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MULTIPLE SOLUTIONS FOR ELLIPTIC PROBLEMS WITH SINGULAR AND SUBLINEAR POTENTIALS

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Abstract. For certain positive numbers μ and λ, we establish the multiplicity of solutions to the problem

$$\begin{cases} -\triangle u = \mu \frac{u}{|x|^2} + \lambda f(u) & \text{a.e. in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open domain in $\mathbb{R}^N$ $(N \geq 3)$ containing the origin with smooth boundary $\partial\Omega$, while $f : \mathbb{R} \to \mathbb{R}$ is continuous, superlinear at zero and sublinear at infinity.

1. Introduction

Consider the problem

$$(1_{\mu,\lambda}) \qquad \begin{cases} -\triangle u = \mu \frac{u}{|x|^2} + g(\lambda, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded open domain with smooth boundary $\partial\Omega$ and $0 \in \Omega$, $g : (0, +\infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $0 \leq \mu < \overline{\mu} = (N-2)^2/4$.

Problem $(1_{0,\lambda})$ has been extensively studied during the last few years (see [7, 8, 10] and references therein), where $g(\lambda, s) = \lambda f(s)$, $f : \mathbb{R} \to \mathbb{R}$ being a continuous function provided with certain growth properties at zero and infinity, respectively.

When $\mu \neq 0$, problem $(1_{\mu,\lambda})$ becomes more delicate due to the presence of the singular potential. Several papers were devoted in order to handle this problem (see [5, 6, 9, 13]). For instance, Ferrero and Gazzola [5], and Ruiz and Willem [13], considered $g(\lambda, s) = |s|^{2^*-2}s + \lambda s$, establishing for certain values of μ and λ the existence of one nontrivial positive solution for $(1_{\mu,\lambda})$. In [9], Montefusco considered $g(\lambda, s) = |s|^{q-2}s$ with either $q \in (1, 2)$, or $q \in (2, 2^*)$, guaranteeing in both cases a nontrivial solution for $(1_{\mu,\lambda})$ whenever $\mu \in (0, \overline{\mu})$ is arbitrarily fixed. Faraci and Livrea [4] exploited Montefusco's result, establishing certain bifurcation theorems which involve the p-Laplacian and its corresponding singular term. Very recently, Chen [2, 3] characterized the exact growth order near the origin of the positive solutions of $(1_{\mu,\lambda})$ in the case when $g(\lambda, s) = s_+^{2^*-1} + \lambda s_+^q$ $(0 < q < 1)$. By means of this construction, Chen was able to obtain multiple solutions of $(1_{\mu,\lambda})$ for certain values of $\lambda > 0$ whenever $0 \leq \mu < \overline{\mu} - 1$ $(N \geq 5)$.

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The purpose of this paper is to obtain *multiple* solutions for the problem

$$(\mathrm{P}_{\mu,\lambda}) \qquad \begin{cases} -\triangle u = \mu \frac{u}{|x|^2} + \lambda f(u) & \text{a.e. in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

(which is nothing but $(1_{\mu,\lambda})$ with $g(\lambda, s) = \lambda f(s)$) when $f : \mathbb{R} \to \mathbb{R}$ is *superlinear at zero*, i.e.

$$\text{(f1)} \qquad \lim_{s\to 0} \frac{f(s)}{s} = 0,$$

and *sublinear at infinity*, i.e.

$$\text{(f2)} \qquad \lim_{|s|\to +\infty} \frac{f(s)}{s} = 0.$$

Denoting by $F(s) = \int_0^s f(t)dt$, we finally assume that

$$\text{(f3)} \qquad \sup_{s\in\mathbb{R}} F(s) > 0.$$

Our main result is

Theorem 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies* (f1), (f2) *and* (f3). *Then for every $\mu \in [0, \overline{\mu})$ there exist an open interval $\Lambda_\mu \subset (0, +\infty)$ and a real number $\sigma_\mu > 0$ such that for every $\lambda \in \Lambda_\mu$ problem $(\mathrm{P}_{\mu,\lambda})$ has at least two distinct, nontrivial weak solutions in $H_0^1(\Omega)$ whose H_0^1-norms are less than σ_μ.*

We emphasize that hypotheses (f3) cannot be omitted. Indeed, if for instance $f \equiv 0$, then (f1) and (f2) clearly hold, but problem $(\mathrm{P}_{\mu,\lambda})$ has only the trivial solution.

