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NONSMOOTH NEUMANN-TYPE PROBLEMS INVOLVING THE *p*-LAPLACIAN

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□ *This paper deals with the problem* $-\Delta_p u + \alpha(x)|u|^{p-2}u = \beta(x)f(|u|)$ *in* Ω, *subjected to the zero Neumann boundary condition, where* $p > 1$, $\Omega \subset \mathbb{R}^N$ *is bounded with smooth boundary,* $\alpha, \beta \in L^\infty(\Omega)$, $\text{essinf}_\Omega \beta > 0$, *and* $f : [0, +\infty) \to \mathbb{R}$ *is a not necessarily continuous nonlinearity that oscillates either at the origin or at the infinity. By using nonsmooth variational methods, we establish in both cases the existence of infinitely many distinct non-negative solutions of the Neumann problem. In our framework,* $\alpha : \Omega \to \mathbb{R}$ *may be a sign-changing or even a nonpositive potential, which is not permitted usually in earlier works.*

Keywords Infinitely many solutions; *p*-Laplacian; Neumann problem; Nonsmooth potential.

AMS Subject Classification 35J60; 35J25; 35J20.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2-boundary $\partial\Omega$ and consider the nonlinear elliptic problem

$$\begin{cases} -\Delta_p u + \alpha(x)u^{p-1} = \beta(x)f(u) & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \qquad (P_0)$$

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where $1 < p < \infty, \triangle_p(\cdot) = \mathrm{div}(|\nabla(\cdot)|^{p-2}\nabla(\cdot))$ is the p-Laplacian operator, ν is the outer unit normal to $\partial\Omega$, and $f \in L^{\infty}_{\mathrm{loc}}([0,\infty))$ is a function with $f(0) = 0, \alpha, \beta \in L^{\infty}(\Omega)$, with $\mathrm{essinf}_{\Omega}\beta > 0$.

In recent years, problem (P_0) has been widely investigated by many authors, see [1–5, 8, 10, 12, 15, 17, 19]. In particular, Ricceri's variational principle (see [18, 19]) and its variants have been successfully applied in various context in order to guarantee the existence of infinitely many weak solutions of (P_0) when the nonlinear term f has an oscillatory behavior (at zero, or at infinity). Marano–Motreanu [12] extended Ricceri's principle to a large class of nondifferentiable functionals, applying their abstract result to a Neumann problem for an elliptic variational-hemivariational inequality that originates from (P_0). By means of [12], Candito [5] studied (P_0) when the nonlinearity f may possess uncountably many discontinuities. Further applications of Ricceri's principle can be found in Anello [1], Cammaroto–Chinnì–Di Bella [4], and Kristály [11].

The common features of the aforementioned papers ([4, 5, 12, 19]) are $p > N$, and $\mathrm{essinf}_{\Omega}\alpha > 0$. The first fact (i.e., $p > N$) has been used in order to apply the compactness of the embedding $W^{1,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$, which was crucial to study (P_0) via Ricceri's principle. However, Anello [1] and Anello–Cordaro [2] considered (P_0) by removing the key inequality $p > N$ where f fulfills certain oscillatory assumptions, and by exploiting an idea of Saint Raymond [20], they still guaranteed the existence of infinitely many solutions for problem (P_0). On the other hand, the hypothesis $\mathrm{essinf}_{\Omega}\alpha > 0$ seemed to be essential in *all* the works cited above (see also [8]).

The aim of our paper is threefold. Considering (P_0) with an oscillatory nonlinearity f (at zero, or at infinity), our contribution can be briefly described as follows:

1. We do *not* require $\mathrm{essinf}_{\Omega}\alpha > 0$; in other words, we allow a *sign-changing* or even a *nonpositive* potential $\alpha \in L^{\infty}(\Omega)$. In order to handle this problem, we carefully control the oscillatory behavior of the nonlinearity f (at zero, or at infinity) by a very natural assumption. Roughly speaking, f oscillates at 0^+ or at $+\infty$ along the curves $s \mapsto C_{\alpha\beta}s^{p-1}$, $s > 0$, where $C_{\alpha\beta} \in [\mathrm{essinf}_{\Omega}\frac{\alpha}{\beta}, \mathrm{esssup}_{\Omega}\frac{\alpha}{\beta}]$. For the precise formulation, see hypotheses (H_0) and (H_∞), respectively. On the other hand, if it happens that $\mathrm{essinf}_{\Omega}\alpha > 0$, earlier results can be deduced from our theorems (see [2, 5, 8, 12, 19]).

2. No relationship between p and N is required; consequently, the variational method used in earlier papers (see [4, 5, 8, 12, 19]) fails. Moreover, our method is completely different than that used by Anello and Cordaro ([1, 2]), and it is based on a nonsmooth critical point theory in the sense of Motreanu–Panagiotopoulos (see [14, Chapter 3]) combined with a careful truncation argument.

3. Our study includes the case where the nonlinear term f has *discontinuities.* Due to this fact, we reformulate the problem (P_0) into a *hemivariational inequality* (see also the works [10, 17] in similar nonsmooth context). By using Clarke's calculus for locally Lipschitz functions, we are able to guarantee the existence of infinitely many distinct solutions for the hemivariational inequality problem [in particular, for problem (P_0)]. Furthermore, we have full information on the L^∞ and $W^{1,p}$-behavior of solutions. Note that our results are new even in the "smooth" context.

In the sequel, we formulate our main results. Before doing this, we emphasize that the solutions of our problem are sought in $W^{1,p}(\Omega)$, which is endowed with the standard norm

$$\|u\|_{W^{1,p}} = \left(\int_\Omega |\nabla u(x)|^p dx + \int_\Omega |u(x)|^p dx \right)^{1/p}.$$

We consider a function $f \in L^\infty_{\mathrm{loc}}([0, +\infty))$ (i.e., locally essentially bounded) that is not necessarily continuous and $f(0) = 0$ [we put $f(s) = 0$ for $s \leq 0$]. In such case, problem (P_0) need *not* have a solution a.e. in Ω. To overcome this inconvenience, we first observe that $F(s) = \int_0^s f(t)dt$, $s \in \mathbb{R}$, becomes a locally Lipschitz function. In particular, it makes sense to consider the generalized directional derivative F^0 of F (see Section 2 for details). Therefore, instead of (P_0), we consider the following nonsmooth problem (hemivariational inequality), denoted by (P): *Find* $u \in W^{1,p}(\Omega)$ *such that*

$$\int_\Omega [|\nabla u|^{p-2}\nabla u \nabla v + \alpha(x)|u|^{p-2}uv]dx + \int_\Omega \beta(x)F^0(u(x); -v(x))dx \geq 0, \quad \forall v \in W^{1,p}(\Omega).$$

Remark 1.1. When f is continuous, then $F^0(u(x); -v(x)) = -f(u(x))v(x)$, thus a non-negative solution $u \in W^{1,p}(\Omega)$ for the hemivariational inequality (P) is a weak solution to the initial problem (P_0). So, in some sense, the solutions of (P) can be considered as generalized solutions of (P_0).

