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ONE-DIMENSIONAL SCALAR FIELD EQUATIONS INVOLVING AN OSCILLATORY NONLINEAR TERM

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Abstract. In this paper we study the equation $-u''+V(x)u = W(x)f(u)$, $x \in \mathbb{R}$, where the nonlinear term f has certain oscillatory behaviour. Via two different variational arguments, we show the existence of infinitely many homoclinic solutions whose norms in an appropriate functional space which involves the potential V tend to zero (resp. at infinity) whenever f oscillates at zero (resp. at infinity). Unlike in classical results, neither symmetry property on f nor periodicity on the potentials V and W are required.

1. **Introduction.** In this paper we study the existence of multiple solutions for the one-dimensional scalar field equation

$$\begin{cases} -u'' + V(x)u = W(x)f(u), & x \in \mathbb{R} \\ u(x) \to 0 \quad \text{as} \quad |x| \to +\infty, \end{cases} \tag{P}$$

where V, W are positive potentials and $f : \mathbb{R} \to \mathbb{R}$ is continuous. It is well-known that certain kinds of solitary waves in nonlinear Klein-Gordon or Schrödinger equations are solutions of (P).

The aim of our paper is to ensure the existence of infinitely many weak solutions of (P) when the nonlinear term f has certain *oscillatory* behaviour and no kind of symmetry. It has been long known that oscillatory nonlinearities can yield infinitely many solutions for Dirichlet problems on *bounded domain*s; see Omari and Zanolin [15], Saint Raymond [18]. However, our study can be fit within the problem raised by Berestycki and Lions [5], who proposed to find classes of *non*-odd nonlinearities guaranteeing the existence of infinitely many solutions for certain nonlinear scalar field equations. Various contributions to this problem can be found in the literature. These results and the techniques employed strongly depend not only on the nonlinearities but also on the potentials and on the space dimension N; for instance, Jones and Küpper [11] required the nonlinear term to be smooth enough and to

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behave like $|s|^\sigma s$ near $\pm\infty$ with $\sigma < 4/(N-2)$, $N > 2$; Coti Zelati and Rabinowitz [8] used a periodicity assumption on the potentials ($N \geq 1$); Bartsch and Willem [4] assumed some sort of convexity on the nonlinearity ($N \geq 2$).

Besides the nonlinearities from the aforementioned papers ([4], [8], [11]), we give a new class of non-odd nonlinearities which can not be recovered by earlier results, offering another contribution to the question raised in [5]. Note, however, that our arguments work only in the case when the space dimension is *one*. This latter fact could seem to be in contrast with the classical literature; indeed, unlike in the one-dimensional case, certain constructions based on (radial) symmetries can be successfully employed in higher dimensional semilinear elliptic problems which reduce the complexity of the studied problem (see, e.g., Bartsch and Willem [4], Conti, Merizzi and Terracini [7], Jones and Küpper [11], Strauss [20]).

Special forms of (P) have been widely investigated in the literature. When $V \equiv 0$ and $W \equiv 1$, Berestycki and Lions [5, Section 6] established a necessary and sufficient condition for the existence of solutions $u \in C^2$ of (P), proving as well the uniqueness of solutions up to translations. Floer and Weinstein [10] studied the existence and behaviour of the solutions of the Gross-Pitaevskii equation, i.e., problem (P) with bounded potential V and cubic nonlinearity f. In the case where f is of pure power type, $V(x) = -p(x) - \lambda$ (with p even and $\lim_{x\to\infty} p(x) = 0$), and $W \equiv 1$, McLeod, Stuart and Troy [14] studied the monotonicity with respect to λ of the L^2-norm of positive solutions for (P). When the potential V is coercive and f satisfies standard mountain-pass assumptions, Rabinowitz [16] proved the existence of a nontrivial solution for (P) (not only in the one dimensional case).

Recently, Bartsch, Pankov and Wang [2] introduced a more general condition on the potential V than the one used by Rabinowitz [16] (see also Bartsch and Wang [3]), namely

($V1$) $V \in L^\infty_{loc}(\mathbb{R})$, $V_0 = \text{essinf}_{\mathbb{R}} V > 0$, and
for any $M > 0$ and any $r > 0$ there holds:

$$meas(\{x \in \mathbb{R} : |x-y| < r,\ V(x) \leq M\}) \to 0 \text{ as } |y| \to +\infty,$$

where $'meas'$ denotes the Lebesgue measure in $\mathbb{R}$.

Under this condition, Bartsch, Liu and Weth [1] proved the existence of an infinite number of sign changing solutions for (P) when f is odd and superlinear. The functional space in [1] is defined as the Hilbert space

$$H_V = \left\{ u \in H^1(\mathbb{R}) : \int_{\mathbb{R}} V(x)u^2 < +\infty \right\},$$

endowed with the inner product $\langle u, v\rangle_V = \int_{\mathbb{R}} (u'v' + V(x)uv)$ for each $u, v \in H_V$.

In the present paper we assume that the potential V fulfills condition ($V1$) while solutions of (P) are being sought in the space H_V. In spite of the fact that the class of potentials which satisfy ($V1$) is large, we emphasize that our intent is not to find/use the most general condition on V in order to study problem (P); assuming ($V1$) on V, our attention will be focused on certain unusual nonlinearities.

As we pointed out before, our nonlinearity $f : \mathbb{R} \to \mathbb{R}$ has certain oscillatory behaviour. Note that for the most familiar oscillatory function $f(s) = \sin s$, problem (P) has only the trivial solution provided that W is suitable small (see Remark 5). This simple example shows that a careful analysis of the oscillatory functions is needed in order to obtain nonzero/multiple solutions for (P). Two cases will be considered:

- f oscillates at 0^+;
- f oscillates at $+\infty$.

In order to describe our results, let (formally) $l = 0^+$, or $l = +\infty$. We will assume:

(S_l) there are two sequences $\{a_k\}$, $\{b_k\}$ in $]0, \infty[$ with $a_k < b_k$ and $b_k \to l$ such that $f(s) \leq 0$ for every $s \in [a_k, b_k]$, and

(F_l) $-\infty < \liminf_{s\to l} \frac{F(s)}{s^2} \leq \limsup_{s\to l} \frac{F(s)}{s^2} = +\infty$,

where $F(s) = \int_0^s f(t)dt$.

Under these conditions, two different kinds of results will be proved in both cases (i.e., when f oscillates at $l = 0^+$, or $l = +\infty$), where we guarantee the existence of infinitely many solutions of (P) whose H_V-norms converge to l. In the first type of problem (see Theorems 2.1 and 2.3), besides some technical assumptions, we consider the case when the function F is *almost even*, i.e.,

($*$) $F(-s) \leq F(s)$ for every $s \geq 0$.

