Abstract

In this paper we study an elliptic problem in $\mathbb{R}^N$ which involves the $p$-Laplacian, $p > N \geq 2$, and the nonlinear term has an oscillatory behavior. By means of a direct variational approach, we establish the existence of infinitely many homoclinic solutions whose $W^{1,p}(\mathbb{R}^N)$-norms tend to zero (to infinity, respectively) whenever the nonlinearity oscillates at zero (at infinity, respectively). The solutions have invariance properties with respect to certain subgroups of the orthogonal group $O(N)$.

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1. Introduction

Various problems from Physics can be well modelled by means of certain equations which involve the $p$-Laplacian operator $\Delta_p$, $p > 1$. For instance, in Fluid Mechanics, the case $p = 2$ ($p > 2$ and $p < 2$, respectively) corresponds to a Newtonian (dilatant and pseudoplastic, respectively) fluid. In order to fix our ideas, let us consider the problem

$$
\begin{align*}
-\Delta_p u + |u|^{p-2}u &= \alpha(x)f(u), & x \in \mathbb{R}^N, \\
u &\in W^{1,p}(\mathbb{R}^N),
\end{align*}
$$

(P)

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where $1 < p < \infty$, $N \geq 2$, $\alpha : \mathbb{R}^N \to \mathbb{R}$ is measurable and $f : [0, \infty[ \to \mathbb{R}$ is a continuous function, $f(0) = 0$.

Problem (P) has been widely studied when $N \geq p$. In the semilinear case (i.e., $N \geq p = 2$), certain solitary waves in the nonlinear Klein–Gordon and Schrödinger equations are solutions of (P); existence and multiplicity of solutions can be found for instance in [1,3,4,8,9,13], and references therein. In the quasilinear case (i.e., $N \geq p \neq 2$), problem (P) was treated in [10,12,16]. The common feature of these papers is the superlinear or the asymptotical linear behavior at infinity of the nonlinear term $f$.

The aim of this paper is twofold. First, we want to handle the case when $p > N$. Although important problems can be treated within this framework (see, for instance [5], where a nonlinear field equation in Quantum Mechanics is considered involving the $p$-Laplacian, for $p = 6$), only a few works are available in this direction, see [11,14,15,20]. Second, instead of some usual assumption on the nonlinear term $f$, we assume that it oscillates at zero or at infinity. As an effect of the oscillatory behavior of $f$ one could expect the existence of infinitely many solutions of problem (P). Indeed, our main results (see Theorems 1.1 and 1.2 below) give sufficient conditions on the oscillatory terms such that problem (P) has infinitely many weak solutions. As a byproduct, these solutions can be constructed in such a way that their norms in $W$ tend to zero (to infinity, respectively) whenever the nonlinearity oscillates at zero (at infinity, respectively), which can be considered as another ‘reflection’ of the oscillation.

In order to describe precisely our results, we recall some basic concepts. The space $W^{1,p}(\mathbb{R}^N)$ is endowed with the standard norm $\|u\|_{W^{1,p}} = (\|\nabla u\|_p^p + \|u\|_p^p)^{1/p}$ where $\| \cdot \|_p$ is the usual norm in $L^p(\mathbb{R}^N)$, $1 < p < \infty$. The space $L^\infty(\mathbb{R}^N)$ is endowed with the usual sup-norm, denoted by $\| \cdot \|_\infty$. Let $N \geq 2$ and define the set

$$G_N = \{ G \subseteq O(N) : G = O(N_1) \times \cdots \times O(N_k), \ k \geq 1,$$

$$N_1 + \cdots + N_k = N, \ N_j \geq 2, \ j = 1, \ldots, k \}.$$

Fix a $G \in G_N$. A function $u : \mathbb{R}^N \to \mathbb{R}$ is called $G$-invariant if $u(gx) = u(x)$ for every $g \in G$ and $x \in \mathbb{R}^N$. In particular, an $O(N)$-invariant function is called radial.