Theorem 1 will be proved by means of a recent three critical point result of Bonanno [1] which is actually a refinement of a general principle of Ricceri [11, 12]. Furthermore, in Section 3 we will give additional information as far as the localization of the interval Λ_μ is concerned (see Remark 2).

2. Preliminaries

Let Ω be a bounded open domain in $\mathbb{R}^N$ $(N \geq 3)$ containing the origin with smooth boundary $\partial\Omega$. The space $H_0^1(\Omega)$ will be endowed by the standard inner product

$$\langle u, v\rangle_{H_0^1} = \int_\Omega \nabla u \nabla v dx, \quad u, v \in H_0^1(\Omega),$$

and by its corresponding norm $\|\cdot\|_{H_0^1}$. The norm of the dual of $H_0^1(\Omega)$ will be denoted by $\|\cdot\|_{H^{-1}}$. We recall Hardy's inequality which states that

$$\text{(1)} \qquad \int_\Omega \frac{u^2(x)}{|x|^2} dx \leq \frac{1}{\overline{\mu}} \|u\|_{H_0^1}^2, \quad u \in H_0^1(\Omega),$$

where $\overline{\mu} = (N-2)^2/4$ (see [6]).

The usual norm on $L^p(\Omega)$ will be denoted by $\|\cdot\|_{L^p}$. The Sobolev embedding constant of the compact embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$, $p \in [1, 2^*)$, will be denoted by $c_p > 0$, i.e. $\|u\|_{L^p} \leq c_p \|u\|_{H_0^1}$, for every $u \in H_0^1(\Omega)$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $F(s) = \int_0^s f(t)dt$. We introduce the energy functional $\mathcal{E}_{\mu,\lambda} : H_0^1(\Omega) \to \mathbb{R}$ associated to $(\mathrm{P}_{\mu,\lambda})$, i.e.

$$\mathcal{E}_{\mu,\lambda}(u) = \Phi_\mu(u) - \lambda J(u), \quad u \in H_0^1(\Omega),$$

where

$$(2)\quad \Phi_\mu(u) = \frac{1}{2}\|u\|^2_{H^1_0} - \frac{\mu}{2}\int_\Omega \frac{u^2(x)}{|x|^2}dx \quad \text{and} \quad J(u) = \int_\Omega F(u(x))dx, \quad u \in H^1_0(\Omega).$$

As long as (f2) is verified, a standard argument shows that $\mathcal{E}_{\mu,\lambda} : H^1_0(\Omega) \to \mathbb{R}$ is of class C^1 and its critical points are exactly the weak solutions of $(\mathrm{P}_{\mu,\lambda})$.

Therefore, it is enough to show the existence of multiple critical points of $\mathcal{E}_{\mu,\lambda} : H^1_0(\Omega) \to \mathbb{R}$ for the parameters μ and λ specified in Theorem 1. This fact will be carried out by means of the following recent critical point result.

Theorem 2 ([1, Theorem 2.1]). *Let X be a separable and reflexive real Banach space, and let $\Phi, J : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals. Assume that there exists $x_0 \in X$ such that $\Phi(x_0) = J(x_0) = 0$ and $\Phi(x) \geq 0$ for every $x \in X$ and that there exists $x_1 \in X$, $\rho > 0$ such that*

(i) $\rho < \Phi(x_1)$;
(ii) $\sup_{\Phi(x)<\rho} J(x) < \rho\frac{J(x_1)}{\Phi(x_1)}$.

Further, put

$$\overline{a} = \frac{\zeta\rho}{\rho\frac{J(x_1)}{\Phi(x_1)} - \sup_{\Phi(x)<\rho} J(x)},$$

with $\zeta > 1$, assume that the functional $\Phi - \lambda J$ is sequentially weakly lower semicontinuous and satisfies the Palais-Smale condition, and

(iii) $\lim_{\|x\|\to+\infty}(\Phi(x) - \lambda J(x)) = +\infty$,

for every $\lambda \in [0, \overline{a}]$.