(Note that if f is *almost odd*, i.e., $-f(-s) \leq f(s)$ for every $s \geq 0$, then ($*$) is fulfilled.) The second type of problem (see Theorems 2.2 and 2.4) faces the case when we avoid completely condition ($*$) but f fulfils certain *sign condition* on $\mathbb{R}^-$. The two types of results are independent, which is shown by some concrete examples.

Our approach is variational; solutions of (P) will be obtained as local minima of the energy functional associated to (P). In the first type of problem, a novel variational principle of Ricceri [17] will be used. In the proofs (of Theorems 2.1 and 2.3) we exploit this principle, which actually gives alternatives to find critical points of certain functionals: either one local minimum or infinitely many critical points. In order to handle the second type of problem, the technique of our proofs is suggested by an idea of Saint Raymond [18]. Precisely, we construct a sequence of subsets in $L^\infty(\mathbb{R})$ such that the *relative* minima of the energy on these sets are actually *local* minima for the energy on the space H_V.

To treat a closely related problem to (P) in $\mathbb{R}^N$ involving the p-Laplacian ($p > N \geq 2$), Kristály [12] recently applied Ricceri's principle, obtaining infinitely many radially symmetric solutions. In [12] the principle of symmetric criticality, as well as the construction of the space of radially symmetric functions, played an indispensable role, due to the higher space-dimension. As we already pointed out, in the one-dimensional case this approach fails. In certain sense, hypothesis ($V1$) on the potential V is devoted to fill this gap. Nevertheless, we believe that by means of a suitable adaptation, other classes of potentials can be considered instead of those which satisfy ($V1$); for instance singular potentials (see Gomes and Sanchez [9]), not necessarily positive potentials (see Sirakov [19]).

In the next section we will state the precise form of our results and give some concrete examples. In the third section we present some auxiliary results, while Sections 4 and 5 are devoted to the proof of our theorems.

2. **Main results.** Let $V : \mathbb{R} \to \mathbb{R}$ satisfying ($V1$), and let the Hilbert space $(H_V, \langle\cdot,\cdot\rangle_V)$ be as in the Introduction. The induced norm will be denoted by $\|\cdot\|_V$. Due to Morrey's Theorem, the embedding $H_V \subset H^1(\mathbb{R}) \equiv W^{1,2}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ is continuous. Moreover, the embedding $H_V \hookrightarrow L^2(\mathbb{R})$ is compact, cf. Bartsch, Pankov and Wang [2]. In the sequel, we denote by $\kappa_\infty > 0$ the best Sobolev embedding constant for $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$; $\|\cdot\|_p$ denotes the usual norm of $L^p(\mathbb{R})$, $p \in [1, \infty]$.

On the potential $W : \mathbb{R} \to \mathbb{R}$ we will assume the following condition:

$(W1)$ $W \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}),\ W \geq 0,\ \|W\|_\infty > 0.$

Two cases will be considered: the nonlinear term $f : \mathbb{R} \to \mathbb{R}$ has oscillation at 0^+ and at $+\infty$, respectively.

2.1. **Oscillation at** 0^+**.** In this subsection, on the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ we will assume that:

(S_{0^+}) there exist two sequences $\{a_k\}$ and $\{b_k\}$ in $]0,\infty[$ with $b_{k+1} < a_k < b_k$, $\lim_{k\to\infty} b_k = 0$ such that

$$f(s) \leq 0 \quad \text{for every } s \in [a_k, b_k], \text{ and}$$

$$(F_{0^+}) \qquad -\infty < \liminf_{s\to 0^+} \frac{F(s)}{s^2} \leq \limsup_{s\to 0^+} \frac{F(s)}{s^2} = +\infty.$$

Recall that $F(s) = \int_0^s f(t)dt$.

Remark 1. Note that assumption (S_{0^+}) is fulfilled, for instance, if

$$\inf\{s > 0 : f(s) < 0\} = 0.$$

Theorem 2.1. *Assume that* $V,\ W : \mathbb{R} \to \mathbb{R}$ *satisfy* $(V1)$ *and* $(W1)$, *respectively, and let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function such that* (S_{0^+}) *and* (F_{0^+}) *are fulfilled with*

(i) $F(-s) \leq F(s)$ *for every* $s \geq 0$.

Assume, in addition, that the sequences $\{a_k\}$ *and* $\{b_k\}$ *from* (S_{0^+}) *satisfy*

(ii) $\lim_{k\to\infty} \frac{a_k}{b_k} = 0$ *and* $\lim_{k\to\infty} \frac{\max_{[0,a_k]} F}{b_k^2} = 0$.

Then, problem (P) *possesses a sequence* $\{u_k\}$ *of weak solutions which satisfy* $\lim_{k\to\infty} \|u_k\|_V = 0$. *In particular,* $\lim_{k\to\infty} \|u_k\|_\infty = 0$.

Remark 2. a.) An easy calculation shows that if f is almost odd, i.e., $-f(-s) \leq f(s)$ for every $s \geq 0$, then (i) is fulfilled. b.) The technical condition (ii) is dispensable in our argument; it is an open question whether it could be removed, keeping only condition (i).

Theorem 2.2. *Assume that* $V,\ W : \mathbb{R} \to \mathbb{R}$ *satisfy* $(V1)$ *and* $(W1)$, *respectively. Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function such that* (S_{0^+}) *and* (F_{0^+}) *are fulfilled and*

(iii) *either* $\sup\{s < 0 :\ f(s) > 0\} = 0$, *or there is a* $\delta > 0$ *such that* $f|_{[-\delta,0]} \equiv 0$.

Then, the conclusions of Theorem 2.1 hold.

Example 1. a.) Let $\{a_k\}$ and $\{b_k\}$ be two sequences in $]0,\infty[$ with $b_{k+1} < a_k < b_k$ and $\lim_{k\to\infty} a_k/b_k = 0$, $\lim_{k\to\infty} b_k = 0$. We introduce the set $A = \{r \in [2,\infty[: \lim_{k\to\infty} a_k^2/b_k^r = 0\}$. Clearly, $2 \in A$. Suppose that $A \neq \{2\}$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(s) = \begin{cases} \frac{1}{2}((1-c)\operatorname{sgn}(s) + c + 1)\varphi_k\left(\frac{|s|-b_{k+1}}{a_k - b_{k+1}}\right), & |s| \in [b_{k+1}, a_k]; \\ 0, & |s| \notin [b_{k+1}, a_k], \end{cases}$$

where $-1 < c < 0$ and $\varphi_k : [0,1] \to [0,\infty[$ is a sequence of continuous functions such that $\varphi_k(0) = \varphi_k(1) = 0$, and there are some positive constants c_1 and c_2 such that

$$c_1(b_k^{r_1} - b_{k+1}^{r_1})(a_k - b_{k+1})^{-1} \leq \int_0^1 \varphi_k(s)ds \leq c_2(b_k^{r_2} - b_{k+1}^{r_2})(a_k - b_{k+1})^{-1}$$

for some $r_1, r_2 \in A \setminus \{2\}$. Now, we can apply Theorem 2.1. Indeed, $-f(-s) \leq f(s)$ for every $s \geq 0$; thus (i) is satisfied (see Remark 2 a.)). Moreover, F is non-decreasing on $\mathbb{R}^+$, while $c_1 b_k^{r_1} \leq F(a_k) = \max_{[0,a_k]} F \leq c_2 b_k^{r_2}$; i.e., hypotheses of Theorem 2.1 are fulfilled. However, Theorem 2.2 cannot be applied because (iii) is not satisfied.

b.) Sequences in case a.) can be chosen as $a_k = k^{-k^{k+1}}$ and $b_k = k^{-k^k}$. In this case $A = [2, \infty[$.

