Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 235 (2007) 366–375
Journal of Differential Equations
www.elsevier.com/locate/jde

Quasilinear elliptic problems in $\mathbb{R}^N$ involving oscillatory nonlinearities

Alexandru Kristály [a,*,1], Gheorghe Moroşanu [b], Stepan Tersian [c,1]

[a] *Babeş-Bolyai University, Department of Economics, 400591 Cluj-Napoca, Romania*
[b] *Central European University, Department of Mathematics, 1051 Budapest, Hungary*
[c] *University of Rousse, Department of Mathematical Analysis, 7012 Rousse, Bulgaria*

Received 19 December 2005; revised 29 November 2006

Available online 27 January 2007

Abstract

In this paper we study an elliptic problem in $\mathbb{R}^N$ which involves the p-Laplacian, $p > N \geqslant 2$, and the nonlinear term has an oscillatory behavior. By means of a direct variational approach, we establish the existence of infinitely many homoclinic solutions whose $W^{1,p}(\mathbb{R}^N)$-norms tend to zero (to infinity, respectively) whenever the nonlinearity oscillates at zero (at infinity, respectively). The solutions have invariance properties with respect to certain subgroups of the orthogonal group $O(N)$.

Keywords: p-Laplacian; Multiple solutions; Oscillatory nonlinearities

1. Introduction

Various problems from Physics can be well modelled by means of certain equations which involve the p-Laplacian operator Δ_p, $p > 1$. For instance, in Fluid Mechanics, the case $p = 2$ ($p > 2$ and $p < 2$, respectively) corresponds to a Newtonian (dilatant and pseudoplastic, respectively) fluid. In order to fix our ideas, let us consider the problem

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = \alpha(x) f(u), & x \in \mathbb{R}^N, \\ u \in W^{1,p}\big(\mathbb{R}^N\big), \end{cases} \tag{P}$$

* Corresponding author.
E-mail address: alexandrukristaly@yahoo.com (A. Kristály).
[1] This article was supported in whole by Central European University (CEU) Special and Extension Programs.

0022-0396/$ – see front matter
doi:10.1016/j.jde.2007.01.012

where $1 < p < \infty$, $N \geqslant 2$, $\alpha : \mathbb{R}^N \to \mathbb{R}$ is measurable and $f : [0, \infty[\to \mathbb{R}$ is a continuous function, $f(0) = 0$.

Problem (P) has been widely studied when $N \geqslant p$. In the semilinear case (i.e., $N \geqslant p = 2$), certain solitary waves in the nonlinear Klein–Gordon and Schrödinger equations are solutions of (P); existence and multiplicity of solutions can be found for instance in [1,3,4,8,9,13], and references therein. In the quasilinear case (i.e., $N \geqslant p \neq 2$), problem (P) was treated in [10,12, 16]. The common feature of these papers is the superlinear or the asymptotical linear behavior at infinity of the nonlinear term f.

The aim of this paper is twofold. First, we want to handle the case when $p > N$. Although important problems can be treated within this framework (see, for instance [5], where a nonlinear field equation in Quantum Mechanics is considered involving the p-Laplacian, for $p = 6$), only a few works are available in this direction, see [11,14,15,20]. Second, instead of some usual assumption on the nonlinear term f, we assume that it *oscillates* at *zero* or at *infinity.* As an effect of the oscillatory behavior of f one could expect the existence of infinitely many solutions of problem (P). Indeed, our main results (see Theorems 1.1 and 1.2 below) give sufficient conditions on the oscillatory terms such that problem (P) has infinitely many weak solutions. As a byproduct, these solutions can be constructed in such a way that their norms in $W^{1,p}(\mathbb{R}^N)$ tend to zero (to infinity, respectively) whenever the nonlinearity oscillates at zero (at infinity, respectively), which can be considered as another 'reflection' of the oscillation.

In order to describe precisely our results, we recall some basic concepts. The space $W^{1,p}(\mathbb{R}^N)$ is endowed with the standard norm $\|u\|_{W^{1,p}} = (\|\nabla u\|_p^p + \|u\|_p^p)^{1/p}$ where $\|\cdot\|_p$ is the usual norm in $L^p(\mathbb{R}^N)$, $1 < p < \infty$. The space $L^\infty(\mathbb{R}^N)$ is endowed with the usual sup-norm, denoted by $\|\cdot\|_\infty$. Let $N \geqslant 2$ and define the set

$$\mathcal{G}_N = \big\{G \subseteq O(N) \colon G = O(N_1) \times \cdots \times O(N_k),\ k \geqslant 1,\\ N_1 + \cdots + N_k = N,\ N_j \geqslant 2,\ j = 1, \ldots, k\big\}.$$

Fix a $G \in \mathcal{G}_N$. A function $u : \mathbb{R}^N \to \mathbb{R}$ is called G-*invariant* if $u(gx) = u(x)$ for every $g \in G$ and $x \in \mathbb{R}^N$. In particular, an $O(N)$-invariant function is called *radial.*

Throughout the paper we assume that the potential $\alpha : \mathbb{R}^N \to \mathbb{R}$ appearing in problem (P) fulfills the following hypothesis:

(H) $\alpha \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is radial, nonnegative, and $\|\alpha\|_\infty > 0$.