Throughout the paper we assume that the potential $\alpha : \mathbb{R}^N \to \mathbb{R}$ appearing in problem (P) fulfills the following hypothesis:

$$(H) \ \alpha \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \text{ is radial, nonnegative, and } \|\alpha\|_\infty > 0.$$

Since we are interested in the case when $p > N$, Morrey’s embedding theorem implies that $W^{1,p}(\mathbb{R}^N)$ is continuously embedded into $L^\infty(\mathbb{R}^N)$, and one can consider continuous representations of the elements from $W^{1,p}(\mathbb{R}^N)$. Moreover, every element $u \in W^{1,p}(\mathbb{R}^N)$ is homoclinic, i.e., $u(x) \to 0$ as $|x| \to \infty$, see [6, p. 167]. Due to Morrey’s theorem and hypothesis $(H)$, the energy functional $E : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ associated with problem (P)

$$E(u) = \frac{1}{p} \|u\|_{W^{1,p}}^p - \int_{\mathbb{R}^N} \alpha(x) F(u(x)) \, dx, \quad u \in W^{1,p}(\mathbb{R}^N), \tag{1}$$

is well defined, where $F(s) = \int_0^s f(t) \, dt$, $s \in \mathbb{R}$. (Since $f(0) = 0$ we may assume that $f$ is extended to the whole real line with zero on $]-\infty, 0]$.)
Now we are in the position to state our first main result which deals with the case when the nonlinearity \( f \) exhibits an oscillation at the origin. More precisely, we assume

\[
(F_0) \quad -\infty < \liminf_{s \to 0^+} \frac{F(s)}{s^p} \leq \limsup_{s \to 0^+} \frac{F(s)}{s^p} = +\infty;
\]

\[ (S_0) \text{ there are two sequences } \{a_k\}, \{b_k\} \text{ such that } 0 < b_{k+1} < a_k < b_k, \lim_{k \to \infty} b_k = 0, \text{ and } f(s) \leq 0 \text{ for every } s \in [a_k, b_k], k \in \mathbb{N}. \]

**Theorem 1.1 (Oscillation at zero).** Let \( p > N \geq 2 \). Let \( \alpha : \mathbb{R}^N \to \mathbb{R} \) be a function which satisfies (H) and \( f : [0, \infty[ \to \mathbb{R} \) be a continuous function such that \((F_0)\), \((S_0)\) are fulfilled.

Then, for every \( G \in \mathcal{G}_N \) there exists a sequence \( \{u_k^G\} \subset W^{1,p}(\mathbb{R}^N) \) of nonnegative, \( G \)-invariant, homoclinic weak solutions of \((P)\) such that

\[
\lim_{k \to \infty} E(u_k^G) = 0 \quad \text{and} \quad \lim_{k \to \infty} \|u_k^G\|_{W^{1,p}} = 0. \tag{2}
\]

**Example 1.1.** Let \( p \) and \( N \) be as in Theorem 1.1. A simple function which satisfies \((F_0)\) and \((S_0)\) is \( f(s) = s^q \sin \frac{1}{s} \) for \( s > 0 \), and \( f(0) = 0 \), where \( \max\{p - 3, 0\} < q < p - 2 \).

Next, we will state the counterpart of Theorem 1.1 when the nonlinearity oscillates at infinity. First, we require similar assumptions as in \((F_0)\) and \((S_0)\), respectively:

\[
(F_\infty) \quad -\infty < \liminf_{s \to +\infty} \frac{F(s)}{s^p} \leq \limsup_{s \to +\infty} \frac{F(s)}{s^p} = +\infty;
\]

\[ (S_\infty) \text{ there are two sequences } \{a_k\}, \{b_k\} \text{ such that } 0 < a_k < b_k < a_{k+1}, \lim_{k \to \infty} b_k = +\infty, \text{ and } f(s) \leq 0 \text{ for every } s \in [a_k, b_k], k \in \mathbb{N}. \]

**Theorem 1.2 (Oscillation at infinity).** Let \( p > N \geq 2 \). Let \( \alpha : \mathbb{R}^N \to \mathbb{R} \) be a potential which satisfies (H) and \( f : [0, \infty[ \to \mathbb{R} \) be a continuous function such that \( f(0) = 0 \), and \((F_\infty)\), \((S_\infty)\) are fulfilled.