Example 2. Let $0 < \alpha < 1 < \beta$ and $f : \mathbb{R} \to \mathbb{R}$ be such that $f(0) = 0$ and $f(s) = |s|^\alpha \max\{0, \sin |s|^{-1}\} + |s|^\beta \min\{0, \sin |s|^{-1}\}$ for $s \neq 0$. Now, we can apply Theorem 2.2 (but not Theorem 2.1).

Remark 3. The weak solutions of (P) are actually the fixed points of the operator $A : H_V \to H_V$, $A(u)(v) = \int_{\mathbb{R}} W(x) f(u) v$. Let $f(s) = s^3 \sin s^{-1}$ if $s \in [-\pi^{-1}, \pi^{-1}] \setminus \{0\}$; $f(s) = 0$ otherwise. In this case, the operator A is uniformly Lipschitz; if $\|W\|_1 < \kappa_\infty^{-2} \min\{1, V_0\}(3\pi^{-2} + \pi^{-1})^{-1}$, then A becomes a contraction, and thus (P) admits only the trivial solution. We notice that f satisfies the assumptions of Theorem 2.2 except hypothesis (F_{0^+}).

2.2. **Oscillation at $+\infty$.** In this subsection, we state the counterparts of Theorems 2.1 and 2.2 when the nonlinearity f has an oscillation at infinity. We assume that

$(S_{+\infty})$ there are sequences $\{a_k\}$ and $\{b_k\}$ in $]0, \infty[$ with $a_k < b_k < a_{k+1}$ and $\lim_{k\to\infty} b_k = +\infty$ such that

$$f(s) \leq 0 \quad \text{for every } s \in [a_k, b_k], \text{ and}$$

$$(F_{+\infty}) \qquad -\infty < \liminf_{s\to+\infty} \frac{F(s)}{s^2} \leq \limsup_{s\to+\infty} \frac{F(s)}{s^2} = +\infty.$$

Remark 4. Assumption $(S_{+\infty})$ is guaranteed, for instance, by the condition

$$\sup\{s > 0 : f(s) < 0\} = +\infty.$$

Theorem 2.3. *Assume that V, $W : \mathbb{R} \to \mathbb{R}$ satisfy $(V1)$ and $(W1)$, respectively, and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $(S_{+\infty})$ and $(F_{+\infty})$ are fulfilled with*

(iv) *$F(-s) \leq F(s)$ for every $s \geq 0$.*

Assume, in addition, that the sequences $\{a_k\}$ and $\{b_k\}$ from $(S_{+\infty})$ satisfy

(v) *$\lim_{k\to\infty} \frac{a_k}{b_k} = 0$ and $\lim_{k\to\infty} \frac{\max_{[0,a_k]} F}{b_k^2} = 0$.*

Then, problem (P) *possesses a sequence $\{u_k\} \subset H_V$ of weak solutions such that $\lim_{k\to\infty} \|u_k\|_V = \infty$.*

Theorem 2.4. *Assume that V, $W : \mathbb{R} \to \mathbb{R}$ satisfy $(V1)$ and $(W1)$, respectively. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $(S_{+\infty})$ and $(F_{+\infty})$ are fulfilled and*

(vi) *there is a non-degenerate interval $I \subset \mathbb{R}^-$ such that $f|_I \geq 0$.*

Then, the conclusion of Theorem 2.3 holds.

Example 3. Let $s_0 > 0$ be fixed and $h \in C^1([s_0, \infty[, \mathbb{R})$ be a coercive, strictly increasing function such that $h(s_0) = 0$. Let $s_k = h^{-1}(k\pi)$ for $k \in \mathbb{N}$. Assume the existence of $\sigma > 0$ such that $\lim_{k\to\infty} \frac{s_{k-1}^{2+\sigma}}{s_k^2} = 0$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(s) = \begin{cases} -\frac{2s_0^3}{s^2+s_0^2}, & s \in]-\infty, -s_0[; \\ s, & s \in [-s_0, 0[; \\ 4(\pi+1)s, & s \in [0, s_0/2[; \\ g(s), & s \in [s_0/2, s_0[; \\ s^{1+\sigma}[(2+\sigma)\sin h(s) + 2sh'(s)\cos h(s)]\sin h(s), & s \in [s_{2k}, s_{2k+1}[; \\ 0, & s \in [s_{2k+1}, s_{2k+2}[, \end{cases}$$

where $g : [s_0/2, s_0] \to \mathbb{R}^+$ is continuous with $g(s_0/2) = 2(\pi+1)s_0$ and $g(s_0) = 0$. Hence, we can apply Theorem 2.3. Indeed, by simple estimations we can verify conditions $(F_{+\infty})$ and (iv), taking into account that $F(s) = (\pi+1)s_0^2/2 + \int_{s_0/2}^{s_0} g(t)dt + s^{2+\sigma}\{\max\{0, \sin h(s)\}\}^2$ if $s \in [s_0, \infty[$. (Note, however, that f is not almost odd, since $-f(-s_0) = s_0 > 0 = f(s_0)$). Setting $a_k = s_{2k-1}$ and $b_k = s_{2k}$ for $k \geq 1$, the assumptions of Theorem 2.3 are fulfilled. However, Theorem 2.4 cannot be applied because $f(s) < 0$ for every $s < 0$.

b.) As a concrete example, set $s_0 = e$, and let $h : [e, \infty[\to \mathbb{R}$ be defined by $h(s) = \ln\ln s$. Then, every $0 < \sigma < 2(e^\pi - 1)$ fulfills case a.).

Example 4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(s) = s^2\sin^2 s - 1$. We can apply Theorem 2.4 (but not Theorem 2.3).