We emphasize that hemivariational inequalities are used in the study of problems with discontinuities (see Chang [6], Gasiński–Papageorgiou [9]), as well as in various engineering problems in which the corresponding energy (Euler) functional is nonsmooth and nonconvex. For various applications, we refer the reader to Motreanu–Panagiotopoulos [14], Naniewicz–Panagiotopoulos [16], and references therein.

Taking into account that f is locally essentially bounded on the whole $\mathbb{R}$ (being 0 on the negative axis), we may define the functions

$$f_l(s) = \lim_{\delta\to 0^+} \text{essinf}_{|t-s|<\delta} f(t) \quad \text{and} \quad f_u(s) = \lim_{\delta\to 0^+} \text{esssup}_{|t-s|<\delta} f(t), \quad s \in \mathbb{R}.$$

One can prove that $\partial F(s) = [f_l(s), f_u(s)]$ for every $s \in \mathbb{R}$ (see [6], [14, Proposition 1.7]), where ∂F denotes the generalized gradient of F (see Section 2).

Now, we are in a position to state our main results. First, we assume

$$\limsup_{s\to 0^+} \frac{pF(s)}{s^p} > \frac{\int_\Omega \alpha(x)\,dx}{\int_\Omega \beta(x)\,dx} \geq \text{essinf}_\Omega \frac{\alpha}{\beta} > \liminf_{s\to 0^+} \frac{f_u(s)}{s^{p-1}}. \qquad (H_0)$$

Note that (H_0) implies an oscillatory behavior of f at zero, while the inequality "$\geq$" always holds.

On the one hand, if $\alpha(x)/\beta(x) = c_0 \in \mathbb{R}$ for a.e. $x \in \Omega$, then f oscillates at zero along the curve $s \mapsto c_0 s^{p-1}$, $s > 0$. In such case, if in addition f is continuous, we have a sequence of *constant* solutions for (P_0), converging to zero, which are roots of the equation $c_0 s^{p-1} = f(s)$, $s > 0$. On the other hand, if $\alpha/\beta \neq$ constant, one cannot have constant solutions for (P_0). The general result—where we do not attach any importance to the relationship between α and β—can be read as follows:

Theorem 1.2. *Let* $\alpha, \beta \in L^\infty(\Omega)$ *with* $\text{essinf}_\Omega \beta > 0$ *and a function* $f \in L^\infty_{\text{loc}}([0, +\infty))$, $f(0) = 0$, *fulfilling* (H_0). *Then* (P) *admits a sequence of distinct non-negative solutions* $\{\tilde{u}_k\}$ *in* $W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *such that*

$$\lim_{k\to\infty} \|\tilde{u}_k\|_{W^{1,p}} = \lim_{k\to\infty} \|\tilde{u}_k\|_\infty = 0.$$

Note that in the case when f oscillates at zero, no assumption is needed on the growth of f at infinity. However, dealing with the case when f oscillates at infinity, we require for it to have a subcritical growth at infinity; namely,

$$\limsup_{s\to\infty} \frac{|f(s)|}{s^{q-1}} < \infty \quad \text{for some } q \in (p, p^*). \qquad (f_\infty)$$

Here and in the sequel, $p^* = pN/(N-p)$ if $N > p$ and $p^* = \infty$ if $p \geq N$. The counterpart of (H_0) at infinity is

$$\limsup_{s\to\infty} \frac{pF(s)}{s^p} > \frac{\int_\Omega \alpha(x)\,dx}{\int_\Omega \beta(x)\,dx} \geq \text{essinf}_\Omega \frac{\alpha}{\beta} > \liminf_{s\to\infty} \frac{f_u(s)}{s^{p-1}}. \qquad (H_\infty)$$

Theorem 1.3. *Let* $\alpha, \beta \in L^\infty(\Omega)$ *with* $\operatorname{essinf}_\Omega \beta > 0$ *and a function* $f \in L^\infty_{loc}([0,\infty))$, $f(0) = 0$, *fulfilling* (f_∞) *and* (H_∞). *Then* (P) *admits a sequence of distinct non-negative solutions* $\{\tilde{w}_k\}$ *in* $W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *such that*

$$\lim_{k\to\infty} \|\tilde{w}_k\|_{W^{1,p}} = \lim_{k\to\infty} \|\tilde{w}_k\|_\infty = \infty.$$

The paper is divided as follows. In the next section, we recall some basic properties of the generalized directional derivative and Clarke generalized gradient of a locally Lipschitz function that will be used throughout the paper. In Sections 3 and 4, we prove Theorems 1.2 and 1.3, respectively. In the last section, we compare our theorems with earlier results via simple examples, emphasizing the applicability of Theorems 1.2 and 1.3.

2. NONSMOOTH CALCULUS: SOME BASIC PROPERTIES OF LOCALLY LIPSCHITZ FUNCTIONS

Let $(X, \|\cdot\|)$ be a real Banach space and X^* its topological dual. A function $h : X \to \mathbb{R}$ is called *locally Lipschitz* if each point $u \in X$ possesses a neighborhood $\mathcal{N}_u$ such that $|h(u_1) - h(u_2)| \le L\|u_1 - u_2\|$ for all $u_1, u_2 \in \mathcal{N}_u$, for a constant $L > 0$ depending on $\mathcal{N}_u$. The *generalized directional derivative* of h at the point $u \in X$ in the direction $z \in X$ is

$$h^0(u; z) = \limsup_{w\to u,\, t\to 0^+} \frac{h(w + tz) - h(w)}{t}.$$

The *Clarke generalized gradient* of h at $u \in X$ is defined by

$$\partial h(u) = \{x^* \in X^* : \langle x^*, z\rangle_X \le h^0(u; z) \text{ for every } z \in X\},$$

which is a nonempty, convex, and w^*-compact subset of X^*, where $\langle\cdot,\cdot\rangle_X$ is the duality pairing between X^* and X.

We list some fundamental properties ofthe generalized directional derivative and gradient that will be used throughout the paper.

Proposition 2.1 (see Clarke [7]).

(i) (*Positive homogeneity*) $h^0(u; cz) = ch^0(u; z)$, *for all* $u, z \in X$, *and* $c > 0$.
(ii) (*Subadditivity*) $h^0(u; z + v) \le h^0(u; z) + h^0(u; v)$ *for all* $u, z, v \in X$.
(iii) $(-h)^0(u; z) = h^0(u; -z)$ *for all* $u, z \in X$.
(iv) *Let* $j : X \to \mathbb{R}$ *be a continuously differentiable function. Then* $\partial j(u) = \{j'(u)\}$, $j^0(u; z)$ *coincides with* $\langle j'(u), z\rangle_X$ *and* $(h + j)^0(u; z) = h^0(u; z) + \langle j'(u), z\rangle_X$ *for all* $u, z \in X$.