Then, for every \( G \in \mathcal{G}_N \) there exists a sequence \( \{u_k^G\} \subset W^{1,p}(\mathbb{R}^N) \) of nonnegative, \( G \)-invariant, homoclinic weak solutions of \((P)\) such that

\[
\lim_{k \to \infty} E(u_k^G) = -\infty \quad \text{and} \quad \lim_{k \to \infty} \|u_k^G\|_{W^{1,p}} = +\infty. \tag{3}
\]

**Example 1.2.** Let \( c \in ]0, 1[ \) and \( q \in ]p - 1, p[ \) be two fixed numbers. The function \( f(s) = s^q(c + \sin s) \) verifies hypotheses \((F_\infty)\) and \((S_\infty)\), respectively.

**Remark 1.1.** It is possible to handle the case when the nonlinear term \( f \) has discontinuities in Theorems 1.1 and 1.2; in such a case, a differential inclusion problem is formulated instead of \((P)\) in order to ‘fill the discontinuity gaps’ of \( f \), see [14].

**Remark 1.2.** In [14], the hypothesis \( \lim_{k \to \infty} \frac{b_k}{a_k} = +\infty \) was indispensable (where \( \{a_k\} \) and \( \{b_k\} \) are the sequences appearing in \((S_0)\) and \((S_\infty)\), respectively). The main advantage of the above results is that one can omit this inconvenient condition, allowing us to include new oscillatory nonlinearities within our framework. Furthermore, we are able to guarantee a sequence of \( G \)-invariant weak solutions of \((P)\) for every \( G \in \mathcal{G}_N \), and not only for \( G = O(N) \) as in [14].

Our approach is based on an elementary variational technic; in the sequel, we will describe it briefly here. By using Morrey’s embedding theorem and hypothesis (H), one can show by a
standard manner that the energy functional \( E \) is of class \( C^1 \) on \( W^{1,p}(\mathbb{R}^N) \) and its critical points are precisely the weak solutions of \((P)\), see [15]. It is well known that \( W^{1,p}(\mathbb{R}^N) \) cannot be compactly embedded into \( L^q(\mathbb{R}^N) \), \( q > 1 \), due to the unboundedness of the domain. However, by Lions theorem (see [17, Théorème III.3]), the fixed point space of \( W^{1,p}(\mathbb{R}^N) \) under the action of \( G \in \mathcal{G}_N \), denoted by \( W^{1,p}_G(\mathbb{R}^N) \), is compactly embedded into \( L^q(\mathbb{R}^N) \) whenever \( p < q < +\infty \).

The restriction of \( E \) to \( W^{1,p}_G(\mathbb{R}^N) \), denoted by \( E_G \), is weakly sequentially lower semicontinuous and its critical points are critical points of \( E \) as well, due to the principle of symmetric criticality of Palais (see [22, Theorem 5.4]). The crucial step in our arguments is the construction of an appropriate sequence of subsets of \( W^{1,p}_G(\mathbb{R}^N) \), proving that the relative minima of \( E_G \) on these sets are actually local minima (thus, critical points) of \( E_G \) on \( W^{1,p}_G(\mathbb{R}^N) \), so \( G \)-invariant weak solutions of \((P)\). Then, a suitable subsequence of critical points of \( E_G \) can be extracted from the aforementioned local minima of \( E_G \) having the properties (2) and (3), respectively. We emphasize that the crucial step described above can be achieved only in the case when \( p > N \) (i.e., when one can apply Morrey’s embedding theorem); in the case \( p \leq N \), a new method should be elaborated in order to obtain similar results as Theorems 1.1 and 1.2, respectively.

Our results complement not only the aforementioned papers ([1,3,4,8,9,13], where the case \( p \leq N \) has been treated) but also some results obtained on bounded domains where elliptic problems with oscillatory nonlinearities have been considered; Dirichlet problems were studied in [2,7,21,24], while Neumann type problems in [18,23]. The common feature of these works is that infinitely many solutions are obtained by means of various methods; for instance, sub–super solution arguments (see [21]; the general variational principle of Ricceri (see [7,18,23]); continuity of certain superposition operators (see [2,24]). Note however that our proofs do not use any abstract argument apart from the compactness result of Lions [17] and the Palais’ principle [22].