Remark 5. Let $f(s) = \sin s$. The operator $A(u)(v) = \int_{\mathbb{R}} W(x)v\sin u$ is uniformly Lipschitz; if $\|W\|_1 < \kappa_\infty^{-2}\min\{1, V_0\}$, then A becomes a contraction, and thus (P) admits only the trivial solution. Note that f satisfies the assumptions of Theorem 2.4 except hypothesis $(F_{+\infty})$.

3. **Preliminaries.** Let $\mathcal{E} : H_V \to \mathbb{R}$ be defined by $\mathcal{E}(u) = \|u\|_V^2/2 - \int_{\mathbb{R}} W(x)F(u)$. Since H_V is continuously embedded into $L^\infty(\mathbb{R})$ and compactly into $L^2(\mathbb{R})$, it is possible to prove in a standard way the sequentially weakly lower semicontinuity of $\mathcal{E}$. Moreover, $\mathcal{E}$ is continuously Gâteaux differentiable with derivative given by $\mathcal{E}'(u)(v) = \langle u, v\rangle_V - \int_{\mathbb{R}} W(x)f(u)v$. Hence, critical points of the energy functional are precisely weak solutions of (P). In the sequel, we will use the continuous representation of every element $u \in H_V \subset H^1(\mathbb{R})$ (see [6]). Moreover, it is well-known that one has $u(x) \to 0$ as $|x| \to +\infty$ whenever $u \in H^1(\mathbb{R})$; in this way, the homoclinicity of solutions of (P) is already guaranteed.

Now, we introduce some notations that will be used in the sequel. Let $\sigma \in]0, 1[$, and define the set

$$A_\sigma = \{\mu > 0 \ : \ W(x) > \sigma\|W\|_\infty \ \text{ a.e. } x \in [x_0 - \mu, x_0 + \mu] \text{ for some } x_0 \in \mathbb{R}\}.$$

Note that $A_\sigma \neq \emptyset$ while $\sup A_\sigma$ is finite and attained on some $\mu_\sigma \in A_\sigma$, due to assumption $(W1)$. Hence, there exists $x_\sigma \in \mathbb{R}$ such that

$$W(x) > \sigma\|W\|_\infty \ \text{ a.e. } x \in [x_\sigma - \mu_\sigma, x_\sigma + \mu_\sigma]. \tag{1}$$

For every $\rho \geq 0$ define

$$w_\rho(x) = \begin{cases} 0, & \text{if } \ |x - x_\sigma| > \mu_\sigma; \\ \rho, & \text{if } \ |x - x_\sigma| \leq \frac{\mu_\sigma}{2}; \\ \frac{2\rho}{\mu_\sigma}(\mu_\sigma - |x - x_\sigma|), & \text{if } \ \frac{\mu_\sigma}{2} < |x - x_\sigma| \leq \mu_\sigma. \end{cases} \tag{2}$$

One can see that w_ρ belongs to H_V because $V \in L^\infty_{loc}(\mathbb{R})$. If we introduce

$$\alpha_\sigma = \left(\frac{4}{\mu_\sigma} + 2\mu_\sigma \sup_{|x-x_\sigma|\le\mu_\sigma} V(x)\right)^{1/2},$$

then an easy calculation shows that

$$\|w_\rho\|_V \le \rho\alpha_\sigma. \tag{3}$$

4. **Proof of Theorems 2.1 and 2.3.** Throughout the proof of Theorems 2.1 and 2.3, we will use the following lemma that is a consequence of the classical Mean Value Theorem and the estimation in (3).

Lemma 4.1. *Let there be two sequences* $\{a_k\}$, $\{b_k\} \subset]0,\infty[$ *such that* $a_k < b_k$, $\lim_{k\to\infty} a_k/b_k = 0$, *and* $f(s) \le 0$ *for every* $s \in [a_k, b_k]$. *Let* $s_k = (b_k/\kappa_\infty)^2 \min\{1, V_0\}$. *Then,*

a) $\max_{[0,b_k]} F = \max_{[0,a_k]} F \equiv F(\overline{s}_k)$ *with* $\overline{s}_k \in [0, a_k]$.
b) $\|w_{\overline{s}_k}\|^2_V < s_k$ *for* k *big enough* ($w_{\overline{s}_k}$ *as in (2)).*

Our main tool is a variational principle of Ricceri ([17]) that can be stated as follows:

Theorem R ([17], Theorem 2.5). *Let* X *be a Hilbert space, and let* $\Phi, \Psi : X \to \mathbb{R}$ *be two sequentially weakly lower semicontinuous, continuously Gâteaux differentiable functionals. Assume that* Ψ *is strongly continuous and coercive. For each* $s > \inf_X \Psi$, *set*

$$\varphi(s) := \inf_{\Psi^s} \frac{\Phi(u) - \inf_{cl_w\Psi^s} \Phi}{s - \Psi(u)}, \tag{4}$$

where $\Psi^s := \{u \in X : \Psi(u) < s\}$ *and* $cl_w\Psi^s$ *is the closure of* Ψ^s *in the weak topology of* X. *Furthermore, set*

$$\delta := \liminf_{s\to(\inf_X \Psi)^+} \varphi(s), \qquad \gamma := \liminf_{s\to+\infty} \varphi(s). \tag{5}$$

Then, the following conclusions hold.

(A) *If* $\delta < +\infty$, *then for every* $\lambda > \delta$, *either*
  (A1) $\Phi + \lambda\Psi$ *possesses a local minimum, which is also a global minimum of* Ψ, *or*
  (A2) *there is a sequence* $\{u_n\}$ *of pairwise distinct critical points of* $\Phi + \lambda\Psi$, *with* $\lim_{n\to+\infty} \Psi(u_n) = \inf_X \Psi$, *weakly converging to a global minimum of* Ψ.
(B) *If* $\gamma < +\infty$, *then for every* $\lambda > \gamma$, *either*
  (B1) $\Phi + \lambda\Psi$ *possesses a global minimum, or*
  (B2) *there is a sequence* $\{u_n\}$ *of critical points of the functional* $\Phi + \lambda\Psi$ *such that* $\lim_{n\to+\infty} \Psi(u_n) = +\infty$.

In our framework $\Psi, \Phi : H_V \to \mathbb{R}$ are defined by

$$\Psi(u) = \|u\|^2_V, \quad \Phi(u) = -\mathcal{F}(u) = -\int_{\mathbb{R}} W(x)F(u), \quad u \in H_V;$$

thus the energy functional becomes $\mathcal{E} = \Psi/2 + \Phi$. Moreover, the function from (4) has the form

$$\varphi(s) := \inf_{\|u\|^2_V < s} \frac{\sup_{\|v\|^2_V \le s} \mathcal{F}(v) - \mathcal{F}(u)}{s - \|u\|^2_V}, \quad s > 0.$$

4.1. **Proof of Theorem 2.1.** Let $\{a_k\}$ and $\{b_k\}$ be as in the hypotheses. Let σ, μ_σ and α_σ be as in Section 3, and recall from (5) that $\delta = \liminf_{s\to 0^+} \varphi(s)$.