(v) (*Lebourg's mean value theorem*) *Let u and v be two points in X. Then there exists a point w in the open segment joining u and v, and $x_w^* \in \partial h(w)$ such that*

$$h(u) - h(v) = \langle x_w^*, u - v \rangle_X.$$

(vi) *If X is finite-dimensional, ∂h is upper semicontinuous in every point of X(∂h is upper semicontinuous in $u \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\partial h(v) \subseteq \partial h(u) + \varepsilon[-1, 1]$ for every $v \in B_X(u, \delta) = \{v \in X : \|v - u\| < \delta\}$).*

A point $u \in X$ is a *critical point* of h if $0 \in \partial h(u)$, that is, $h^0(u; z) \geq 0$ for all $z \in X$ (see Chang [6]).

3. PROOF OF THEOREM 1.2

Throughout this section, we assume the hypotheses of Theorem 1.2 are fulfilled. The standard norm of $L^q(\Omega)$ will be denoted by $\|\cdot\|_q$, $q \in [1, \infty]$.

Because $f(0) = 0$, we may put $f(s) = F(s) = 0$ for every $s \leq 0$. Let $\tilde{s} > 0$ be arbitrarily fixed, and let $\tilde{f}(s) = f(\min(s, \tilde{s}))$ for $s \geq 0$, and $\tilde{f}(s) = f(0) = 0$ for $s < 0$. Define $\widetilde{F}(s) = \int_0^s \tilde{f}(t)dt$, $s \in \mathbb{R}$. It is clear that $\widetilde{F}$ is locally Lipschitz, and an elementary calculation shows that

$$\widetilde{F}^0(s; t) = F^0(s; t) \quad \text{for every } s \in [0, \tilde{s}) \text{ and } t \in \mathbb{R}. \tag{3.1}$$

Due to (H_0), one can fix $c_0 \in \mathbb{R}$ such that

$$\operatorname{essinf}_\Omega \frac{\alpha}{\beta} > c_0 > \liminf_{s \to 0^+} \frac{f_u(s)}{s^{p-1}}. \tag{3.2}$$

In particular, there is a sequence $\{s_k\} \subset (0, \tilde{s})$ converging (decreasingly) to 0, such that

$$\tilde{f}_u(s_k) = f_u(s_k) < c_0 s_k^{p-1}. \tag{3.3}$$

Let us define the functions

$$g(s) = \tilde{f}(s) - c_0 s_+^{p-1} \quad \text{and} \quad G(s) = \int_0^s g(t)dt = \widetilde{F}(s) - \frac{c_0}{p} s_+^p, \quad s \in \mathbb{R}, \tag{3.4}$$

where $s_+ = \max(s, 0)$. It is clear that $g \in L^\infty_{\text{loc}}(\mathbb{R})$ and G is locally Lipschitz.

Because $g_u(s_k) < 0$, see (3.3), for $\varepsilon_k = -g_u(s_k) > 0$ there exists $\delta_k > 0$ such that $\partial G(s) \subseteq \partial G(s_k) + \varepsilon_k[-1, 1]$ for all $s \in \mathbb{R}$ with $|s - s_k| \leq \delta_k$, see Proposition 2.1(vi). Note that $\partial G(s_k) = [g_l(s_k), g_u(s_k)]$; consequently,

$$\partial G(s) \subset (-\infty, 0] \quad \text{for every } s \in [a_k, b_k], \tag{3.5}$$

where $a_k = s_k - \min(\delta_k, s_k/2)$ and $b_k = s_k + \min(\delta_k, s_k/2)$. Clearly, $a_k > 0$ for every $k \in \mathbb{N}$, and $\lim_{k\to\infty} b_k = 0$. Moreover, one can assume without loosing the generality that $b_{k+1} < a_k < b_k$ for every $k \in \mathbb{N}$.

On the other hand, on account of (3.4) and (H_0), we have

$$\limsup_{s\to 0^+} \frac{pG(s)}{s^p} > \frac{\int_\Omega \alpha(x)\,dx}{\int_\Omega \beta(x)\,dx} - c_0. \tag{3.6}$$

The functional $\mathcal{G} : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{G}(u) = \int_\Omega \beta(x) G(u(x))\,dx, \quad u \in W^{1,p}(\Omega),$$

is well-defined, locally Lipschitz, and (see [7])

$$\mathcal{G}^0(u; v) \leq \int_\Omega \beta(x) G^0(u(x); v(x))\,dx, \quad u, v \in W^{1,p}(\Omega). \tag{3.7}$$

Due to the choice of c_0, see (3.2), if $\lambda(x) = \alpha(x) - c_0\beta(x)$, one has

$$\text{essinf}_\Omega \lambda \geq \text{essinf}_\Omega \left(\frac{\alpha}{\beta} - c_0\right) \text{essinf}_\Omega \beta > 0.$$

Therefore, the norm

$$\|u\|_\lambda = \left(\int_\Omega |\nabla u(x)|^p\,dx + \int_\Omega \lambda(x)|u(x)|^p\,dx\right)^{1/p} \tag{3.8}$$

is equivalent to the standard norm $\|\cdot\|_{W^{1,p}}$. Finally, we define the functional $\mathcal{E} : W^{1,p}(\Omega) \to \mathbb{R}$ by

$$\mathcal{E}(u) = \frac{1}{p}\|u\|_\lambda^p - \mathcal{G}(u)$$

which is clearly locally Lipschitz.

Let us fix a number $r < 0$ arbitrarily, and for every $k \in \mathbb{N}$, consider the set

$$S_k = \{u \in W^{1,p}(\Omega) : r \leq u(x) \leq b_k \text{ a.e. } x \in \Omega\},$$

where b_k is from (3.5).