Then there is an open interval $\Lambda \subseteq [0, \overline{a}]$ and a number $\sigma > 0$ such that for each $\lambda \in \Lambda$, the equation $\Phi'(x) - \lambda J'(x) = 0$ admits at least three solutions in X having norm less than σ.

3. Proof of Theorem 1

Through this section, we suppose that the assumptions of Theorem 1 are fulfilled. In order to conclude the proof, we apply Theorem 2 by choosing $X = H^1_0(\Omega)$ as well as $\Phi = \Phi_\mu$ and J as in (2). Due to (1) we have at once that $\Phi_\mu(u) \geq 0$ for every $\mu \in [0, \overline{\mu})$ and $u \in X$.

Lemma 1. *For every $\mu \in [0, \overline{\mu}]$ and $\lambda \in \mathbb{R}$, the functional $\mathcal{E}_{\mu,\lambda}$ is sequentially weakly lower semicontinuous on $H^1_0(\Omega)$.*

Proof. Due to (f2), there exists $c > 0$ such that

$$(3)\qquad |f(s)| \leq c(1 + |s|), \ s \in \mathbb{R}.$$

Thus, the sequentially weak continuity of J is achieved in a standard way, by means of the compact embedding $H^1_0(\Omega) \hookrightarrow L^2(\Omega)$. On the other hand, using the concentration-compactness principle, Montefusco proved in [9, Theorem 3.2] the sequentially weakly lower semicontinuity of Φ_μ for every $\mu \in [0, \overline{\mu}]$. □

Lemma 2. *For every $\mu \in [0, \overline{\mu})$ and $\lambda \in \mathbb{R}$, the functional $\mathcal{E}_{\mu,\lambda}$ is coercive and satisfies the Palais-Smale condition.*

Proof. Let us fix $\mu \in [0, \overline{\mu})$ and $\lambda \in \mathbb{R}$ arbitrarily. By (f2), there exists $\delta = \delta(\mu, \lambda) > 0$ such that for every $|s| > \delta$ one has

$$|f(s)| < \frac{1}{2}(1 - \mu/\overline{\mu})c_2^{-2}(1 + |\lambda|)^{-1}|s|.$$

After integration, we obtain

$$|F(s)| \leq \frac{1}{2}(1 - \mu/\overline{\mu})c_2^{-2}(1 + |\lambda|)^{-1}|s|^2 + \max_{|t|\leq\delta}|f(t)||s| \quad \text{for all } s \in \mathbb{R}.$$

Thus, for every $u \in H_0^1(\Omega)$ we have

$$\begin{aligned} \mathcal{E}_{\mu,\lambda}(u) &\geq \frac{1}{2}(1 - \mu/\overline{\mu})\|u\|_{H_0^1}^2 - |\lambda| \int_\Omega |F(u(x))|dx \\ &\geq \frac{1}{2}(1 - \mu/\overline{\mu})(1 + |\lambda|)^{-1}\|u\|_{H_0^1}^2 - c_1|\lambda| \max_{|t|\leq\delta}|f(t)|\|u\|_{H_0^1}. \end{aligned}$$

If $\|u\|_{H_0^1} \to +\infty$ we conclude that $\mathcal{E}_{\mu,\lambda}(u) \to +\infty$ as well, i.e. $\mathcal{E}_{\mu,\lambda}$ is coercive.