Lemma 4.2. $\delta = 0$.

Proof. By definition, $\delta \geq 0$. Suppose that $\delta > 0$. By (F_{0^+}), there exist two positive numbers $\underline{M}$ and ϱ such that $F(s) > -\underline{M}s^2$ for every $s \in]0, \varrho[$. Furthermore, let s_k, $\overline{s}_k$ be as in Lemma 4.1, and let $\overline{w}_k = w_{\overline{s}_k}$ be as in (2). By (ii) and condition $\lim_{k\to\infty} \frac{\overline{s}_k}{b_k} = 0$ $(0 \leq \overline{s}_k \leq a_k)$, there exists $k_0 \in \mathbb{N}$ such that

$$\frac{F(\overline{s}_k)}{b_k^2} + \frac{1}{\|W\|_1}\left(\frac{\delta}{2}\alpha_\sigma^2 + 2\mu_\sigma\|W\|_\infty\underline{M}\right)\frac{\overline{s}_k^2}{b_k^2} < \frac{\delta}{2}\frac{\min\{1, V_0\}}{\kappa_\infty^2\|W\|_1} \tag{6}$$

for $k > k_0$.

Let $v \in H_V$ be arbitrarily fixed with $\|v\|_V^2 \leq s_k$. Due to the continuous embedding of H_V into $L^\infty(\mathbb{R})$, we have $\|v\|_\infty \leq b_k$. In view of (i) and Lemma 4.1, we obtain

$$F(v(x)) \leq \max_{[-b_k, b_k]} F = \max_{[0,b_k]} F = F(\overline{s}_k), \quad \text{for every } x \in \mathbb{R}.$$

Since $0 \leq \overline{w}_k(x) \leq \overline{s}_k < \varrho$ for large $k \in \mathbb{N}$ and for all $x \in \mathbb{R}$, taking into account (6) and $\|\overline{w}_k\|_V^2 \leq \overline{s}_k^2\alpha_\sigma^2$, it follows that

$$\begin{aligned}
\sup_{\|v\|_V^2 \leq s_k} \mathcal{F}(v) - \mathcal{F}(\overline{w}_k) &= \sup_{\|v\|_V^2 \leq s_k} \int_{\mathbb{R}} W(x)F(v)dx - \int_{\mathbb{R}} W(x)F(\overline{w}_k)dx \\
&\leq \|W\|_1 F(\overline{s}_k) + 2\mu_\sigma\|W\|_\infty\underline{M}\overline{s}_k^2 \\
&< \frac{\delta}{2}(s_k - \overline{s}_k^2\alpha_\sigma^2) \\
&\leq \frac{\delta}{2}(s_k - \|\overline{w}_k\|_V^2).
\end{aligned}$$

Since $\|\overline{w}_k\|_V^2 < s_k$ (cf. Lemma 4.1), and $s_k \to 0$ as $k \to \infty$, we obtain

$$\delta \leq \liminf_{k\to\infty} \varphi(s_k) \leq \liminf_{k\to\infty} \frac{\sup_{\|v\|_V^2 \leq s_k} \mathcal{F}(v) - \mathcal{F}(\overline{w}_k)}{s_k - \|\overline{w}_k\|_V^2} \leq \frac{\delta}{2},$$

a contradiction. This proves our claim. □

Lemma 4.3. *0 is not a local minimum of* $\mathcal{E} = \Psi/2 + \Phi$.

Proof. Let $\underline{M}$ and ϱ be as in the proof of Lemma 4.2, and let $\overline{M} > 0$ be such that

$$\sigma\mu_\sigma\overline{M}\|W\|_\infty - \alpha_\sigma^2/2 - \underline{M}\|W\|_1 > 0.$$

By using again (F_{0^+}), we deduce the existence of a sequence $\{\tilde{s}_k\} \subset]0, \varrho[$ converging to zero such that $F(\tilde{s}_k) > \overline{M}\tilde{s}_k^2$. Let $\tilde{w}_k \equiv w_{\tilde{s}_k}$ be as in (2). Taking into account (3) and (1), we have

$$\begin{aligned}
\mathcal{E}(\tilde{w}_k) &= \|\tilde{w}_k\|_V^2/2 - \int_{\mathbb{R}} W(x)F(\tilde{w}_k) \\
&\leq \frac{\alpha_\sigma^2}{2}\tilde{s}_k^2 - \int_{|x-x_\sigma| \leq \frac{\mu_\sigma}{2}} W(x)F(\tilde{s}_k) - \int_{\frac{\mu_\sigma}{2} \leq |x-x_\sigma| \leq \mu_\sigma} W(x)F(\tilde{w}_k) \\
&\leq \tilde{s}_k^2\left(\frac{\alpha_\sigma^2}{2} - \sigma\mu_\sigma\overline{M}\|W\|_\infty + \underline{M}\|W\|_1\right) < 0 = \mathcal{E}(0).
\end{aligned}$$

Since $\|\tilde{w}_k\|_V \to 0$ as $k \to \infty$, 0 is not a local minimum of $\mathcal{E}$, as claimed. □

Proof of Theorem 2.1. Applying Theorem R (A), with $\lambda = 1/2$ (due to Lemma 4.2), we can exclude condition ($A1$) (due to Lemma 4.3). Therefore there exists a sequence of pairwise distinct critical points of $\mathcal{E} = \Psi/2 + \Phi$ converging to zero in H_V. □

Remark 6. A closer inspection of the proof of Lemma 4.2 allows us to replace hypothesis $\lim_{k\to\infty} \frac{\max_{[0,a_k]} F}{b_k^2} = 0$ from (ii) with a weaker but more technical condition. More specifically, it is enough to require that $\limsup_{k\to\infty} \frac{\max_{[0,a_k]} F}{b_k^2} < \frac{\min\{1,V_0\}}{2\kappa_\infty^2 \|W\|_1}$. Note that in this latter case we are able to prove that $0 \leq \delta < 1/2$, which is enough to apply Theorem R (A) with $\lambda = 1/2$.

4.2. **Proof of Theorem 2.3.** Let $\{a_k\}$ and $\{b_k\}$ be from Theorem 2.3, and let σ, μ_σ and α_σ be as in Section 3.