2. Variational setting

First of all, note that assumption \((S_0)\) implies that \( f(0) \leq 0 \). Consequently, \( f(0) = 0 \), due to the left inequality of \((F_0)\). Therefore, in both cases (i.e., in Theorems 1.1 and 1.2) we may extend continuously the function \( f \) to the whole real line with value zero on the interval \([-\infty,0] \).

Note that the functional \( E \), defined in (1), is of class \( C^1 \) on \( W^{1,p}(\mathbb{R}^N) \), see [15, Proposition 2.1]. Moreover, its critical points are precisely the weak solutions of \((P)\). Thus, in order to prove the theorem, it is enough to find a sequence of distinct critical points of \( E \) with the required properties.

To do this, fix \( G = O(N_1) \times \cdots \times O(N_k) \in \mathcal{G}_N \). The action of \( G \) on the space \( W^{1,p}(\mathbb{R}^N) \) will be defined by

\[(gu)(x) = u(g_1^{-1}x_1, \ldots, g_k^{-1}x_k)\]

for every \( u \in W^{1,p}(\mathbb{R}^N) \), \( g = (g_1, \ldots, g_k) \), \( g_j \in O(N_j) \), \( x = (x_1, \ldots, x_k) \), \( x_j \in \mathbb{R}^{N_j} \), \( j = 1, \ldots, k \). Let us denote by \( E_G \) the restriction of the energy functional \( E \) to the subspace of \( G \)-invariant functions of \( W^{1,p}(\mathbb{R}^N) \), i.e.

\[W^{1,p}_G(\mathbb{R}^N) \overset{\text{def}}{=} \{ u \in W^{1,p}(\mathbb{R}^N) : u(g_1x_1, \ldots, g_kx_k) = u(x_1, \ldots, x_k) \text{ for all } g_j \in O(N_j), x_j \in \mathbb{R}^{N_j}, j = 1, \ldots, k \} \text{.}\]
Since $\alpha$ is radial, the functional $E$ is $G$-invariant. Due to the principle of symmetric criticality of Palais, the critical points of $E_G$ are critical points of $E$ as well. (Note that the fixed point space of the action $G$ on the space $W^{1,p}(\mathbb{R}^N)$ is exactly $W^{1,p}_G(\mathbb{R}^N)$.)

**Proposition 2.1.** Functional $E_G$ is sequentially weakly lower semicontinuous on $W^{1,p}_G(\mathbb{R}^N)$.

**Proof.** The function $\|\cdot\|_{W^{1,p}}^p$ is clearly sequentially weakly lower semicontinuous on $W^{1,p}_G(\mathbb{R}^N)$, see [6, p. 35]. Let us prove that the function $F: W^{1,p}_G(\mathbb{R}^N) \rightarrow \mathbb{R}$, defined by $F(u) = \int_{\mathbb{R}^N} \alpha(x) F(u(x)) \, dx$, is sequentially weakly continuous.

Suppose the contrary, i.e., let $\{u_n\} \subset W^{1,p}_G(\mathbb{R}^N)$ be a sequence which converges weakly to $u \in W^{1,p}_G(\mathbb{R}^N)$ but $F_G(u_n) \nrightarrow F_G(u)$ as $n \rightarrow \infty$. Therefore, up to a subsequence, one can find a number $\varepsilon_0 > 0$ such that

$$0 < \varepsilon_0 \leq |F_G(u_n) - F_G(u)|$$

for every $n \in \mathbb{N}$, and by Lions theorem, $u_n$ converges strongly to $u$ in $L^q(\mathbb{R}^N)$, for some fixed $q \in ]p, +\infty[$. For every $n \in \mathbb{N}$ one has $0 < \theta_n < 1$ such that

$$0 < \varepsilon_0 \leq \left| \int_{\mathbb{R}^N} \alpha(x) F(u + \theta_n(u_n - u)) \, dx \right| \leq \|\alpha\| \max\{r, b_k\} \|u_n - u\|_q,$$

where $q' = q(q - 1)^{-1}$ and $M_n = \|u\|_\infty + \|u_n\|_\infty$. Note that $\sup_{n \in \mathbb{N}} M_n < +\infty$; indeed, $W^{1,p}_G(\mathbb{R}^N)$ is continuously embedded into $L^\infty(\mathbb{R}^N)$. Letting $n \rightarrow \infty$, in the above relation the right-hand side tends to 0, a contradiction. \[\square\]

Now, we are going to prove our main results.