Now, let $\{u_n\}$ be a sequence in $H_0^1(\Omega)$ such that $\{\mathcal{E}_{\mu,\lambda}(u_n)\}$ is bounded and $\|\mathcal{E}'_{\mu,\lambda}(u_n)\|_{H^{-1}} \to 0$. Since $\mathcal{E}_{\mu,\lambda}$ is coercive, the sequence $\{u_n\}$ is bounded. Up to a subsequence, we may suppose that $u_n \to u$ weakly in $H_0^1(\Omega)$, and $u_n \to u$ strongly in $L^2(\Omega)$ for some $u \in H_0^1(\Omega)$. On the other hand, we have

$$\begin{aligned} (1 - \mu/\overline{\mu})\|u_n - u\|_{H_0^1}^2 &\leq \|u_n - u\|_{H_0^1}^2 - \mu \int_\Omega \frac{(u_n(x) - u(x))^2}{|x|^2}dx \\ &= \mathcal{E}'_{\mu,\lambda}(u_n)(u_n - u) + \mathcal{E}'_{\mu,\lambda}(u)(u - u_n) \\ &\quad + \lambda \int_\Omega [f(u_n(x)) - f(u(x))](u_n(x) - u(x))dx. \end{aligned}$$

It is clear the first two terms from the last expression tend to 0, while by means of (3) one has

$$\begin{aligned} &\int_\Omega |f(u_n(x)) - f(u(x))||u_n(x) - u(x)|dx \\ &\leq c[2(\text{meas}\Omega)^{1/2} + \|u_n\|_{L^2} + \|u\|_{L^2}]\|u_n - u\|_{L^2} \to 0 \end{aligned}$$

as $n \to \infty$. Thus we proved that $\|u_n - u\|_{H_0^1} \to 0$. □

Lemma 3. *For every* $\mu \in [0, \overline{\mu})$,

$$\lim_{\rho\to 0^+} \frac{\sup\{J(u) : \Phi_\mu(u) < \rho\}}{\rho} = 0.$$

Proof. Fix $\mu \in [0, \overline{\mu})$. Due to (f1), for an arbitrarily small $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that

$$|f(s)| < \frac{\varepsilon}{4}(1 - \mu/\overline{\mu})c_2^{-2}|s| \quad \text{for all } |s| < \delta.$$

For a fixed $p \in (2, 2^*)$, combining (3) with the above inequality, one has

$$|F(s)| \leq \frac{\varepsilon}{4}(1 - \mu/\overline{\mu})c_2^{-2}|s|^2 + c(1 + \delta)\delta^{1-p}|s|^p \quad \text{for all } s \in \mathbb{R}. \tag{4}$$

For $\rho > 0$ define the sets

$$S_\rho^1 = \{u \in H_0^1(\Omega) : \Phi_\mu(u) < \rho\}; \quad S_\rho^2 = \{u \in H_0^1(\Omega) : (1 - \mu/\overline{\mu})\|u\|_{H_0^1}^2 < 2\rho\}.$$

Thanks to (1), $S^1_\rho \subseteq S^2_\rho$. Moreover, by using (4), for every $u \in S^2_\rho$ we have

$$J(u) \leq \frac{\varepsilon}{2}\rho + c(1+\delta)\delta^{1-p}c_p^p 2^{p/2}(1-\mu/\overline{\mu})^{-p/2}\rho^{p/2} \equiv \frac{\varepsilon}{2}\rho + c'\rho^{p/2}.$$

Thus there exists $\rho(\varepsilon) > 0$ such that for every $0 < \rho < \rho(\varepsilon)$

$$0 \leq \frac{\sup_{u\in S^1_\rho} J(u)}{\rho} \leq \frac{\sup_{u\in S^2_\rho} J(u)}{\rho} \leq \frac{\varepsilon}{2} + c'\rho^{\frac{p-2}{2}} < \varepsilon,$$

which completes the proof. □

Let $s_0 \in \mathbb{R}$ such that $F(s_0) > 0$; see (f3). Also choose $R_0 > 0$ in such a way that $R_0 < \operatorname{dist}(0, \partial\Omega)$. For $\sigma \in (0,1)$ define

$$u_\sigma(x) = \begin{cases} 0, & \text{if} \quad x \in \mathbb{R}^N \setminus B_N(0,R_0), \\ s_0, & \text{if} \quad x \in B_N(0,\sigma R_0), \\ \frac{s_0}{R_0(1-\sigma)}(R_0 - |x|), & \text{if} \quad x \in B_N(0,R_0) \setminus B_N(0,\sigma R_0), \end{cases}$$

where $B_N(0,r)$ denotes the N-dimensional open ball with center 0 and radius $r > 0$. It is clear that $u_\sigma \in H^1_0(\Omega)$. Moreover, denoting by ω_N the volume of the N-dimensional unit ball, one has

$$\|u_\sigma\|^2_{H^1_0} = s_0^2(1-\sigma)^{-2}(1-\sigma^N)\omega_N R_0^{N-2}$$

and

$$J(u_\sigma) \geq [F(s_0)\sigma^N - \max_{|t|\leq|s_0|}|F(t)|(1-\sigma^N)]\omega_N R_0^N.$$