Since we are interested in the case when $p > N$, Morrey's embedding theorem implies that $W^{1,p}(\mathbb{R}^N)$ is continuously embedded into $L^\infty(\mathbb{R}^N)$, and one can consider continuous representations of the elements from $W^{1,p}(\mathbb{R}^N)$. Moreover, every element $u \in W^{1,p}(\mathbb{R}^N)$ is homoclinic, i.e., $u(x) \to 0$ as $|x| \to \infty$, see [6, p. 167]. Due to Morrey's theorem and hypothesis (H), the energy functional $\mathcal{E} : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ associated with problem (P)

$$\mathcal{E}(u) = \frac{1}{p}\|u\|_{W^{1,p}}^p - \int\limits_{\mathbb{R}^N} \alpha(x) F\big(u(x)\big)\, dx, \quad u \in W^{1,p}\big(\mathbb{R}^N\big), \tag{1}$$

is well defined, where $F(s) = \int_0^s f(t)\, dt$, $s \in \mathbb{R}$. (Since $f(0) = 0$ we may assume that f is extended to the whole real line with zero on $]-\infty, 0]$.)

Now we are in the position to state our first main result which deals with the case when the nonlinearity f exhibits an oscillation at the *origin*. More precisely, we assume

(F_0) $-\infty < \liminf_{s\to 0^+} \frac{F(s)}{s^p} \leqslant \limsup_{s\to 0^+} \frac{F(s)}{s^p} = +\infty$;
(S_0) there are two sequences $\{a_k\}$, $\{b_k\}$ such that $0 < b_{k+1} < a_k < b_k$, $\lim_{k\to\infty} b_k = 0$, and $f(s) \leqslant 0$ for every $s \in [a_k, b_k]$, $k \in \mathbb{N}$.

Theorem 1.1 *(Oscillation at zero). Let $p > N \geqslant 2$. Let $\alpha : \mathbb{R}^N \to \mathbb{R}$ be a function which satisfies (H) and $f : [0,\infty[\, \to \mathbb{R}$ be a continuous function such that (F_0), (S_0) are fulfilled.*

Then, for every $G \in \mathcal{G}_N$ there exists a sequence $\{u_k^G\} \subset W^{1,p}(\mathbb{R}^N)$ of nonnegative, G-invariant, homoclinic weak solutions of (P) *such that*

$$\lim_{k\to\infty} \mathcal{E}(u_k^G) = 0 \quad \text{and} \quad \lim_{k\to\infty} \|u_k^G\|_{W^{1,p}} = 0. \tag{2}$$

Example 1.1. Let p and N be as in Theorem 1.1. A simple function which satisfies (F_0) and (S_0) is $f(s) = s^q \sin \frac{1}{s}$ for $s > 0$, and $f(0) = 0$, where $\max\{p-3, 0\} < q < p-2$.

Next, we will state the counterpart of Theorem 1.1 when the nonlinearity oscillates at *infinity*. First, we require similar assumptions as in (F_0) and (S_0), respectively:

(F_∞) $-\infty < \liminf_{s\to +\infty} \frac{F(s)}{s^p} \leqslant \limsup_{s\to +\infty} \frac{F(s)}{s^p} = +\infty$;
(S_∞) there are two sequences $\{a_k\}$, $\{b_k\}$ such that $0 < a_k < b_k < a_{k+1}$, $\lim_{k\to\infty} b_k = +\infty$, and $f(s) \leqslant 0$ for every $s \in [a_k, b_k]$, $k \in \mathbb{N}$.

Theorem 1.2 *(Oscillation at infinity). Let $p > N \geqslant 2$. Let $\alpha : \mathbb{R}^N \to \mathbb{R}$ be a potential which satisfies (H) and $f : [0,\infty[\, \to \mathbb{R}$ be a continuous function such that $f(0) = 0$, and (F_∞), (S_∞) are fulfilled.*

Then, for every $G \in \mathcal{G}_N$ there exists a sequence $\{u_k^G\} \subset W^{1,p}(\mathbb{R}^N)$ of nonnegative, G-invariant, homoclinic weak solutions of (P) *such that*

$$\lim_{k\to\infty} \mathcal{E}(u_k^G) = -\infty \quad \text{and} \quad \lim_{k\to\infty} \|u_k^G\|_{W^{1,p}} = +\infty. \tag{3}$$

Example 1.2. Let $c \in\,]0,1[$ and $q \in\,]p-1, p]$ be two fixed numbers. The function $f : [0,\infty[\, \to \mathbb{R}$ defined by $f(s) = s^q(c + \sin s)$ verifies hypotheses (F_∞) and (S_∞), respectively.