### 3. Proof of Theorem 1.1

Fix a number $r < 0$ and define the set

$$W_k = \{ u \in W^{1,p}_G(\mathbb{R}^N) : r \leq u(x) \leq b_k \text{ for every } x \in \mathbb{R}^N \},$$

where the sequence $\{b_k\}$ appears in $(S_0)$.

**Claim 3.1.** Functional $E_G$ is bounded from below on $W_k$ and its infimum on $W_k$ is attained.

**Proof.** The set $W_k$ is convex. Moreover, it is closed in $W^{1,p}_G(\mathbb{R}^N)$ due to the continuity of the embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$. Consequently, the set $W_k$ is weakly closed. Moreover,

$$E_G(u) = \frac{1}{p} \|u\|_{W^{1,p}}^p - \int_{\mathbb{R}^N} \alpha(x) F(u(x)) \, dx \geq -\|\alpha\|_1 \max_{r, b_k} F \quad \text{for } u \in W_k.$$
Thus, \( E_G \) is bounded from below on \( W_k \). Let \( \eta_k = \inf_{W_k} E_G \), and \( \{u_n\} \) be a sequence in \( W_k \) such that \( \eta_k \leq E_G(u_n) \leq \eta_k + 1/n \) for all \( n \in \mathbb{N} \). Then,

\[
\frac{1}{p} \|u_n\|_{W^{1,p}}^p \leq \eta_k + 1 + \|\alpha\|_1 \max_{[r,bk]} F
\]

for all \( n \in \mathbb{N} \), i.e. \( \{u_n\} \) is bounded in \( W^{1,p}_G(\mathbb{R}^N) \). So, up to a subsequence, \( \{u_n\} \) weakly converges in \( W^{1,p}_G(\mathbb{R}^N) \) to some \( u^G_k \in W_k \). By the sequentially weakly lower semicontinuity of \( E_G \), cf. Proposition 2.1, we conclude that \( E_G(u^G_k) = \eta_k = \inf_{W_k} E_G \).

**Claim 3.2.** Let \( u^G_k \in W_k \) be such that \( E_G(u^G_k) = \inf_{W_k} E_G \). Then, \( 0 \leq u^G_k(x) \leq a_k \) for all \( x \in \mathbb{R}^N \).

**Proof.** Let \( A = \{x \in \mathbb{R}^N : u^G_k(x) \notin [0,a_k]\} \) and suppose that \( A \neq \emptyset \). Thus, \( \text{meas}(A) > 0 \) due to the continuity of \( u^G_k \). Define

\[
h(s) = \begin{cases} 
0, & \text{if } s < 0; \\
, & \text{if } s \in [0,a_k]; \\
ak, & \text{if } s > a_k.
\end{cases}
\]

Set \( v_k = h \circ u^G_k \). Since \( h \) is uniformly Lipschitz and \( h(0) = 0 \), \( v_k \) belongs to \( W^{1,p}(\mathbb{R}^N) \), cf. [19] or [24, Lemma 3.4]. Moreover, \( v_k \) is \( G \)-invariant because of \( u^G_k \); thus \( v_k \in W^{1,p}_G(\mathbb{R}^N) \). In addition, \( v_k \in W_k \). Denoting by

\[
A_1 = \{x \in A : u^G_k(x) < 0\} \quad \text{and} \quad A_2 = \{x \in A : u^G_k(x) > a_k\},
\]

we have that \( v_k(x) = u^G_k(x) \) for all \( x \in \mathbb{R}^N \setminus A \), \( v_k(x) = 0 \) for all \( x \in A_1 \) and \( v_k(x) = a_k \) for all \( x \in A_2 \). Then,

\[
E_G(v_k) - E_G(u^G_k) = -\frac{1}{p} \int_A |\nabla u^G_k|^p + \frac{1}{p} \int_A [|v_k|^p - |u^G_k|^p] - \int_A \alpha(x)[F(v_k) - F(u^G_k)]
\]

\[
= -\frac{1}{p} \int_A |\nabla u^G_k|^p - \frac{1}{p} \int_{A_1} |u^G_k|^p + \frac{1}{p} \int_{A_2} [a_k^p - (u^G_k)^p]
\]

\[
- \int_{A_2} \alpha(x)[F(a_k) - F(u^G_k)].
\]