Proposition 3.1. *The functional $\mathscr{E}$ is bounded from below on S_k and its infimum m_k on S_k is attained at $\tilde{u}_k \in S_k$. Moreover, $m_k = \mathscr{E}(\tilde{u}_k) < 0$ for every $k \in \mathbb{N}$.*

Proof. Note that for every $u \in S_k$, we have

$$\mathscr{E}(u) = \frac{1}{p}\|u\|_\lambda^p - \mathscr{G}(u) \geq -\|\beta\|_1 \max_{[r,b_k]} G > -\infty.$$

It is clear that S_k is convex and closed, thus weakly closed in $W^{1,p}(\Omega)$. Let $m_k = \inf_{S_k} \mathscr{E}$, and $\{u_n\}$ be a sequence in S_k such that $m_k \leq \mathscr{E}(u_n) \leq m_k + 1/n$ for all $n \in \mathbb{N}$. Then,

$$\frac{\|u_n\|_\lambda^p}{p} \leq m_k + 1 + \|\beta\|_1 \max_{[r,b_k]} G$$

for all $n \in \mathbb{N}$, i.e., $\{u_n\}$ is bounded in $W^{1,p}(\Omega)$. So, up to a subsequence, $\{u_n\}$ weakly converges in $W^{1,p}(\Omega)$ to some $\tilde{u}_k \in S_k$. Applying Lebourg's mean value theorem [see Proposition 2.1(v)], and the subcritical growth of the function g, one can easily conclude that $\mathscr{G}$ is a sequentially weakly continuous function (here, we use the theorem of Rellich–Kondrachov); consequently, $\mathscr{E}$ is sequentially weakly lower semicontinuous, which implies that $\mathscr{E}(\tilde{u}_k) = m_k = \inf_{S_k} \mathscr{E}$.

Now, we prove that $m_k < 0$ for every $k \in \mathbb{N}$. Due to (3.6), we may choose a sequence $\{\tilde{s}_k\} \subset (0, \tilde{s})$ such that $\tilde{s}_k \leq b_k$ for every $k \in \mathbb{N}$, $\lim_{k\to\infty} \tilde{s}_k = 0$, and

$$\frac{pG(\tilde{s}_k)}{\tilde{s}_k^p} > \frac{\int_\Omega \alpha(x)\,dx}{\int_\Omega \beta(x)\,dx} - c_0.$$

Let $w_k(x) = \tilde{s}_k$ for every $k \in \mathbb{N}$ and $x \in \Omega$. Then $w_k \in S_k$ and on account of the above inequality and $\lambda(x) = \alpha(x) - c_0\beta(x)$, one has

$$\mathscr{E}(w_k) = \frac{1}{p}\tilde{s}_k^p \int_\Omega \lambda(x)\,dx - G(\tilde{s}_k)\int_\Omega \beta(x)\,dx < 0.$$

Consequently, $m_k = \inf_{S_k} \mathscr{E} \leq \mathscr{E}(w_k) < 0$. □

Proposition 3.2. $0 \leq \tilde{u}_k(x) \leq a_k$ *a.e.* $x \in \Omega$. [*The number a_k is from* (3.5).]

Proof. Let $W = \{x \in \Omega : \tilde{u}_k(x) \notin [0, a_k]\}$ and suppose that meas$(W) > 0$. Define the function $h(s) = \min(s_+, a_k)$ and set $\tilde{w}_k = h \circ \tilde{u}_k$. Due to Marcus–Mizel [13], $\tilde{w}_k$ belongs to $W^{1,p}(\Omega)$ (as h is Lipschitz continuous). Moreover $\tilde{w}_k \in S_k$. We introduce the following two sets

$$W_1 = \{x \in W : \tilde{u}_k(x) < 0\} \quad \text{and} \quad W_2 = \{x \in W : \tilde{u}_k(x) > a_k\}.$$

Then, $W = W_1 \cup W_2$, and we have that $\tilde{w}_k(x) = \tilde{u}_k(x)$ for all $x \in \Omega \setminus W$, $\tilde{w}_k(x) = 0$ for all $x \in W_1$, and $\tilde{w}_k(x) = a_k$ for all $x \in W_2$. Moreover,

$$\begin{aligned}
&\mathscr{E}(\tilde{w}_k) - \mathscr{E}(\tilde{u}_k) \\
&\quad = -\frac{1}{p}\int_W |\nabla \tilde{u}_k|^p dx + \frac{1}{p}\int_W \lambda(x)[|\tilde{w}_k|^p - |\tilde{u}_k|^p]dx \\
&\qquad - \int_W \beta(x)[G(\tilde{w}_k) - G(\tilde{u}_k)]dx \\
&\quad = -\frac{1}{p}\int_W |\nabla \tilde{u}_k|^p dx - \frac{1}{p}\int_{W_1} \lambda(x)|\tilde{u}_k|^p dx + \frac{1}{p}\int_{W_2} \lambda(x)[a_k^p - \tilde{u}_k^p]dx \\
&\qquad - \int_{W_1} \beta(x)[G(0) - G(\tilde{u}_k(x))]dx - \int_{W_2} \beta(x)[G(a_k) - G(\tilde{u}_k(x))]dx.
\end{aligned}$$

Note that $\int_{W_1} \beta(x)[G(0) - G(\tilde{u}_k(x))]dx = 0$. Next, applying Lebourg's mean value theorem, we obtain

$$\int_{W_2} \beta(x)[G(a_k) - G(\tilde{u}_k(x))]dx = \int_{W_2} \beta(x)\xi_k(x)(a_k - \tilde{u}_k(x))dx,$$

where $\xi_k(x) \in \partial G(\tau_k(x))$ for some $\tau_k(x) \in [a_k, \tilde{u}_k(x)] \subseteq [a_k, b_k]$, a.e. $x \in W_2$. Due to (3.5), we have $\xi_k(x) \leq 0$ for a.e. $x \in W_2$; consequently,

$$\int_{W_2} \beta(x)[G(a_k) - G(\tilde{u}_k(x))]dx \geq 0.$$

In conclusion, every term of the above expression is nonpositive. On the other hand, because $\mathscr{E}(\tilde{w}_k) \geq \mathscr{E}(\tilde{u}_k) = \inf_{S_k} \mathscr{E}$, then every term should be zero. In particular,

$$\int_{W_1} \lambda(x)|\tilde{u}_k|^p = \int_{W_2} \lambda(x)[a_k^p - \tilde{u}_k^p] = 0.$$

These equalities imply that $\text{meas}(W_1) = \text{meas}(W_2) = 0$, so $\text{meas}(W) = 0$. □

Proposition 3.3. $\lim_{k\to\infty} m_k = 0$.

Proof. The upper semicontinuity of the multivalued function ∂G implies that it maps compact sets into bounded sets. In particular,

$$M_G := \max_{s\in[0,a_1]} \{|\xi| : \xi \in \partial G(s)\} < +\infty.$$

Applying Lebourg's mean value theorem, for every $s \in [0, a_1]$, we have

$$|G(s)| = |G(s) - G(0)| \leq M_G s. \tag{3.9}$$

Using Proposition 3.2, we have

$$\begin{aligned} m_k = \mathcal{E}(\tilde{u}_k) &\geq -\int_\Omega \beta(x) G(\tilde{u}_k(x))dx \\ &\geq -M_G \int_\Omega \beta(x)\tilde{u}_k(x)dx \geq -M_G\|\beta\|_1 a_k. \end{aligned}$$

Because $\lim_{k\to\infty} a_k = 0$, thus $\lim_{k\to\infty} m_k \geq 0$. On the other hand, $m_k < 0$ for every $k \in \mathbb{N}$, see Proposition 3.1, which concludes our proof. □

Combining Propositions 3.1 and 3.3, one can see that there are infinitely many distinct elements $\tilde{u}_k$. The following result is the "core" of this section.