Remark 1.1. It is possible to handle the case when the nonlinear term f has *discontinuities* in Theorems 1.1 and 1.2; in such a case, a differential inclusion problem is formulated instead of (P) in order to 'fill the discontinuity gaps' of f, see [14].

Remark 1.2. In [14], the hypothesis $\lim_{k\to\infty} \frac{b_k}{a_k} = +\infty$ was indispensable (where $\{a_k\}$ and $\{b_k\}$ are the sequences appearing in (S_0) and (S_∞), respectively). The main advantage of the above results is that one can omit this inconvenient condition, allowing us to include new oscillatory nonlinearities within our framework. Furthermore, we are able to guarantee a sequence of G-invariant weak solutions of (P) for every $G \in \mathcal{G}_N$, and not only for $G = O(N)$ as in [14].

Our approach is based on an elementary variational technic; in the sequel, we will describe it briefly here. By using Morrey's embedding theorem and hypothesis (H), one can show by a standard manner that the energy functional $\mathcal{E}$ is of class $\mathcal{C}^1$ on $W^{1,p}(\mathbb{R}^N)$ and its critical points are precisely the weak solutions of (P), see [15]. It is well known that $W^{1,p}(\mathbb{R}^N)$ cannot be compactly embedded into $L^q(\mathbb{R}^N)$, $q > 1$, due to the unboundedness of the domain. However, by Lions theorem (see [17, Théorème III.3]), the fixed point space of $W^{1,p}(\mathbb{R}^N)$ under the action of $G \in \mathcal{G}_N$, denoted by $W_G^{1,p}(\mathbb{R}^N)$, is compactly embedded into $L^q(\mathbb{R}^N)$ whenever $p < q < +\infty$. The restriction of $\mathcal{E}$ to $W_G^{1,p}(\mathbb{R}^N)$, denoted by $\mathcal{E}_G$, is weakly sequentially lower semicontinuous and its critical points are critical points of $\mathcal{E}$ as well, due to the principle of symmetric criticality of Palais (see [22, Theorem 5.4]). The crucial step in our arguments is the construction of an appropriate sequence of subsets of $W_G^{1,p}(\mathbb{R}^N)$, proving that the relative minima of $\mathcal{E}_G$ on these sets are actually local minima (thus, critical points) of $\mathcal{E}_G$ on $W_G^{1,p}(\mathbb{R}^N)$, so G-invariant weak solutions of (P). Then, a suitable subsequence of critical points of $\mathcal{E}_G$ can be extracted from the aforementioned local minima of $\mathcal{E}_G$ having the properties (2) and (3), respectively. We emphasize that the crucial step described above can be achieved only in the case when $p > N$ (i.e., when one can apply Morrey's embedding theorem); in the case $p \leqslant N$, a new method should be elaborated in order to obtain similar results as Theorems 1.1 and 1.2, respectively.

Our results complement not only the aforementioned papers ([1,3,4,8,9,13], where the case $p \leqslant N$ has been treated) but also some results obtained on *bounded* domains where elliptic problems with oscillatory nonlinearities have been considered; Dirichlet problems were studied in [2,7,21,24], while Neumann type problems in [18,23]. The common feature of these works is that infinitely many solutions are obtained by means of various methods; for instance, sub–super solution arguments (see [21]); the general variational principle of Ricceri (see [7,18,23]); continuity of certain superposition operators (see [2,24]). Note however that our proofs do not use any abstract argument apart from the compactness result of Lions [17] and the Palais' principle [22].

2. Variational setting

First of all, note that assumption (S_0) implies that $f(0) \leqslant 0$. Consequently, $f(0) = 0$, due to the left inequality of (F_0). Therefore, in both cases (i.e., in Theorems 1.1 and 1.2) we may extend continuously the function f to the whole real line with value zero on the interval $]-\infty, 0]$.

Note that the functional $\mathcal{E}$, defined in (1), is of class C^1 on $W^{1,p}(\mathbb{R}^N)$, see [15, Proposition 2.1]. Moreover, its critical points are precisely the weak solutions of (P). Thus, in order to prove the theorem, it is enough to find a sequence of distinct critical points of $\mathcal{E}$ with the required properties.

