t^{-\alpha}F(tu,tv) - F(u,v) \in \partial_t(\tau^{-\alpha}F(\tau u,\tau v))(t-1)$$
$$\subseteq \tau^{-\alpha-1}[-\alpha F(\tau u,\tau v) + \tau u\partial_1 F(\tau u,\tau v) + \tau v\partial_2 F(\tau u,\tau v)](t-1).$$

Thus there exist $w^\tau_i\in\partial_i F(\tau u,\tau v)$ $(i\in\{1,2\})$, such that

$$t^{-\alpha}F(tu,tv) - F(u,v) = -\tau^{-\alpha-1}[\alpha F(\tau u,\tau v) + w^\tau_1(-\tau u) + w^\tau_2(-\tau v)](t-1).$$

Using (F^4_α), we have

$$t^{-\alpha}F(tu,tv) - F(u,v) \geq -\tau^{-\alpha-1}[\alpha F(\tau u,\tau v) + F^0_1(\tau u,\tau v;-\tau u)$$
$$+ F^0_2(\tau u,\tau v;-\tau v)](t-1) \geq 0.$$

This leads exactly to (27).

Now, we choose an element $u_0\in E$ such that $\|u_0\|_E = 1$. Due to (F^5), $\int_{\mathbb{R}^N} F(u_0,u_0)\mathrm{d}x > 0$. Moreover, by (27) we get

$$\mathcal{J}_E(tu_0,tu_0) = t^2 - \int_{\mathbb{R}^N} F(tu_0,tu_0)\mathrm{d}x$$
$$\leq t^2 - t^\alpha\int_{\mathbb{R}^N} F(u_0,u_0)\mathrm{d}x \to -\infty$$

as $t\to\infty$, because $\alpha > 2$. Thus, choosing $t_0 > \rho/\sqrt{2}$ large enough and denoting by $e = t_0u_0\in E$, we are led to (26).

(II) *When F is non-negative and (F^4_ν) holds for some $\nu \in]2, 2^*[$.*

We show that

$$F(tu, tv) \geq t^2 F(u, v) + \frac{c_2}{\nu - 2}(|u|^\nu + |v|^\nu)(t^\nu - t^2) \tag{29}$$

for all $t > 1$ and $(u, v) \in \mathbb{R}^2$. To this end, we fix again $(u, v) \in \mathbb{R}^2$. We define $K :]0, \infty[\to \mathbb{R}$ by

$$K(t) = t^{-2}F(tu, tv) - \frac{c_2}{\nu - 2}(|u|^\nu + |v|^\nu)t^{\nu-2}.$$

It is clear that K is locally Lipschitz and a similar calculation to that in (28) shows that

$$\partial_t K(t) \subseteq t^{-3}\{-2F(tu, tv) + t[\partial_1 F(tu, tv)u + \partial_2 F(tu, tv)v]\} - c_2 t^{\nu-3}(|u|^\nu + |v|^\nu).$$

Using again Lebourg's mean value theorem, for all $t > 1$ there exists $\tau \in]1, t[$ such that

$$K(t) - K(1) \in \partial_t K(\tau)(t - 1).$$

Moreover, there exists $w_i^\tau \in \partial_i F(\tau u, \tau v)$ $(i \in \{1, 2\})$, such that

$$\begin{aligned} K(t) - K(1) = {} & -\tau^{-3}[2F(\tau u, \tau v) + w_1^\tau(-\tau u) + w_2^\tau(-\tau v) \\ & + c_2(|\tau u|^\nu + |\tau v|^\nu)](t - 1). \end{aligned}$$

From (F^4_ν), we have

$$\begin{aligned} K(t) - K(1) \geq {} & -\tau^{-3}[2F(\tau u, \tau v) + F_1^0(\tau u, \tau v; -\tau u) + F_2^0(\tau u, \tau v; -\tau v) \\ & + c_2(|\tau u|^\nu + |\tau v|^\nu)](t - 1) \geq 0, \end{aligned}$$

which leads us to (29).