Proposition 3.4. *$\tilde{u}_k$ is a solution for (P) for every $k \in \mathbb{N}$.*

Proof. Let ψ_{S_k} be the indicator function of the set S_k [i.e., $\psi_{S_k}(u) = 0$ if $u \in S_k$, and $\psi_{S_k}(u) = +\infty$ otherwise] and define the functional $I_k : W^{1,p}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ by $I_k = \mathcal{E} + \psi_{S_k}$. Because $\tilde{u}_k$ is a minimum point of $\mathcal{E}$ relative to the set S_k, see Proposition 3.1, it will be a critical point of I_k in the sense of Motreanu–Panagiotopoulos, see [14, Chapter 3]. Consequently,

$$\mathcal{E}^0(\tilde{u}_k; w - \tilde{u}_k) + \psi_{S_k}(w) - \psi_{S_k}(\tilde{u}_k) \geq 0, \quad \forall w \in W^{1,p}(\Omega).$$

In particular,

$$\mathcal{E}^0(\tilde{u}_k; w - \tilde{u}_k) \geq 0, \quad \forall w \in S_k.$$

Therefore, using Proposition 2.1(iii), (iv), for every $w \in S_k$ we have

$$\begin{aligned} 0 &\leq \mathcal{E}^0(\tilde{u}_k; w - \tilde{u}_k) \\ &= \int_\Omega [|\nabla \tilde{u}_k|^{p-2}\nabla\tilde{u}_k \nabla(w - \tilde{u}_k) + \lambda(x)\tilde{u}_k^{p-1}(w - \tilde{u}_k)]dx + (-\mathcal{G})^0(\tilde{u}_k; w - \tilde{u}_k) \\ &= \int_\Omega [|\nabla \tilde{u}_k|^{p-2}\nabla\tilde{u}_k \nabla(w - \tilde{u}_k) + \lambda(x)\tilde{u}_k^{p-1}(w - \tilde{u}_k)]dx + \mathcal{G}^0(\tilde{u}_k; -w + \tilde{u}_k). \end{aligned}$$

Taking into account relation (3.7), for every $w \in S_k$ we obtain

$$0 \leq \int_\Omega [|\nabla \tilde{u}_k|^{p-2}\nabla \tilde{u}_k \nabla(w - \tilde{u}_k) + \lambda(x)\tilde{u}_k^{p-1}(w - \tilde{u}_k)]dx + \int_\Omega \beta(x)G^0(\tilde{u}_k(x); -w(x) + \tilde{u}_k(x))dx. \tag{3.10}$$

Let us define the function $h(s) = \min(b_k, \max(s, r))$, and fix $\varepsilon > 0$ and $v \in W^{1,p}(\Omega)$ arbitrarily. Because h is Lipschitz continuous, $w_k = h \circ (\tilde{u}_k + \varepsilon v)$ belongs to $W^{1,p}(\Omega)$, see Marcus–Mizel [13]. The explicit expression of w_k is

$$w_k(x) = \begin{cases} r, & \text{if } x \in \{\tilde{u}_k + \varepsilon v < r\} \\ \tilde{u}_k(x) + \varepsilon v(x), & \text{if } x \in \{r \leq \tilde{u}_k + \varepsilon v < b_k\} \\ b_k & \text{if } x \in \{b_k \leq \tilde{u}_k + \varepsilon v\}. \end{cases}.$$

Therefore, $w_k \in S_k$. Considering $w = w_k$ as a test function in (3.10), we obtain

$$\begin{aligned} 0 \leq &-\int_{\{\tilde{u}_k+\varepsilon v<r\}} |\nabla \tilde{u}_k|^p + \int_{\{\tilde{u}_k+\varepsilon v<r\}} \lambda(x)\tilde{u}_k^{p-1}(r - \tilde{u}_k) \\ &+ \int_{\{\tilde{u}_k+\varepsilon v<r\}} \beta(x)G^0(\tilde{u}_k(x); -r + \tilde{u}_k(x))dx \\ &+ \varepsilon \int_{\{r\leq\tilde{u}_k+\varepsilon v<b_k\}} |\nabla \tilde{u}_k|^{p-2}\nabla \tilde{u}_k \nabla v + \varepsilon \int_{\{r\leq\tilde{u}_k+\varepsilon v<b_k\}} \lambda(x)\tilde{u}_k^{p-1} v \\ &+ \int_{\{r\leq\tilde{u}_k+\varepsilon v<b_k\}} \beta(x)G^0(\tilde{u}_k(x); -\varepsilon v(x))dx \\ &- \int_{\{b_k\leq\tilde{u}_k+\varepsilon v\}} |\nabla \tilde{u}_k|^p + \int_{\{b_k\leq\tilde{u}_k+\varepsilon v\}} \lambda(x)\tilde{u}_k^{p-1}(b_k - \tilde{u}_k) \\ &+ \int_{\{b_k\leq\tilde{u}_k+\varepsilon v\}} \beta(x)G^0(\tilde{u}_k(x); -b_k + \tilde{u}_k(x))dx. \end{aligned}$$

After a suitable rearrangement of the terms of the above inequality, using as well the positive homogeneity and subadditivity of $G^0(s;\cdot)$, see Proposition 2.1(i), (ii), we obtain that

$$\begin{aligned} 0 \leq\ &\varepsilon \int_\Omega |\nabla \tilde{u}_k|^{p-2}\nabla \tilde{u}_k \nabla v + \varepsilon \int_\Omega \lambda(x)\tilde{u}_k^{p-1} v + \varepsilon \int_\Omega \beta(x)G^0(\tilde{u}_k(x); -v(x))dx \\ &+ \int_{\{\tilde{u}_k+\varepsilon v<r\}} \lambda(x)|r|^{p-2}r(r - \tilde{u}_k - \varepsilon v) - \int_{\{b_k\leq\tilde{u}_k+\varepsilon v\}} \lambda(x)b_k^{p-1}(\tilde{u}_k + \varepsilon v - b_k) \end{aligned}$$