To do this, fix $G = O(N_1) \times \cdots \times O(N_k) \in \mathcal{G}_N$. The action of G on the space $W^{1,p}(\mathbb{R}^N)$ will be defined by

$$(gu)(x) = u\big(g_1^{-1}x_1, \ldots, g_k^{-1}x_k\big)$$

for every $u \in W^{1,p}(\mathbb{R}^N)$, $g = (g_1, \ldots, g_k)$, $g_j \in O(N_j)$, $x = (x_1, \ldots, x_k)$, $x_j \in \mathbb{R}^{N_j}$, $j = 1, \ldots, k$. Let us denote by $\mathcal{E}_G$ the restriction of the energy functional $\mathcal{E}$ to the subspace of G-invariant functions of $W^{1,p}(\mathbb{R}^N)$, i.e.

$$W_G^{1,p}\big(\mathbb{R}^N\big) \stackrel{\text{def}}{=} \big\{u \in W^{1,p}\big(\mathbb{R}^N\big)\colon u(g_1x_1, \ldots, g_kx_k) = u(x_1, \ldots, x_k)$$
$$\text{for all } g_j \in O(N_j),\ x_j \in \mathbb{R}^{N_j},\ j = 1, \ldots, k\big\}.$$

Since α is radial, the functional $\mathcal{E}$ is G-invariant. Due to the principle of symmetric criticality of Palais, the critical points of $\mathcal{E}_G$ are critical points of $\mathcal{E}$ as well. (Note that the fixed point space of the action G on the space $W^{1,p}(\mathbb{R}^N)$ is exactly $W^{1,p}_G(\mathbb{R}^N)$.)

Proposition 2.1. *Functional $\mathcal{E}_G$ is sequentially weakly lower semicontinuous on $W^{1,p}_G(\mathbb{R}^N)$.*

Proof. The function $\|\cdot\|^p_{W^{1,p}}$ is clearly sequentially weakly lower semicontinuous on $W^{1,p}_G(\mathbb{R}^N)$, see [6, p. 35]. Let us prove that the function $\mathcal{F}: W^{1,p}_G(\mathbb{R}^N) \to \mathbb{R}$, defined by $\mathcal{F}(u) = \int_{\mathbb{R}^N} \alpha(x)F(u(x))\,dx$, is sequentially weakly continuous.

Suppose the contrary, i.e., let $\{u_n\} \subset W^{1,p}_G(\mathbb{R}^N)$ be a sequence which converges weakly to $u \in W^{1,p}_G(\mathbb{R}^N)$ but $\mathcal{F}_G(u_n) \nrightarrow \mathcal{F}_G(u)$ as $n \to \infty$. Therefore, up to a subsequence, one can find a number $\varepsilon_0 > 0$ such that

$$0 < \varepsilon_0 \leqslant \left|\mathcal{F}_G(u_n) - \mathcal{F}_G(u)\right| \quad \text{for every } n \in \mathbb{N},$$

and by Lions theorem, u_n converges strongly to u in $L^q(\mathbb{R}^N)$, for some fixed $q \in]p, +\infty[$. For every $n \in \mathbb{N}$ one has $0 < \theta_n < 1$ such that

$$\begin{aligned} 0 < \varepsilon_0 \leqslant \left|\mathcal{F}_G(u_n) - \mathcal{F}_G(u)\right| &\leqslant \int\limits_{\mathbb{R}^N} \alpha(x)\left|f\big(u + \theta_n(u_n - u)\big)\right| \cdot |u_n - u|\,dx \\ &\leqslant \|\alpha\|_{q'} \max\big\{\left|f(s)\right|:\ 0 \leqslant s \leqslant M_n\big\}\|u_n - u\|_q, \end{aligned}$$

where $q' = q(q-1)^{-1}$ and $M_n = \|u\|_\infty + \|u_n\|_\infty$. Note that $\sup_{n\in\mathbb{N}} M_n < +\infty$; indeed, $W^{1,p}_G(\mathbb{R}^N)$ is continuously embedded into $L^\infty(\mathbb{R}^N)$. Letting $n \to \infty$, in the above relation the right-hand side tends to 0, a contradiction. □

Now, we are going to prove our main results.

3. Proof of Theorem 1.1

Fix a number $r < 0$ and define the set

$$W_k = \big\{u \in W^{1,p}_G\big(\mathbb{R}^N\big):\ r \leqslant u(x) \leqslant b_k \text{ for every } x \in \mathbb{R}^N\big\},$$

where the sequence $\{b_k\}$ appears in (S_0).

Claim 3.1. *Functional $\mathcal{E}_G$ is bounded from below on W_k and its infimum on W_k is attained.*

Proof. The set W_k is convex. Moreover, it is closed in $W^{1,p}_G(\mathbb{R}^N)$ due to the continuity of the embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$. Consequently, the set W_k is weakly closed. Moreover,

$$\mathcal{E}_G(u) = \frac{1}{p}\|u\|^p_{W^{1,p}} - \int\limits_{\mathbb{R}^N} \alpha(x)F\big(u(x)\big)\,dx \geqslant -\|\alpha\|_1 \max_{[r,b_k]} F \quad \text{for } u \in W_k.$$