By \((S_0)\), one has that \( F(s) \leq F(a_k) \) for every \( s \in [a_k,b_k] \). Using this fact, we observe that every term of the right-hand side of the above expression is non-positive. On the other hand, since \( E_G(v_k) \geq E_G(u^G_k) = \inf_{W_k} E_G \), then in particular,

\[
\int_A |\nabla u^G_k|^p = 0,
\]
By the first equality we obtain the existence of a positive measured subset \( B \) of \( A \) and a constant \( M \in \mathbb{R} \) such that \( u_k^G = M \) on the set \( B \). Then, either \( B \subset A_1 \) or \( B \subset A_2 \). If \( B \subset A_1 \), then

\[
0 = \int_{A_1} |u_k^G|^p \geq \int_B |u_k^G|^p = |M|^p \text{meas}(B) > 0,
\]
a contradiction. If \( B \subset A_2 \), then

\[
0 = \int_{A_2} \left[ a_k^p - (u_k^G)^p \right] \leq \int_B \left[ a_k^p - (u_k^G)^p \right] = \left[ a_k^p - M^p \right] \text{meas}(B) < 0,
\]
a contradiction. This shows that \( A \) has zero measure, therefore, \( A = \emptyset \). \( \square \)

**Claim 3.3.** Let \( u_k^G \in W_k \) be such that \( \mathcal{E}_G(u_k^G) = \inf_{W_k} \mathcal{E}_G \). Then \( u_k^G \) is a local minimum point of \( \mathcal{E}_G \) in \( W_{1,p}(\mathbb{R}^N) \).

**Proof.** Indeed, otherwise there would be a sequence \( \{u_n\} \subset W_{1,p}(\mathbb{R}^N) \) which converges to \( u_k^G \) and \( \mathcal{E}_G(u_n) < \mathcal{E}_G(u_k^G) = \inf_{W_k} \mathcal{E}_G \) for all \( n \in \mathbb{N} \). From this inequality it follows that \( u_n \notin W_k \) for any \( n \in \mathbb{N} \). Since \( u_n \to u_k^G \) in \( W_{1,p}(\mathbb{R}^N) \), then due to Morrey’s theorem, \( u_n \to u_k^G \) in \( L^\infty(\mathbb{R}^N) \) as well. In particular, for every \( 0 < \delta < \min\{-r, b_k - a_k\}/2 \), there exists \( n_\delta \in \mathbb{N} \) such that \( \|u_n - u_k^G\|_\infty < \delta \) for every \( n \geq n_\delta \). By using Claim 3.2 and taking into account the choice of the number \( \delta \), we conclude that

\[
r < u_n(x) < b_k \text{ for all } x \in \mathbb{R}^N, \ n \geq n_\delta,
\]
which clearly contradicts the fact \( u_n \notin W_k \). \( \square \)

**Claim 3.4.** Let \( \eta_k = \inf_{W_k} \mathcal{E}_G = \mathcal{E}_G(u_k^G) \). Then \( \eta_k < 0 \) for all \( k \in \mathbb{N} \) and \( \lim_{k \to \infty} \eta_k = 0 \).