$$
\begin{aligned}
&- \int_{\{\tilde{u}_k+\varepsilon v<r\}} |\nabla \tilde{u}_k|^p - \int_{\{\tilde{u}_k+\varepsilon v<r\}} \lambda(x)\big(\tilde{u}_k^{p-1} - |r|^{p-2}r\big)(\tilde{u}_k - r) \\
&- \varepsilon \int_{\{\tilde{u}_k+\varepsilon v<r\}} |\nabla \tilde{u}_k|^{p-2}\nabla \tilde{u}_k \nabla v - \varepsilon \int_{\{\tilde{u}_k+\varepsilon v<r\}} \lambda(x)\big(\tilde{u}_k^{p-1} - |r|^{p-2}r\big)v \\
&- \int_{\{b_k\le\tilde{u}_k+\varepsilon v\}} |\nabla \tilde{u}_k|^p + \int_{\{b_k\le\tilde{u}_k+\varepsilon v\}} \lambda(x)\big(b_k^{p-1} - \tilde{u}_k^{p-1}\big)(\tilde{u}_k - b_k) \\
&- \varepsilon \int_{\{b_k\le\tilde{u}_k+\varepsilon v\}} |\nabla \tilde{u}_k|^{p-2}\nabla \tilde{u}_k \nabla v + \varepsilon \int_{\{b_k\le\tilde{u}_k+\varepsilon v\}} \lambda(x)\big(b_k^{p-1} - \tilde{u}_k^{p-1}\big)v \\
&+ \int_{\{\tilde{u}_k+\varepsilon v<r\}} \beta(x)G^0(\tilde{u}_k(x); \tilde{u}_k(x) + \varepsilon v(x) - r)\,dx \\
&+ \int_{\{b_k\le\tilde{u}_k+\varepsilon v\}} \beta(x)G^0(\tilde{u}_k(x); \tilde{u}_k(x) + \varepsilon v(x) - b_k)\,dx.
\end{aligned}
$$

First, because $r < 0$, one has

$$
\begin{aligned}
\int_{\{\tilde{u}_k+\varepsilon v<r\}} \lambda(x)|r|^{p-2}r(r - \tilde{u}_k - \varepsilon v) &\le 0 \\
&\le \int_{\{\tilde{u}_k+\varepsilon v<r\}} \lambda(x)\big(\tilde{u}_k^{p-1} - |r|^{p-2}r\big)(\tilde{u}_k - r).
\end{aligned}
$$

Second, we have

$$
\begin{aligned}
&\int_{\{b_k\le\tilde{u}_k+\varepsilon v\}} \lambda(x)b_k^{p-1}(\tilde{u}_k + \varepsilon v - b_k) \ge 0 \\
&\qquad\qquad \ge \int_{\{b_k\le\tilde{u}_k+\varepsilon v\}} \lambda(x)\big(b_k^{p-1} - \tilde{u}_k^{p-1}\big)(\tilde{u}_k - b_k).
\end{aligned}
$$

Third, because by Proposition 3.2 we have $\tilde{u}_k(x) \in [0, a_k]$ a.e. $x \in \Omega$, one can find $C_k > 0$ such that $|G^0(\tilde{u}_k(x); s)| \le C_k|s|$ for every $s \in \mathbb{R}$ and a.e. $x \in \Omega$. Thus,

$$
\begin{aligned}
&\int_{\{\tilde{u}_k+\varepsilon v<r\}} \beta(x)G^0(\tilde{u}_k(x); \tilde{u}_k(x) + \varepsilon v(x) - r)\,dx \\
&\quad \le C_k \int_{\{\tilde{u}_k+\varepsilon v<r\}} \beta(x)(r - \tilde{u}_k(x) - \varepsilon v(x))\,dx \\
&\quad \le -\varepsilon C_k \int_{\{\tilde{u}_k+\varepsilon v<r\}} \beta(x)v(x)\,dx.
\end{aligned}
$$

In a similar way, we have

$$\begin{aligned}\int_{\{b_k\le \tilde{u}_k+\varepsilon v\}} &\beta(x)G^0(\tilde{u}_k(x);\tilde{u}_k(x)+\varepsilon v(x)-b_k)\,dx\\ &\le C_k\int_{\{b_k\le \tilde{u}_k+\varepsilon v\}}\beta(x)(\tilde{u}_k(x)+\varepsilon v(x)-b_k)\,dx\\ &\le \varepsilon C_k\int_{\{b_k\le \tilde{u}_k+\varepsilon v\}}\beta(x)v(x)\,dx.\end{aligned}$$

Taking into account the above estimates and dividing by $\varepsilon>0$, we obtain that

$$\begin{aligned}0\le{}&\int_\Omega|\nabla\tilde{u}_k|^{p-2}\nabla\tilde{u}_k\nabla v+\int_\Omega\lambda(x)\tilde{u}_k^{p-1}v+\int_\Omega\beta(x)G^0(\tilde{u}_k(x);-v(x))\,dx\\ &-\int_{\{\tilde{u}_k+\varepsilon v<r\}}|\nabla\tilde{u}_k|^{p-2}\nabla\tilde{u}_k\nabla v-\int_{\{\tilde{u}_k+\varepsilon v<r\}}\lambda(x)\big(\tilde{u}_k^{p-1}-|r|^{p-2}r\big)v\\ &-\int_{\{b_k\le\tilde{u}_k+\varepsilon v\}}|\nabla\tilde{u}_k|^{p-2}\nabla\tilde{u}_k\nabla v+\int_{\{b_k\le\tilde{u}_k+\varepsilon v\}}\lambda(x)\big(b_k^{p-1}-\tilde{u}_k^{p-1}\big)v\\ &-C_k\int_{\{\tilde{u}_k+\varepsilon v<r\}}\beta(x)v(x)\,dx+C_k\int_{\{b_k\le\tilde{u}_k+\varepsilon v\}}\beta(x)v(x)\,dx.\end{aligned}$$

Now, letting $\varepsilon\to0^+$, and taking into account Proposition 3.2 [i.e., $0\le\tilde{u}_k(x)\le a_k$ a.e. $x\in\Omega$], we have $\operatorname{meas}(\{\tilde{u}_k+\varepsilon v<r\})\to0$ and $\operatorname{meas}(\{b_k\le\tilde{u}_k+\varepsilon v\})\to0$, respectively. Consequently, the above inequality reduces to

$$0\le\int_\Omega|\nabla\tilde{u}_k|^{p-2}\nabla\tilde{u}_k\nabla v+\int_\Omega\lambda(x)\tilde{u}_k^{p-1}v+\int_\Omega\beta(x)G^0(\tilde{u}_k(x);-v(x))\,dx.$$

Due to (3.4), (3.1), and to the fact that $\lambda(x)=\alpha(x)-c_0\beta(x)$, we obtain

$$0\le\int_\Omega|\nabla\tilde{u}_k|^{p-2}\nabla\tilde{u}_k\nabla v+\int_\Omega\alpha(x)\tilde{u}_k^{p-1}v+\int_\Omega\beta(x)F^0(\tilde{u}_k(x);-v(x))\,dx.$$

Because $v\in W^{1,p}(\Omega)$ was arbitrarily chosen, $\tilde{u}_k$ is a non-negative solution for (P). The proof is complete. □

In order to conclude the proof of Theorem 1.2, we prove:

Proposition 3.5. $\lim_{k\to\infty}\|\tilde{u}_k\|_\infty=\lim_{k\to\infty}\|\tilde{u}_k\|_{W^{1,p}}=0$.