Let $u_0 \in E$ such that $\|u_0\|_E = 1$. Thus, due to the choice of ν and (29), we have

$$\begin{aligned} \mathcal{J}_E(tu_0, tu_0) &= t^2 - \int_{\mathbb{R}^N} F(tu_0, tu_0)\mathrm{d}x \\ &\leq t^2 - \int_{\mathbb{R}^N} t^2 F(u_0, u_0)\mathrm{d}x - \frac{2c_2}{\nu - 2}(t^\nu - t^2)\int_{\mathbb{R}^N} |u_0|^\nu \mathrm{d}x \\ &\leq t^2 - \frac{2c_2}{\nu - 2}(t^\nu - t^2)\|u_0\|_\nu^\nu \to -\infty, \end{aligned}$$

as $t \to \infty$. Further we proceed as above. □

4. Proof of Theorems

Proof of Theorem 1.1. It is clear that $E = H^1_r(\mathbb{R}^N)$ (introduced in (1)) is a closed subspace of $H^1(\mathbb{R}^N)$, which is compactly embedded in $L^p(\mathbb{R}^N)$ (see [17]). Choosing $X = E \times E$ and $h = \mathcal{J}_E$, the geometric conditions (i) and (ii) from Theorem 2.1 are verified for $\mathcal{J}_E$, due to Proposition 3.2(i). Let $\eta > 0$ and $e \in E$ be the corresponding elements from (25) and (26). Defining $c \in \mathbb{R}$ like in Theorem 2.1 for the element $(e, e) \in E \times E$, we have that $c \geq \eta$. By Proposition 3.1(i), $\mathcal{J}_E$ satisfies $(PS)_c$, so also $(C)_c$. Hence there exists at least one critical point $(u_1, v_1) \in E \times E$ of $\mathcal{J}_E$, the critical value $c = \mathcal{J}_E(u_1, v_1)$ being strictly positive, which means that (u_1, v_1) cannot be (0,0). It is standard to see that $O(N)$ acts linear isometrically on $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ and $\mathcal{J}$ is $O(N)$-invariant. In view of Proposition 2.4 and relation (10), we may conclude that (u_1, v_1) will be a critical point of $\mathcal{J}$ on the whole space $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Consequently, by Proposition 2.3, this element will be a weak solution of (S).

When $N = 4$ or $N \geq 6$, the space $E = H^1_{G_N}(\mathbb{R}^N)$ (introduced in (2)) is a closed subspace of $H^1(\mathbb{R}^N)$ and it is compactly embedded in $L^p(\mathbb{R}^N)$ (see [2, p. 455–457] or [19, p. 20]). Moreover, the only radial function of $H^1_{G_N}(\mathbb{R}^N)$ is 0 and the subgroup G_N of $O(N)$ acts linear isometrically on $H^1(\mathbb{R}^N)$. Since F is even, $\mathcal{J}$ is G_N-invariant (see for detail [2, p. 456]). Now, following the proof of the first part for $E = H^1_{G_N}(\mathbb{R}^N)$ instead of $H^1_r(\mathbb{R}^N)$ and for G_N instead of $O(N)$, we obtain a weak solution $(u_2, v_2) \neq (0, 0)$ of (S), with $u_2, v_2 \in H^1_{G_N}(\mathbb{R}^N)$. Clearly, $(u_1, v_1) \neq (u_2, v_2)$, which completes the proof. □

Proof of Theorem 1.2. The framework is the same as in Theorem 1.1, using (ii) instead of (i) from Propositions 3.1 and 3.2. □

Remark 4.1. We mention that our arguments also work for non-autonomous functions $F : \mathbb{R}^N \times \mathbb{R}^2 \to \mathbb{R}$, providing that F is radial in the variable $x \in \mathbb{R}^N$.

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