Lemma 4.4. $\gamma = 0$, *where* $\gamma = \liminf_{s\to+\infty} \varphi(s)$ *is from (5).*

Proof. It is clear that $\gamma \geq 0$. Suppose that $\gamma > 0$. Let s_k, $\overline{s}_k$ be as in Lemma 4.1 and let $\overline{w}_k = w_{\overline{s}_k}$ be as in (2). Moreover, by ($F_{+\infty}$) there exist two positive numbers $\underline{M}$ and ϱ such that $F(s) > -\underline{M}s^2$ for every $s > \varrho$. Due to hypothesis (v), for enough large natural numbers k we have

$$\frac{F(\overline{s}_k)}{b_k^2} + \frac{1}{\|W\|_1}\left(\frac{\gamma}{2}\alpha_\sigma^2 + 2\mu_\sigma\|W\|_\infty \underline{M}\right)\frac{\overline{s}_k^2}{b_k^2} + \frac{\max_{[0,\varrho]}|F|}{b_k^2} < \frac{\gamma}{2}\frac{\min\{1,V_0\}}{\kappa_\infty^2\|W\|_1},$$

since $\overline{s}_k/b_k \to 0$, and $b_k \to \infty$ as $k \to \infty$.

In a similar way as in the proof of Lemma 4.2, using (iv) and the above relation, one has

$$\begin{aligned}
\sup_{\|v\|_V^2 \leq s_k} \mathcal{F}(v) - \mathcal{F}(\overline{w}_k) &= \sup_{\|v\|_V^2 \leq s_k} \int_{\mathbb{R}} W(x)F(v)dx - \int_{\mathbb{R}} W(x)F(\overline{w}_k)dx \\
&\leq \|W\|_1 F(\overline{s}_k) + 2\mu_\sigma\|W\|_\infty \underline{M}\overline{s}_k^2 + \|W\|_1 \max_{[0,\varrho]}|F| \\
&< \frac{\gamma}{2}(s_k - \|\overline{w}_k\|_V^2).
\end{aligned}$$

Since $s_k \to +\infty$,

$$\gamma \leq \liminf_{k\to\infty} \varphi(s_k) \leq \liminf_{k\to\infty} \frac{\sup_{\|v\|_V^2 \leq s_k} \mathcal{F}(v) - \mathcal{F}(\overline{w}_k)}{s_k - \|\overline{w}_k\|_V^2} \leq \frac{\gamma}{2},$$

which contradicts $\gamma > 0$. □

Lemma 4.5. $\mathcal{E} = \Psi/2 + \Phi$ *is not bounded from below on* H_V.

Proof. Let $\underline{M}$ and ϱ be as in the proof of Lemma 4.4, and let $\overline{M} > 0$ be such that

$$\sigma\mu_\sigma\overline{M}\|W\|_\infty - \alpha_\sigma^2/2 - \underline{M}\|W\|_1 > 0.$$

By using the second part of ($F_{+\infty}$), we find a sequence $\{\tilde{s}_k\}$ which tends to $+\infty$ such that $F(\tilde{s}_k) > \overline{M}\tilde{s}_k^2$. Let $\tilde{w}_k \equiv w_{\tilde{s}_k}$ be as in (2). Thanks to (1), one has

$$\mathcal{E}(\tilde{w}_k) \leq \frac{\alpha_\sigma^2}{2}\tilde{s}_k^2 - \sigma\mu_\sigma\overline{M}\|W\|_\infty\tilde{s}_k^2 - \int_{\frac{\mu_\sigma}{2} \leq |x-x_\sigma| \leq \mu_\sigma} W(x)F(\tilde{w}_k).$$

Set $X_\sigma = \{x \in \mathbb{R} : \frac{\mu_\sigma}{2} \leq |x - x_\sigma| \leq \mu_\sigma\}$. Note that for large $k \in \mathbb{N}$ we have

$$\begin{aligned}\int_{X_\sigma} W(x)F(\tilde{w}_k) &= \int_{X_\sigma \cap \{\tilde{w}_k(x) > \varrho\}} W(x)F(\tilde{w}_k) + \int_{X_\sigma \cap \{\tilde{w}_k(x) \leq \varrho\}} W(x)F(\tilde{w}_k) \\ &\geq -\underline{M} \int_{X_\sigma \cap \{\tilde{w}_k(x) > \varrho\}} W(x)|\tilde{w}_k|^2 - \|W\|_1 \max_{[0,\varrho]} |F| \\ &\geq -\underline{M}\|W\|_1 \tilde{s}_k^2 - \|W\|_1 \max_{[0,\varrho]} |F|.\end{aligned}$$

Thus,

$$\mathcal{E}(\tilde{w}_k) \leq \tilde{s}_k^2 \left(\frac{\alpha_\sigma^2}{2} - \sigma\mu_\sigma \overline{M}\|W\|_\infty + \underline{M}\|W\|_1 \right) + \|W\|_1 \max_{[0,\varrho]} |F|,$$

which proves that $\inf_{H_V} \mathcal{E} = -\infty$. □

Proof of Theorem 2.3. In Theorem R (B) we can choose $\lambda = 1/2$ (due to Lemma 4.4). Moreover, thanks to Lemma 4.5 the alternative $(B1)$ can be excluded; then there exists a sequence $\{u_n\}$ of critical points of $\mathcal{E} = \Psi/2 + \Phi$ such that $\|u_n\|_V \to +\infty$ as $n \to \infty$. □

Remark 7. It is possible to relax hypothesis $\lim_{k\to\infty} \frac{\max_{[0,a_k]} F}{b_k^2} = 0$ from (v) in a same way as we described in Remark 6.

5. **Proof of Theorems 2.2 and 2.4.**

5.1. **Proof of Theorem 2.2.** Let us suppose first that $\sup\{s < 0 : f(s) > 0\} = 0$ is fulfilled in (iii). Then, one can deduce the existence of a monotone sequence $\{c_k\}$ such that $c_k < 0$, $f(c_k) > 0$ and $c_k \to 0$ as $k \to \infty$. Moreover, let $\{d_k\}$ be another sequence such that $d_k < c_k < d_{k+1}$ with $f(s) > 0$ for $s \in [d_k, c_k]$. Then, simultaneously with (S_{0^+}) we have

$$F(s) \leq F(c_k),\ s \in [d_k, c_k]; \tag{7}$$

$$F(s) \leq F(a_k),\ s \in [a_k, b_k]. \tag{8}$$

Note that if the second alternative of (iii) is fulfilled, i.e., there is a $\delta > 0$ such that $f|_{[-\delta,0]} \equiv 0$, then relation (7) becomes trivial for any sequences $\{c_k\}$, $\{d_k\}$ with the properties above.