**Proof.** Since \( \|\alpha\|_\infty > 0 \), then for every number \( 0 < \theta < 1 \) one can find a measurable set \( S_\theta \) with positive measure such that \( \alpha(x) > \theta \|\alpha\|_\infty \) for a.e. \( x \in S_\theta \). Moreover, since \( \alpha \) is radial, one can assume that \( S_\theta \) is \( O(N) \)-invariant, i.e., \( gS_\theta = S_\theta \) for every \( g \in O(N) \). For simplicity, let us fix \( \theta = 1/p \), and denote \( S_0 = S_{1/p} \). Then, one can find \( x_0 \in \mathbb{R}^N \) and \( \mu_0 > 0 \), with \( \mu_0 < |x_0| \) whenever \( x_0 \neq 0 \), such that

\[
\text{meas}\{S_0 \setminus \{x \in \mathbb{R}^N: |x| - |x_0| \leq \mu_0/2\}\} = 0.
\]

Define for every \( s > 0 \) the function \( w_s: \mathbb{R}^N \to \mathbb{R} \) by

\[
w_s(x) = \begin{cases} 0, & \text{if } |x| - |x_0| \geq \mu_0; \\ s, & \text{if } |x| - |x_0| \leq \mu_0/2; \\ \frac{2s}{\mu_0} (\mu_0 - |x| - |x_0|), & \text{if } \mu_0/2 < |x| - |x_0| < \mu_0. \end{cases}
\]
It is clear that \( w_s \in W^{1,p}(\mathbb{R}^N) \) and it is radial, thus in particular, \( w_s \in W^{1,p}_G(\mathbb{R}^N) \). Moreover, a simple estimation shows that

\[
\|w_s\|_{W^{1,p}}^p \leq s^p\left(1 + \frac{2p}{\mu_0}\right)\text{meas}\{x \in \mathbb{R}^N : |x| - |x_0| \leq \mu_0\} =: s^pC(x_0, \mu_0).
\]

By \((F_0)\) there exist two positive numbers \( l \) and \( \varrho \) such that \( F(s) > -ls^p \) for every \( s \in [0, \varrho[. \) Let \( L > 0 \) be such that

\[
L\|\alpha\|_\infty \text{meas } S_0 - l\|\alpha\|_1 p - C(x_0, \mu_0) > 0. \tag{5}
\]

By using the right-hand side of \((F_0)\), there exists a sequence \( \{s_k\} \subset ]0, \varrho[\) converging to zero, and \( F(s_k) > ls_k^p \). Let \( \{s_k\} \) be a decreasing subsequence of \( \{s_k\} \) such that \( s_k \leq b_k \) for all \( k \in \mathbb{N} \) and \( w_k \equiv w_{s_k} \) as in \((4)\). It is clear that \( w_k \in W_k \). Moreover, due to \((5)\), one has

\[
\mathcal{E}_G(w_k) \leq s_k^p\left(\frac{1}{p}C(x_0, \mu_0) + l\|\alpha\|_1 - \frac{1}{p}L\|\alpha\|_\infty \text{meas } S_0\right) < 0.
\]

Thus, \( \eta_k = \inf_{W_k} \mathcal{E}_G \leq \mathcal{E}_G(w_k) < 0 \).

Now we will prove that \( \eta_k \to 0 \) as \( k \to \infty \). Due to Claim 3.2, for every \( x \in \mathbb{R}^N \) one has

\[
|F(u_k^G(x))| \leq \int_0^{a_k} |f| \leq a_k \max_{[0, a_k]} |f| \leq a_k \max_{[0, a_1]} |f|. \tag{6}
\]

Then

\[
0 > \eta_k = \mathcal{E}_G(u_k^G) \geq -\int_{\mathbb{R}^N} \alpha(x)F(u_k^G) \geq -\|\alpha\|_1 \max_{[0, a_1]} |f| a_k.
\]

Since the sequence \( \{a_k\} \) tends to zero, then \( \eta_k \to 0 \) as \( k \to \infty \). \( \square \)

**Proof of Theorem 1.1 concluded.** Since \( u_k^G \) are local minima of \( \mathcal{E}_G \) (cf. Claim 3.3), they are critical points of \( \mathcal{E}_G \), thus \( G \)-invariant weak solutions of \((P)\). Due to Claim 3.4, there are infinitely many distinct \( u_k^G \). Moreover, due to \((6)\), we have

\[
\frac{1}{p}\|u_k^G\|_{W^{1,p}}^p = \int_{\mathbb{R}^N} \alpha(x)F(u_k^G) + \eta_k \leq \|\alpha\|_1 \max_{[0, a_1]} |f| a_k,
\]

which proves that \( \|u_k^G\|_{W^{1,p}} \to 0 \) as \( k \to \infty \). \( \square \)
4. Proof of Theorem 1.2