Thus, $\mathcal{E}_G$ is bounded from below on W_k. Let $\eta_k = \inf_{W_k} \mathcal{E}_G$, and $\{u_n\}$ be a sequence in W_k such that $\eta_k \leqslant \mathcal{E}_G(u_n) \leqslant \eta_k + 1/n$ for all $n \in \mathbb{N}$. Then,

$$\frac{1}{p}\|u_n\|_{W^{1,p}}^p \leqslant \eta_k + 1 + \|\alpha\|_1 \max_{[r,b_k]} F$$

for all $n \in \mathbb{N}$, i.e. $\{u_n\}$ is bounded in $W_G^{1,p}(\mathbb{R}^N)$. So, up to a subsequence, $\{u_n\}$ weakly converges in $W_G^{1,p}(\mathbb{R}^N)$ to some $u_k^G \in W_k$. By the sequentially weakly lower semicontinuity of $\mathcal{E}_G$, cf. Proposition 2.1, we conclude that $\mathcal{E}_G(u_k^G) = \eta_k = \inf_{W_k} \mathcal{E}_G$. $\square$

Claim 3.2. *Let $u_k^G \in W_k$ be such that $\mathcal{E}_G(u_k^G) = \inf_{W_k} \mathcal{E}_G$. Then, $0 \leqslant u_k^G(x) \leqslant a_k$ for all $x \in \mathbb{R}^N$.*

Proof. Let $A = \{x \in \mathbb{R}^N\colon u_k^G(x) \notin [0, a_k]\}$ and suppose that $A \neq \emptyset$. Thus, $\mathrm{meas}(A) > 0$ due to the continuity of u_k^G. Define

$$h(s) = \begin{cases} 0, & \text{if } s < 0; \\ s, & \text{if } s \in [0, a_k]; \\ a_k, & \text{if } s > a_k. \end{cases}$$

Set $v_k = h \circ u_k^G$. Since h is uniformly Lipschitz and $h(0) = 0$, v_k belongs to $W^{1,p}(\mathbb{R}^N)$, cf. [19] or [24, Lemma 3.4]. Moreover, v_k is G-invariant because of u_k^G; thus $v_k \in W_G^{1,p}(\mathbb{R}^N)$. In addition, $v_k \in W_k$. Denoting by

$$A_1 = \{x \in A\colon u_k^G(x) < 0\} \quad \text{and} \quad A_2 = \{x \in A\colon u_k^G(x) > a_k\},$$

we have that $v_k(x) = u_k^G(x)$ for all $x \in \mathbb{R}^N \setminus A$, $v_k(x) = 0$ for all $x \in A_1$ and $v_k(x) = a_k$ for all $x \in A_2$. Then,

$$\begin{aligned}
\mathcal{E}_G(v_k) - \mathcal{E}_G(u_k^G) &= -\frac{1}{p}\int_A |\nabla u_k^G|^p + \frac{1}{p}\int_A \big[|v_k|^p - |u_k^G|^p\big] - \int_A \alpha(x)\big[F(v_k) - F(u_k^G)\big] \\
&= -\frac{1}{p}\int_A |\nabla u_k^G|^p - \frac{1}{p}\int_{A_1} |u_k^G|^p + \frac{1}{p}\int_{A_2} \big[a_k^p - (u_k^G)^p\big] \\
&\quad - \int_{A_2} \alpha(x)\big[F(a_k) - F(u_k^G)\big].
\end{aligned}$$

By (S_0), one has that $F(s) \leqslant F(a_k)$ for every $s \in [a_k, b_k]$. Using this fact, we observe that every term of the right-hand side of the above expression is non-positive. On the other hand, since $\mathcal{E}_G(v_k) \geqslant \mathcal{E}_G(u_k^G) = \inf_{W_k} \mathcal{E}_G$, then in particular,

$$\int_A |\nabla u_k^G|^p = 0,$$

$$\int_{A_1} |u_k^G|^p = \int_{A_2} \left[a_k^p - \left(u_k^G\right)^p\right] = 0.$$

By the first equality we obtain the existence of a positive measured subset B of A and a constant $M \in \mathbb{R}$ such that $u_k^G = M$ on the set B. Then, either $B \subset A_1$ or $B \subset A_2$. If $B \subset A_1$, then

$$0 = \int_{A_1} |u_k^G|^p \geqslant \int_B |u_k^G|^p = |M|^p \operatorname{meas}(B) > 0,$$

a contradiction. If $B \subset A_2$, then

$$0 = \int_{A_2} \left[a_k^p - \left(u_k^G\right)^p\right] \leqslant \int_B \left[a_k^p - \left(u_k^G\right)^p\right] = \left[a_k^p - M^p\right] \operatorname{meas}(B) < 0,$$

a contradiction. This shows that A has zero measure, therefore, $A = \emptyset$. $\square$

Claim 3.3. *Let $u_k^G \in W_k$ be such that $\mathcal{E}_G(u_k^G) = \inf_{W_k} \mathcal{E}_G$. Then u_k^G is a local minimum point of $\mathcal{E}_G$ in $W_G^{1,p}(\mathbb{R}^N)$.*