Define the set

$$S_k = \{u \in H_V : d_k \leq u(x) \leq b_k \text{ for every } x \in \mathbb{R}\}.$$

Lemma 5.1. *The energy functional $\mathcal{E}$ is bounded from below on S_k, and its infimum on S_k is attained.*

Proof. It is clear that S_k is convex. Moreover, it is closed in H_V due to the continuity of the embedding $H_V \hookrightarrow L^\infty(\mathbb{R})$; then S_k is weakly closed. Since

$$\mathcal{E}(u) = \|u\|_V^2/2 - \int_{\mathbb{R}} W(x)F(u) \geq -\|W\|_1 \max_{[d_k,b_k]} F \qquad \text{for } u \in S_k,$$

$\mathcal{E}$ is bounded from below on S_k. Let $\gamma_k = \inf_{S_k} \mathcal{E}$, and let $\{u_n\}$ be a sequence in S_k such that $\gamma_k \leq \mathcal{E}(u_n) \leq \gamma_k + 1/n$ for all $n \in \mathbb{N}$. Then,

$$\|u_n\|_V^2/2 \leq \gamma_k + 1 + \|W\|_1 \max_{[d_k,b_k]} F$$

for all $n \in \mathbb{N}$; i.e., $\{u_n\}$ is bounded in H_V. So, up to a subsequence, $\{u_n\}$ weakly converges in H_V to some $\tilde{u}_k \in S_k$. By the sequentially weakly lower semicontinuity of $\mathcal{E}$, we conclude that $\mathcal{E}(\tilde{u}_k) = \inf_{S_k} \mathcal{E}$. □

Lemma 5.2. *Let $\tilde{u}_k \in S_k$ be such that $\mathcal{E}(\tilde{u}_k) = \inf_{S_k} \mathcal{E}$. Then, $c_k \le \tilde{u}_k(x) \le a_k$ for all $x \in \mathbb{R}$.*

Proof. Let $X = \{x \in \mathbb{R} : \tilde{u}_k(x) \notin [c_k, a_k]\}$, and suppose that $X \neq \emptyset$. Thus, $meas(X) > 0$ due to the continuity of $\tilde{u}_k$. Define

$$h(x) = \begin{cases} c_k, & \text{if} \quad x < c_k; \\ x, & \text{if} \quad x \in [c_k, a_k]; \\ a_k, & \text{if} \quad x > a_k. \end{cases}$$

Set $\tilde{v}_k = h \circ \tilde{u}_k$. Due to [13], $\tilde{v}_k$ belongs to $H^1(\mathbb{R})$ (since h is uniformly Lipschitz and $h(0) = 0$). Moreover, $\tilde{v}_k \in H_V$, since $\int_{\mathbb{R}} V(x)\tilde{v}_k^2 \le \int_{\mathbb{R}} V(x)\tilde{u}_k^2 < +\infty$. In addition, $\tilde{v}_k \in S_k$. Denoting by

$$X_1 = \{x \in X : \tilde{u}_k(x) < c_k\} \qquad \text{and} \qquad X_2 = \{x \in X : \tilde{u}_k(x) > a_k\},$$

we have that $\tilde{v}_k(x) = \tilde{u}_k(x)$ for all $x \in \mathbb{R} \setminus X$; $\tilde{v}_k(x) = c_k$ for all $x \in X_1$; and $\tilde{v}_k(x) = a_k$ for all $x \in X_2$. Then,

$$\begin{aligned} &\mathcal{E}(\tilde{v}_k) - \mathcal{E}(\tilde{u}_k) \\ &\quad = -\frac{1}{2}\int_X (\tilde{u}_k')^2 + \frac{1}{2}\int_X V(x)[\tilde{v}_k^2 - \tilde{u}_k^2] - \int_X W(x)[F(\tilde{v}_k) - F(\tilde{u}_k)] \\ &\quad = -\frac{1}{2}\int_X (\tilde{u}_k')^2 + \frac{1}{2}\int_{X_1} V(x)[c_k^2 - \tilde{u}_k^2] + \frac{1}{2}\int_{X_2} V(x)[a_k^2 - \tilde{u}_k^2] \\ &\qquad - \int_{X_1} W(x)[F(c_k) - F(\tilde{u}_k)] - \int_{X_2} W(x)[F(a_k) - F(\tilde{u}_k)]. \end{aligned}$$

From (7) and (8) we obtain that every term of the above expression is not positive. On the other hand, since $\mathcal{E}(\tilde{v}_k) \ge \mathcal{E}(\tilde{u}_k) = \inf_{S_k} \mathcal{E}$, then in particular

$$\int_X (\tilde{u}_k')^2 = 0, \tag{9}$$

$$\int_{X_1} V(x)[c_k^2 - \tilde{u}_k^2] = \int_{X_2} V(x)[a_k^2 - \tilde{u}_k^2] = 0.$$

By (9) we obtain the existence of a positive measured subset Y of X and a constant C such that $\tilde{u}_k = C$ on Y. Then, either $Y \subset X_1$ or $Y \subset X_2$. Assume that the first case occurs (analogously if $Y \subset X_2$). So,

$$0 = \int_{X_1} V(x)[c_k^2 - \tilde{u}_k^2] \le \int_Y V(x)[c_k^2 - C^2] \le V_0[c_k^2 - C^2]meas(Y) < 0,$$

a contradiction. This shows that X has zero measure; therefore, $X = \emptyset$. □

Lemma 5.3. *Let $\tilde{u}_k \in S_k$ be such that $\mathcal{E}(\tilde{u}_k) = \inf_{S_k} \mathcal{E}$. Then $\tilde{u}_k$ is a local minimum of $\mathcal{E}$ in H_V.*

Proof. Suppose the contrary. Then there exists a sequence $\{u_n\} \subset H_V$ which converges to $\tilde{u}_k$ while $\mathcal{E}(u_n) < \mathcal{E}(\tilde{u}_k)$ for all $n \in \mathbb{N}$. From the latter, it follows that $u_n \notin S_k$ for any $n \in \mathbb{N}$. Since $u_n \to \tilde{u}_k$ in H_V, then $u_n \to \tilde{u}_k$ in $L^\infty(\mathbb{R})$ as well. In particular, for every $0 < \varepsilon < \min\{c_k - d_k, b_k - a_k\}/2$, there exists $n_\varepsilon \in \mathbb{N}$ such that

$\|u_n - \tilde{u}_k\|_\infty < \varepsilon$ for every $n \geq n_\varepsilon$. By using Lemma 5.2 and taking into account the choice of the number ε, we conclude that

$$d_k < u_n(x) < b_k \quad \text{for all } x \in \mathbb{R},\ n \geq n_\varepsilon,$$

which clearly contradicts the fact $u_n \notin S_k$. □

Lemma 5.4. *Let* $\gamma_k = \inf_{S_k} \mathcal{E} = \mathcal{E}(\tilde{u}_k)$. *Then,* $\gamma_k < 0$ *for all* $k \in \mathbb{N}$ *and* $\lim_{k\to\infty} \gamma_k = 0$.