Proof. Because $0\le\tilde{u}_k(x)\le a_k$ for every $k\in\mathbb{N}$ and a.e. $x\in\Omega$ (cf. Proposition 3.2), and $\lim_{k\to\infty}a_k=0$, we have $\lim_{k\to\infty}\|\tilde{u}_k\|_\infty=0$.

Note that

$$\frac{\|\tilde{u}_k\|_\lambda^p}{p} = m_k + \int_\Omega \beta(x)G(\tilde{u}_k(x))dx.$$

Using inequality (3.9), we have $|\int_\Omega \beta(x)G(\tilde{u}_k(x))dx| \leq \|\beta\|_1 M_G a_k$. Combining this relation and Proposition 3.3, we obtain that $\lim_{k\to\infty}\|\tilde{u}_k\|_{W^{1,p}} = 0$. □

4. PROOF OF THEOREM 1.3

The proof of Theorem 1.3 is similar to that of Theorem 1.2; consequently, we only outline the proof. We assume the hypotheses of Theorem 1.3 are fulfilled. Because $f(0) = 0$, we may put $f(s) = F(s) = 0$ for every $s \leq 0$. Due to (H_∞), one can fix $c_\infty \in \mathbb{R}$ such that

$$\text{essinf}_\Omega \frac{\alpha}{\beta} > c_\infty > \liminf_{s\to\infty} \frac{f_u(s)}{s^{p-1}}.$$

In particular, there is a sequence $\{s_k\} \subset (0, \infty)$ converging (increasingly) to $+\infty$, such that

$$f_u(s_k) < c_\infty s_k^{p-1}. \tag{4.1}$$

We define the functions

$$\begin{aligned} g(s) &= f(s) - c_\infty s_+^{p-1} \quad \text{and} \\ G(s) &= \int_0^s g(t)dt = F(s) - \frac{c_\infty}{p} s_+^p, \quad s \in \mathbb{R}. \end{aligned} \tag{4.2}$$

It is clear that $g \in L^\infty_{\text{loc}}(\mathbb{R})$ and G is locally Lipschitz. A similar reason as in the previous section—using (4.1) instead of (3.3)—shows that there are two sequences $\{a_k\}$ and $\{b_k\}$ with positive terms, both converging to $+\infty$, such that

$$\partial G(s) \subset (-\infty, 0] \quad \text{for every } s \in [a_k, b_k], \tag{4.3}$$

and $a_k < b_k < a_{k+1}$ for every $k \in \mathbb{N}$.

On the other hand, on account of (4.2) and (H_∞), we have

$$\limsup_{s\to\infty} \frac{pG(s)}{s^p} > \frac{\int_\Omega \alpha(x)dx}{\int_\Omega \beta(x)dx} - c_\infty. \tag{4.4}$$

The functional $\mathscr{G} : W^{1,p}(\Omega) \to \mathbb{R}$, the function $\lambda(x) = \alpha(x) - c_\infty\beta(x)$, the norm $\|\cdot\|_\lambda$, and the functional $\mathscr{E} : W^{1,p}(\Omega) \to \mathbb{R}$ will be defined in the same way as in the previous section. We emphasize that both $\mathcal{G}$ and $\mathcal{E}$ are well defined and locally Lipschitz. Indeed, on account of (f_∞), one can find a constant $C_q > 0$ such that

$$|\xi| \leq C_q(1 + |s|^{q-1}), \quad \forall \xi \in \partial G(s), \; s \in \mathbb{R}, \tag{4.5}$$

where $q < p^*$. The local Lipschitzianness of $\mathcal{G}$ and $\mathcal{E}$ follows in a standard way.

Let us fix a number $r < 0$ arbitrarily, and for every $k \in \mathbb{N}$, consider the set

$$T_k = \{u \in W^{1,p}(\Omega) : r \leq u(x) \leq b_k \text{ a.e. } x \in \Omega\},$$

where b_k is from (4.3).

Proposition 4.1. *The functional $\mathcal{E}$ is bounded from below on T_k and its infimum $\tilde{m}_k$ on T_k is attained at $\tilde{w}_k \in T_k$. Moreover, $\lim_{k\to\infty} \tilde{m}_k = -\infty$.*

Proof. The first part is exactly the same as in Proposition 3.1; the sequentially weakly lower semicontinuity of $\mathcal{E}$ follows from hypothesis (f_∞) [in particular, from relation (4.5)] and Rellich–Kondrachov theorem.

Let us prove $\lim_{k\to\infty} \tilde{m}_k = -\infty$. Due to (4.4), we may choose a number $M_\infty \in \mathbb{R}$ and an increasing sequence $\{\tilde{s}_k\} \subset (0, \infty)$ such that $\lim_{k\to\infty} \tilde{s}_k = \infty$, and

$$\frac{pG(\tilde{s}_k)}{\tilde{s}_k^p} > M_\infty > \frac{\int_\Omega \alpha(x)\,dx}{\int_\Omega \beta(x)\,dx} - c_\infty. \tag{4.6}$$

Because $\lim_{k\to\infty} b_k = \infty$, one can fix a subsequence $\{b_{n_k}\}$ of $\{b_k\}$ such that $\tilde{s}_k \leq b_{n_k}$ for every $k \in \mathbb{N}$. Let $w_k(x) = \tilde{s}_k$ for every $k \in \mathbb{N}$ and $x \in \Omega$. Then $w_k \in T_{n_k}$ and on account of the first inequality from (4.6), one has

$$\begin{aligned}
\mathcal{E}(w_k) &= \frac{1}{p}\tilde{s}_k^p \int_\Omega \lambda(x)\,dx - G(\tilde{s}_k)\int_\Omega \beta(x)\,dx \\
&< \frac{1}{p}\tilde{s}_k^p \left(\frac{\int_\Omega \alpha(x)\,dx}{\int_\Omega \beta(x)\,dx} - c_\infty - M_\infty\right)\int_\Omega \beta(x)\,dx.
\end{aligned}$$

Because $\frac{\int_\Omega \alpha(x)\,dx}{\int_\Omega \beta(x)\,dx} - c_\infty - M_\infty < 0$, see the second inequality from (4.6), we have

$$\lim_{k\to\infty} \tilde{m}_{n_k} = \lim_{k\to\infty} \inf_{T_{n_k}} \mathcal{E} \leq \lim_{k\to\infty} \mathcal{E}(w_k) = -\infty.$$

Because the sequence $\{\tilde{m}_k\}$ is nonincreasing, we conclude the proof. □

On account of Proposition 4.1, one can find infinitely many distinct elements $\tilde{w}_k$. Moreover, using (4.3) instead of (3.5), one can prove a result analogous to Proposition 3.2 establishing that $0 \leq \tilde{w}_k(x) \leq a_k$ for a.e. $x \in \Omega$ and $k \in \mathbb{N}$, the proof being similar to the one of Proposition 3.2. A similar reasoning as in Proposition 3.4 shows that $\tilde{w}_k$ is a solution for (P) for every $k \in \mathbb{N}$. In order to conclude the proof of Theorem 1.3, it remains to prove:

Proposition 4.2. $\lim_{k\to\infty}\|\tilde{w}_k\|_\infty = \lim_{k\to\infty}\|\tilde{w}_k\|_{W^{1,p}} = \infty$.