Let \( r < 0 \) and define similarly as in Section 3 the set

\[
W_k = \{ u \in W^{1,p}_G : r \leq u(x) \leq b_k \text{ for every } x \in \mathbb{R}^N \}.
\]

The first part of the proof is similar to that of Theorem 1.1. Indeed, we can prove that the functional \( E_G \) is bounded from below on \( W_k \) and its infimum on \( W_k \) is attained (see Claim 3.1). Moreover, if \( u^G_k \in W_k \) is chosen such that \( E_G(u^G_k) = \inf_{W_k} E_G \), then \( 0 \leq u^G_k(x) \leq a_k \) for all \( x \in \mathbb{R}^N \) (see Claim 3.2), and \( u^G_k \) is a local minimum point of \( E_G \) in \( W^{1,p}_G(\mathbb{R}^N) \) (see Claim 3.3). Instead of Claim 3.4, we prove Claim 4.1.

**Claim 4.1.** Let \( \delta_k = \inf_{W_k} E_G = E_G(u^G_k) \). Then \( \lim_{k \to \infty} \delta_k = -\infty \).

**Proof.** By \((F_\infty)\) there exist two positive numbers \( l \) and \( \varrho \) such that \( F(s) > -ls^p \) for every \( s > \varrho \).

Let \( L > 0 \) be such that

\[
L\|\alpha\|_\infty \operatorname{meas} S_0 - l\|\alpha\|_1 p - C(x_0, \mu_0) > 0,
\]

where \( S_0, \mu_0, x_0 \) and \( C(x_0, \mu_0) \) are from the previous section. Due to the right part of \((F_\infty)\), there exists a sequence \( \{s_k\} \) which tends to \(+\infty\) such that \( F(s_k) > Ls_k^p \). Let \( \{b_{l_k}\} \) be an increasing subsequence of \( \{b_k\} \) such that \( s_k \leq b_{l_k} \) for all \( k \in \mathbb{N} \) and \( w_k = w_{s_k} \) as in (4). It is clear that \( w_k \in W_k \). Moreover, one can deduce that

\[
E_G(w_k) \leq s_k^p \left( \frac{1}{p} C(x_0, \mu_0) + l\|\alpha\|_1 - \frac{1}{p} L\|\alpha\|_\infty \operatorname{meas} S_0 \right) + \|\alpha\|_1 \max_{[0, \varrho]} |F|.
\]

The term near \( s_k^p \) in the above estimation is strictly negative due to (7), which proves that \( \delta_k = \inf_{W_k} E_G \leq E_G(w_k) \to -\infty \) as \( k \to \infty \). Since the sequence \( \{\delta_k\} \) is non-increasing, we are done. \( \Box \)

**Proof of Theorem 1.2 concluded.** It is clear that we have infinitely many pairwise distinct local minimum points \( u^G_k \) of \( E_G \) with \( u^G_k \in W_k \). Now we will prove that \( \|u^G_k\|_{W^{1,p}_G} \to +\infty \) as \( k \to \infty \).

Let us assume the contrary. Therefore, there is a subsequence \( \{u^G_{n_k}\} \) of \( \{u^G_k\} \) which is bounded in \( W^{1,p}_G(\mathbb{R}^N) \). Thus, it is also bounded in \( L^\infty(\mathbb{R}^N) \). In particular we can find \( m_0 \in \mathbb{N} \) such that \( u^G_{n_k} \in W_{m_0} \) for all \( k \in \mathbb{N} \). For every \( n_k \geq m_0 \) one has

\[
\delta_{m_0} \geq \delta_{n_k} = \inf_{W_{n_k}} E_G = E_G(u^G_{n_k}) \geq \inf_{W_{m_0}} E_G = \delta_{m_0},
\]

which proves that \( \delta_{n_k} = \delta_{m_0} \) for all \( n_k \geq m_0 \), contradicting Claim 4.1. This concludes our proof. \( \Box \)
References