Proof. Indeed, otherwise there would be a sequence $\{u_n\} \subset W_G^{1,p}(\mathbb{R}^N)$ which converges to u_k^G and $\mathcal{E}_G(u_n) < \mathcal{E}_G(u_k^G) = \inf_{W_k} \mathcal{E}_G$ for all $n \in \mathbb{N}$. From this inequality it follows that $u_n \notin W_k$ for any $n \in \mathbb{N}$. Since $u_n \to u_k^G$ in $W_G^{1,p}(\mathbb{R}^N)$, then due to Morrey's theorem, $u_n \to u_k^G$ in $L^\infty(\mathbb{R}^N)$ as well. In particular, for every $0 < \delta < \min\{-r, b_k - a_k\}/2$, there exists $n_\delta \in \mathbb{N}$ such that $\|u_n - u_k^G\|_\infty < \delta$ for every $n \geqslant n_\delta$. By using Claim 3.2 and taking into account the choice of the number δ, we conclude that

$$r < u_n(x) < b_k \quad \text{for all } x \in \mathbb{R}^N,\ n \geqslant n_\delta,$$

which clearly contradicts the fact $u_n \notin W_k$. $\square$

Claim 3.4. *Let $\eta_k = \inf_{W_k} \mathcal{E}_G = \mathcal{E}_G(u_k^G)$. Then $\eta_k < 0$ for all $k \in \mathbb{N}$ and* $\lim_{k\to\infty} \eta_k = 0$.

Proof. Since $\|\alpha\|_\infty > 0$, then for every number $0 < \theta < 1$ one can find a measurable set S_θ with positive measure such that $\alpha(x) > \theta\|\alpha\|_\infty$ for a.e. $x \in S_\theta$. Moreover, since α is radial, one can assume that S_θ is $O(N)$-invariant, i.e., $gS_\theta = S_\theta$ for every $g \in O(N)$. For simplicity, let us fix $\theta = 1/p$, and denote $S_0 = S_{1/p}$. Then, one can find $x_0 \in \mathbb{R}^N$ and $\mu_0 > 0$, with $\mu_0 < |x_0|$ whenever $x_0 \neq 0$, such that

$$\operatorname{meas}\left\{S_0 \setminus \left\{x \in \mathbb{R}^N\colon\ \left||x| - |x_0|\right| \leqslant \mu_0/2\right\}\right\} = 0.$$

Define for every $s > 0$ the function $w_s : \mathbb{R}^N \to \mathbb{R}$ by

$$w_s(x) = \begin{cases} 0, & \text{if } ||x| - |x_0|| \geqslant \mu_0; \\ s, & \text{if } ||x| - |x_0|| \leqslant \mu_0/2; \\ \frac{2s}{\mu_0}(\mu_0 - ||x| - |x_0||), & \text{if } \mu_0/2 < ||x| - |x_0|| < \mu_0. \end{cases} \tag{4}$$

It is clear that $w_s \in W^{1,p}(\mathbb{R}^N)$ and it is radial, thus in particular, $w_s \in W_G^{1,p}(\mathbb{R}^N)$. Moreover, a simple estimation shows that

$$\|w_s\|_{W^{1,p}}^p \leqslant s^p\left(1+\frac{2^p}{\mu_0^p}\right)\operatorname{meas}\bigl\{x \in \mathbb{R}^N\colon \bigl||x|-|x_0|\bigr| \leqslant \mu_0\bigr\} =: s^p C(x_0,\mu_0).$$

By (F_0) there exist two positive numbers l and ϱ such that $F(s) > -ls^p$ for every $s \in]0,\varrho[$. Let $L > 0$ be such that

$$L\|\alpha\|_\infty \operatorname{meas} S_0 - l\|\alpha\|_1 p - C(x_0,\mu_0) > 0. \tag{5}$$

By using the right-hand side of (F_0), there exists a sequence $\{s_k\} \subset]0,\varrho[$ converging to zero, and $F(s_k) > Ls_k^p$. Let $\{s_{l_k}\}$ be a decreasing subsequence of $\{s_k\}$ such that $s_{l_k} \leqslant b_k$ for all $k \in \mathbb{N}$ and $w_k \equiv w_{s_{l_k}}$ as in (4). It is clear that $w_k \in W_k$. Moreover, due to (5), one has

$$\mathcal{E}_G(w_k) \leqslant s_{l_k}^p\left(\frac{1}{p}C(x_0,\mu_0) + l\|\alpha\|_1 - \frac{1}{p}L\|\alpha\|_\infty \operatorname{meas} S_0\right) < 0.$$

Thus, $\eta_k = \inf_{W_k} \mathcal{E}_G \leqslant \mathcal{E}_G(w_k) < 0$.