Proof. By (F_{0+}) there exist two positive numbers $\underline{M}$ and ϱ such that $F(s) > -\underline{M}s^2$ for every $s \in]0, \varrho[$. Let $\overline{M} > 0$ be such that

$$\sigma\mu_\sigma \overline{M}\|W\|_\infty - \alpha_\sigma^2/2 - \underline{M}\|W\|_1 > 0,$$

where σ, μ_σ and α_σ are as in Section 3. With this choice of $\overline{M}$, by using (F_{0+}) again there exists a sequence $\{s_k\} \subset]0, \varrho[$ converging to zero such that $F(s_k) > \overline{M}s_k^2$. Let $\{s_{l_k}\}$ be a decreasing subsequence of $\{s_k\}$ such that $s_{l_k} \leq b_k$ for all $k \in \mathbb{N}$ and $w_k \equiv w_{s_{l_k}}$, as in (2). It is clear that $w_k \in S_k$. A similar calculation as in the proof of Lemma 4.3 shows that

$$\begin{aligned} \mathcal{E}(w_k) &= \|w_k\|_V^2/2 - \int_{\mathbb{R}} W(x)F(w_k) \\ &\leq s_{l_k}^2 \left(\frac{\alpha_\sigma^2}{2} - \sigma\mu_\sigma \overline{M}\|W\|_\infty + \underline{M}\|W\|_1 \right) < 0. \end{aligned}$$

Thus, $\gamma_k = \inf_{S_k} \mathcal{E} \leq \mathcal{E}(w_k) < 0$.

Now we will prove that $\gamma_k \to 0$ as $k \to \infty$. One has

$$0 > \gamma_k = \|\tilde{u}_k\|_V^2/2 - \int_{\mathbb{R}} W(x)F(\tilde{u}_k) \geq -\|W\|_1 \max_{[d_1,b_1]} |f|\ \|\tilde{u}_k\|_\infty.$$

Taking into account that $\|\tilde{u}_k\|_\infty \leq \max\{|c_k|, a_k\}$, and that the sequences $\{c_k\}, \{a_k\}$ tend to zero, then $\gamma_k \to 0$. □

Proof of Theorem 2.2. Since $\tilde{u}_k$ are local minima of $\mathcal{E}$ (cf. Lemma 5.3), they are critical points of $\mathcal{E}$ and thus weak solutions of (P). Due to Lemma 5.4, there are infinitely pairwise distinct $\tilde{u}_k$. Moreover,

$$\|\tilde{u}_k\|_V^2/2 = \int_{\mathbb{R}} W(x)F(\tilde{u}_k) + \gamma_k \leq \|W\|_1 \max_{[d_1,b_1]} |f|\ \max\{|c_k|, a_k\},$$

which proves that $\|\tilde{u}_k\|_V \to 0$. □

5.2. **Proof of Theorem 2.4.** By $(S_{+\infty})$ we deduce that $F(s) \leq F(a_k),\ s \in [a_k, b_k]$. Moreover, by applying (vi) we can fix $d < c < 0$ such that $F(s) \leq F(c),\ s \in [d, c]$.

Define the set

$$T_k = \{u \in H_V : d \leq u(x) \leq b_k \text{ for every } x \in \mathbb{R}\}.$$

The proofs of the next three lemmas are the same as in Subsection 5.1.

Lemma 5.5. *The energy functional* $\mathcal{E}$ *is bounded from below on* T_k, *and its infimum on* T_k *is attained.*

Lemma 5.6. *Let* $\tilde{z}_k \in T_k$ *be such that* $\mathcal{E}(\tilde{z}_k) = \inf_{T_k} \mathcal{E}$. *Then,* $c \leq \tilde{z}_k(x) \leq a_k$ *for all* $x \in \mathbb{R}$.

Lemma 5.7. *Let* $\tilde{z}_k \in T_k$ *be such that* $\mathcal{E}(\tilde{z}_k) = \inf_{T_k} \mathcal{E}$. *Then* $\tilde{z}_k$ *is a local minimum of* $\mathcal{E}$ *in* H_V.

Lemma 5.8. *Let* $\delta_k = \inf_{T_k} \mathcal{E}$. *Then* $\lim_{k\to\infty} \delta_k = -\infty$.

Proof. By $(F_{+\infty})$ there exist two positive numbers $\underline{M}$ and ϱ such that $F(s) > -\underline{M}s^2$ for every $s > \varrho$. Let $\overline{M} > 0$ be such that

$$\sigma\mu_\sigma\overline{M}\|W\|_\infty - \alpha_\sigma^2/2 - \underline{M}\|W\|_1 > 0,$$

where σ, μ_σ and α_σ are as in Section 3. By using the second part of $(F_{+\infty})$, there exists a sequence $\{s_k\}$ which tends to $+\infty$ such that $F(s_k) > \overline{M}s_k^2$. Let $\{b_{l_k}\}$ be an increasing subsequence of $\{b_k\}$ such that $s_k \leq b_{l_k}$ for all $k \in \mathbb{N}$ and $w_k \equiv w_{s_k}$, as in (2). It is clear that $w_k \in T_{l_k}$. Moreover, in a similar way as in the proof of Lemma 4.5, one can deduce that

$$\mathcal{E}(w_k) \leq s_k^2\left(\frac{\alpha_\sigma^2}{2} - \sigma\mu_\sigma\overline{M}\|W\|_\infty + \underline{M}\|W\|_1\right) + \|W\|_1 \max_{[0,\varrho]}|F|,$$

which proves that $\delta_{l_k} = \inf_{T_{l_k}} \mathcal{E} \leq \mathcal{E}(w_k) \to -\infty$ as $k \to \infty$. Since the sequence $\{\delta_k\}$ is non-increasing, our claim follows. □

Proof of Theorem 2.4. Due to Lemmas 5.7 and 5.8, there are infinitely pairwise distinct local minima $\tilde{z}_k$ of $\mathcal{E}$ with $\tilde{z}_k \in T_k$. Now we will prove that $\|\tilde{z}_k\|_V \to \infty$. Assume for contradiction that there is a subsequence $\{\tilde{z}_{n_k}\}$ of $\{\tilde{z}_k\}$ which is bounded in H_V. Thus, it is bounded in $L^\infty(\mathbb{R})$ as well. In particular we can find $m_0 \in \mathbb{N}$ such that $\tilde{z}_{n_k} \in T_{m_0}$ for all $k \in \mathbb{N}$. For every $n_k \geq m_0$ one has

$$\delta_{m_0} \geq \delta_{n_k} = \inf_{T_{n_k}} \mathcal{E} = \mathcal{E}(\tilde{z}_{n_k}) \geq \inf_{T_{m_0}} \mathcal{E} = \delta_{m_0},$$

which proves that $\delta_{n_k} = \delta_{m_0}$ for all $n_k \geq m_0$, contradicting Lemma 5.8. □

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