For σ close enough to 1, the right-hand side of the last inequality becomes strictly positive; choose such a number, say σ_0.

Proof of Theorem 1 *completed.* Fix $\mu \in [0,\overline{\mu})$. Due to Lemma 3, we may choose $\rho_0 > 0$ such that

$$2\rho_0 < (1-\mu/\overline{\mu})\|u_{\sigma_0}\|^2_{H^1_0},$$
$$\frac{\sup\{J(u) : \Phi_\mu(u) < \rho_0\}}{\rho_0} < \frac{2; [F(s_0)\sigma_0^N - \max_{|t|\leq|s_0|}|F(t)|(1-\sigma_0^N)]\omega_N R_0^N}{\|u_{\sigma_0}\|^2_{H^1_0}}.$$

By choosing $x_1 = u_{\sigma_0}$, hypotheses (i) and (ii) of Theorem 2 are verified. Define

$$\overline{a} = \overline{a}_\mu = \frac{1+\rho_0}{\frac{J(u_{\sigma_0})}{\Phi_\mu(u_{\sigma_0})} - \frac{\sup\{J(u):\Phi_\mu(u)<\rho_0\}}{\rho_0}}. \tag{5}$$

Taking into account Lemmas 1 and 2, and putting $x_0 = 0$, all the assumptions of Theorem 2 are verified. Thus there exist an open interval $\Lambda_\mu \subset [0,\overline{a}_\mu]$ and a number $\sigma_\mu > 0$ such that for each $\lambda \in \Lambda_\mu$, the equation $\mathcal{E}'_{\mu,\lambda}(u) \equiv \Phi'_\mu(u) - \lambda J'(u) = 0$ admits at least three solutions in $H^1_0(\Omega)$ having H^1_0-norm less than σ_μ. Since one of them may be the trivial one ($f(0) = 0$, see (f1)), we still have at least two distinct, nontrivial solutions of $(\mathrm{P}_{\mu,\lambda})$ with the required properties. □

Remark 1. Since $f(0) = 0$, one can consider the continuous function $s \mapsto f(s_+)$ instead of f, obtaining nonpositive solutions of the problem $(\mathrm{P}_{\mu,\lambda})$. Here, $s_+ = \max\{s,0\}$. Moreover, if f is locally Lipschitz, the solutions belong to $C^2(\Omega \setminus \{0\})$.

Remark 2. It is important to know explicit estimations of the intervals Λ_μ, $\mu \in [0,\overline{\mu})$, guaranteed by Theorem 1. In order to give such an estimation, let us fix s_0, R_0, and σ_0 as in Section 3. Let $\mu \in [0,\overline{\mu})$. Based on Lemma 3, one can assume that $\rho_0 < 1$ and

$$\frac{\sup\{J(u) : \Phi_\mu(u) < \rho_0\}}{\rho_0} < \frac{J(u_{\sigma_0})}{2\Phi_\mu(u_{\sigma_0})}.$$

Thus, according to (5), one has $\overline{a}_\mu < \frac{4\Phi_\mu(u_{\sigma_0})}{J(u_{\sigma_0})}$. In conclusion, we have

$$\Lambda_\mu \subset \left[0, 2\left(1-\frac{\mu}{\overline{\mu}}\right)\left(\frac{s_0}{R_0}\right)^2 \frac{(1-\sigma_0)^{-2}(1-\sigma_0^N)}{F(s_0)\sigma_0^N - \max_{|t|\leq|s_0|}|F(t)|(1-\sigma_0^N)}\right].$$

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