Proof. Arguing by contradiction, assume that there exists a subsequence $\{\tilde{w}_{n_k}\}$ of $\{\tilde{w}_k\}$ such that $\|\tilde{w}_{n_k}\|_\infty \leq M$ for some $M > 0$. In particular, $\{\tilde{w}_{n_k}\} \subset T_l$ for some $l \in \mathbb{N}$. Therefore, for every $n_k \geq l$, we have

$$\tilde{m}_l \geq \tilde{m}_{n_k} = \inf_{T_{n_k}} \mathscr{E} = \mathscr{E}(\tilde{w}_{n_k}) \geq \inf_{T_l} \mathscr{E} = \tilde{m}_l.$$

Consequently, $\tilde{m}_{n_k} = \tilde{m}_l$ for every $n_k \geq l$, which contradicts Proposition 4.1. This concludes the first part of the proof.

Now, if $p > N$, combining the first part with the continuous embedding $W^{1,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$, we have $\lim_{k\to\infty}\|\tilde{w}_k\|_{W^{1,p}} = \infty$. If $p \leq N$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous where q appears in (f_∞). On account of Lebourg's mean value theorem, (4.5), and Hölder's inequality, there exists $C' > 0$ such that

$$|\mathscr{G}(u)| \leq C'\big(\|u\|_{W^{1,p}} + \|u\|^q_{W^{1,p}}\big), \quad \forall u \in W^{1,p}(\Omega).$$

Let us assume that there exists a subsequence $\{\tilde{w}_{n_k}\}$ of $\{\tilde{w}_k\}$ such that for some $M > 0$, we have $\|\tilde{w}_{n_k}\|_{W^{1,p}} \leq M$. In particular, due to the above inequality, the sequence $\{\frac{\|\tilde{w}_{n_k}\|^p_\lambda}{p} - \mathscr{G}(\tilde{w}_{n_k})\}$ is bounded. But

$$\tilde{m}_{n_k} = \mathscr{E}(\tilde{w}_{n_k}) = \frac{\|\tilde{w}_{n_k}\|^p_\lambda}{p} - \mathscr{G}(\tilde{w}_{n_k}),$$

i.e., the sequence $\{\tilde{m}_{n_k}\}$ is also bounded. This fact contradicts Proposition 4.1. □

5. EXAMPLES AND COMPARISONS WITH EARLIER WORKS

In this section, we give two simple examples of functions that verify our hypotheses. Furthermore, we emphasize that [12, Theorem 2.2], [19, Theorem 2] follow from our Theorem 1.2 in a very natural way.

Example 5.1. Let $f : [0, +\infty) \to \mathbb{R}$ be defined by $f(0) = 0$ and

$$f(s) = -s^{p-1} \sin \frac{1}{s^{p-1}} + \frac{p-1}{p} \cos \frac{1}{s^{p-1}} + h(s), \quad s > 0,$$

where $h \in L^\infty_{\text{loc}}([0, +\infty))$ satisfies $|h(s)| \leq c_h s^{p-1}$ for all $s \geq 0$ with $c_h > 0$. Then (H_0) is verified for every $\alpha, \beta \in L^\infty(\Omega)$, with $\text{essinf}_\Omega \beta > 0$ and

$$\int_\Omega \alpha(x)\, dx < (1 - c_h)\|\beta\|_1. \tag{5.1}$$

[In particular, if α is nonpositive and $c_h \leq 1$, (5.1) is evidently fulfilled.] We observe that $F(0) = 0$ and $F(s) = -\frac{s^p}{p} \sin \frac{1}{s^{p-1}} + \int_0^s h(t)\, dt, \quad s > 0$. Consequently, $\limsup_{s\to 0^+} \frac{pF(s)}{s^p} \geq 1 - c_h$ and $\liminf_{s\to 0^+} \frac{f_u(s)}{s^{p-1}} = -\infty$, which justifies the assertion above, and one can apply Theorem 1.2.

Remark 5.2. The function from Example 5.1 with $h = 0$ appears in Marano–Motreanu [12, p. 119]; in particular, we may choose $c_h = 0$ in this case. In order to obtain the same conclusion as we did [the existence of infinitely many solutions for (P)], the authors in [12] assumed that $p > N$, $\text{essinf}_\Omega \alpha > 0$, and

$$\|\alpha\|_1 = \frac{1}{c^p} < \|\beta\|_1, \tag{5.2}$$

where $c = \sup\{\|u\|_\alpha^{-1}\|u\|_\infty : u \in W^{1,p}(\Omega), u \neq 0\}$ and $\|\cdot\|_\alpha$ comes from (3.8). Because $c_h = 0$, it can be seen that (5.1) is weaker than (5.2), and they coincide when $\alpha \geq 0$.

Example 5.3. Let $f : [0, \infty) \to \mathbb{R}$ be defined by $f(s) = s^{[p]+1} \sin s + h(s)$, where $[p]$ denotes the integer part of $p \in \mathbb{R}$ and $h \in L^\infty_{\text{loc}}([0, +\infty))$ is as in Example 5.1. Then (H_∞) and (f_∞) are verified for every $\alpha, \beta \in L^\infty(\Omega)$ with $\text{essinf}_\Omega \beta > 0$ and $N < p(p+2)/2$, respectively. (No further assumption is needed for α.) Indeed, a simple computation shows that $\limsup_{s\to\infty} \frac{pF(s)}{s^p} = +\infty$ and $\liminf_{s\to\infty} \frac{f_u(s)}{s^{p-1}} = \liminf_{s\to\infty} \frac{f(s)}{s^{p-1}} = -\infty$. Therefore, (H_∞) is trivially verified, and $N < p(p+2)/2$ implies (f_∞); thus, one can apply Theorem 1.3. If $h = 0$, the function f is continuous, so the solutions of (P) will be weak solutions for (P_0).

Remark 5.4. The function from Example 5.3 with $h = 0$ appeared in Faraci–Kristály [8], where the hypotheses $p > N$ and $\text{essinf}_\Omega \alpha > 0$ were indispensable.

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