Now we will prove that $\eta_k \to 0$ as $k \to \infty$. Due to Claim 3.2, for every $x \in \mathbb{R}^N$ one has

$$\bigl|F\bigl(u_k^G(x)\bigr)\bigr| \leqslant \int_0^{a_k} |f| \leqslant a_k \max_{[0,a_k]} |f| \leqslant a_k \max_{[0,a_1]} |f|. \tag{6}$$

Then

$$0 > \eta_k = \mathcal{E}_G\bigl(u_k^G\bigr) \geqslant -\int_{\mathbb{R}^N} \alpha(x)F\bigl(u_k^G\bigr) \geqslant -\|\alpha\|_1 \max_{[0,a_1]} |f|\, a_k.$$

Since the sequence $\{a_k\}$ tends to zero, then $\eta_k \to 0$ as $k \to \infty$. $\square$

Proof of Theorem 1.1 concluded. Since u_k^G are local minima of $\mathcal{E}_G$ (cf. Claim 3.3), they are critical points of $\mathcal{E}_G$, thus G-invariant weak solutions of (P). Due to Claim 3.4, there are infinitely many distinct u_k^G. Moreover, due to (6), we have

$$\frac{1}{p}\bigl\|u_k^G\bigr\|_{W^{1,p}}^p = \int_{\mathbb{R}^N} \alpha(x)F\bigl(u_k^G\bigr) + \eta_k \leqslant \|\alpha\|_1 \max_{[0,a_1]} |f| a_k,$$

which proves that $\|u_k^G\|_{W^{1,p}} \to 0$ as $k \to \infty$. $\square$

4. Proof of Theorem 1.2

Let $r < 0$ and define similarly as in Section 3 the set

$$W_k = \{u \in W_G^{1,p}\colon\ r \leqslant u(x) \leqslant b_k \text{ for every } x \in \mathbb{R}^N\}.$$

The first part of the proof is similar to that of Theorem 1.1. Indeed, we can prove that the functional $\mathcal{E}_G$ is bounded from below on W_k and its infimum on W_k is attained (see Claim 3.1). Moreover, if $u_k^G \in W_k$ is chosen such that $\mathcal{E}_G(u_k^G) = \inf_{W_k} \mathcal{E}_G$, then $0 \leqslant u_k^G(x) \leqslant a_k$ for all $x \in \mathbb{R}^N$ (see Claim 3.2), and u_k^G is a local minimum point of $\mathcal{E}_G$ in $W_G^{1,p}(\mathbb{R}^N)$ (see Claim 3.3). Instead of Claim 3.4, we prove

Claim 4.1. *Let $\delta_k = \inf_{W_k} \mathcal{E}_G = \mathcal{E}_G(u_k^G)$. Then $\lim_{k\to\infty} \delta_k = -\infty$.*

Proof. By (F_∞) there exist two positive numbers l and ϱ such that $F(s) > -ls^p$ for every $s > \varrho$. Let $L > 0$ be such that

$$L\|\alpha\|_\infty \operatorname{meas} S_0 - l\|\alpha\|_1 p - C(x_0, \mu_0) > 0, \tag{7}$$

where S_0, μ_0, x_0 and $C(x_0, \mu_0)$ are from the previous section. Due to the right part of (F_∞), there exists a sequence $\{s_k\}$ which tends to $+\infty$ such that $F(s_k) > Ls_k^p$. Let $\{b_{l_k}\}$ be an increasing subsequence of $\{b_k\}$ such that $s_k \leqslant b_{l_k}$ for all $k \in \mathbb{N}$ and $w_k \equiv w_{s_k}$ as in (4). It is clear that $w_k \in W_{l_k}$. Moreover, one can deduce that

$$\mathcal{E}_G(w_k) \leqslant s_k^p\left(\frac{1}{p}C(x_0, \mu_0) + l\|\alpha\|_1 - \frac{1}{p}L\|\alpha\|_\infty \operatorname{meas} S_0\right) + \|\alpha\|_1 \max_{[0,\varrho]} |F|.$$

The term near s_k^p in the above estimation is strictly negative due to (7), which proves that $\delta_{l_k} = \inf_{W_{l_k}} \mathcal{E}_G \leqslant \mathcal{E}_G(w_k) \to -\infty$ as $k \to \infty$. Since the sequence $\{\delta_k\}$ is non-increasing, we are done. $\square$

Proof of Theorem 1.2 concluded. It is clear that we have infinitely many pairwise distinct local minimum points u_k^G of $\mathcal{E}_G$ with $u_k^G \in W_k$. Now we will prove that $\|u_k^G\|_{W^{1,p}} \to +\infty$ as $k \to \infty$. Let us assume the contrary. Therefore, there is a subsequence $\{u_{n_k}^G\}$ of $\{u_k^G\}$ which is bounded in $W_G^{1,p}(\mathbb{R}^N)$. Thus, it is also bounded in $L^\infty(\mathbb{R}^N)$. In particular we can find $m_0 \in \mathbb{N}$ such that $u_{n_k}^G \in W_{m_0}$ for all $k \in \mathbb{N}$. For every $n_k \geqslant m_0$ one has

$$\delta_{m_0} \geqslant \delta_{n_k} = \inf_{W_{n_k}} \mathcal{E}_G = \mathcal{E}_G(u_{n_k}^G) \geqslant \inf_{W_{m_0}} \mathcal{E}_G = \delta_{m_0},$$

which proves that $\delta_{n_k} = \delta_{m_0}$ for all $n_k \geqslant m_0$, contradicting Claim 4.1. This concludes our proof. $\square$

References

[1] S. Adachi, A positive solution of a nonhomogeneous elliptic equation in $\mathbb{R}^N$ with G-invariant nonlinearity, Comm. Partial Differential Equations 27 (2002) 1–22.

[2] G. Anello, G. Cordaro, Infinitely many arbitrarily small positive solutions for the Dirichlet problem involving the p-Laplacian, Proc. Roy. Soc. Edinburgh Sect. A 132 (2002) 511–519.

[3] T. Bartsch, Y. Ding, On a nonlinear Schrödinger equation with periodic potential, Math. Ann. 313 (1999) 15–37.

[4] T. Bartsch, M. Willem, Infinitely many non-radial solutions of a Euclidean scalar field equations, J. Funct. Anal. 117 (1993) 447–460.

[5] V. Benci, D. Fortunato, L. Pisani, Solitons in several space dimensions: Derrick's problem and infinitely many solutions, Arch. Ration. Mech. Anal. 154 (2000) 297–324.

[6] H. Brézis, Analyse Fonctionnelle – Théorie et Applications, Masson, Paris, 1992.

[7] F. Cammaroto, A. Chinnì, B. Di Bella, Infinitely many solutions for the Dirichlet problem involving the p-Laplacian, Nonlinear Anal. 61 (2005) 41–49.

[8] V. Coti Zelati, P.H. Rabinowitz, Homoclinic type solutions for a semilinear elliptic PDE on $\mathbb{R}^n$, Comm. Pure Appl. Math. 45 (1992) 1217–1269.

[9] Y. Ding, S. Luan, Multiple solutions for a class of nonlinear Schrödinger equations, J. Differential Equations 207 (2004) 423–457.

[10] P. Drábek, Y.X. Huang, Multiple positive solutions of quasilinear elliptic equations in $\mathbb{R}^n$, Nonlinear Anal. 37 (1999) 457–466.

[11] P. Drábek, C.G. Simader, Nonlinear eigenvalue problems for quasilinear equations in unbounded domains, Math. Nachr. 203 (1999) 5–30.

[12] B. Franchi, E. Lanconelli, J. Serrin, Existence and uniqueness of nonnegative solutions of quasilinear equations in $\mathbb{R}^n$, Adv. Math. 118 (1996) 177–243.

[13] F.A. van Heerden, Homoclinic solutions for a semilinear elliptic equation with an asymptotically linear nonlinearity, Calc. Var. Partial Differential Equations 20 (2004) 431–455.

[14] A. Kristály, Infinitely many solutions for a differential inclusion problem in $\mathbb{R}^N$, J. Differential Equations 220 (2006) 511–530.

[15] A. Kristály, Cs. Varga, On a class of quasilinear eigenvalue problems in $\mathbb{R}^N$, Math. Nachr. 278 (2005) 1756–1765.

[16] G. Li, S. Yan, Eigenvalue problems for quasilinear elliptic equations on $\mathbb{R}^N$, Comm. Partial Differential Equations 14 (1989) 1291–1314.

[17] P.-L. Lions, Symétrie et compacité dans les espaces de Sobolev, J. Funct. Anal. 49 (1982) 315–334.

[18] S.A. Marano, D. Motreanu, Infinitely many critical points of non-differentiable functions and applications to a Neumann-type problem involving the p-Laplacian, J. Differential Equations 182 (2002) 108–120.

[19] M. Marcus, V. Mizel, Every superposition operator mapping one Sobolev space into another is continuous, J. Funct. Anal. 33 (1979) 217–229.

[20] E. Montefusco, V. Rădulescu, Nonlinear eigenvalue problems for quasilinear operators on unbounded domains, NoDEA Nonlinear Differential Equations Appl. 8 (2001) 481–497.

[21] Z. Omari, F. Zanolin, Infinitely many solutions of a quasilinear elliptic problem with an oscillatory potential, Comm. Partial Differential Equations 21 (1996) 721–733.

[22] R. Palais, The principle of symmetric criticality, Comm. Math. Phys. 69 (1979) 19–30.

[23] B. Ricceri, Infinitely many solutions of the Neumann problem for elliptic equations involving the p-Laplacian, Bull. London Math. Soc. 33 (2001) 331–340.

[24] J. Saint Raymond, On the multiplicity of the solutions of the equation $-\Delta u = \lambda f(u)$, J. Differential Equations 180 (2002) 65–88.