Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 325 (2007) 975–986
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

A nonsmooth principle of symmetric criticality and variational–hemivariational inequalities ☆

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Received 25 November 2005
Available online 23 March 2006
Submitted by M.A. Noor

Abstract

In this paper we prove the principle of symmetric criticality for Motreanu–Panagiotopoulos type functionals, i.e., for convex, proper, lower semicontinuous functionals which are perturbed by a locally Lipschitz function. By means of this principle a variational–hemivariational inequality is studied on certain type of unbounded strips.

Keywords: Motreanu–Panagiotopoulos type functional; Principle of symmetric criticality; Variational–hemivariational inequalities; Unbounded strip

1. Introduction

In order to develop a realistic model for physical phenomena from mechanics and engineering, P.D. Panagiotopoulos developed the theory of the *hemivariational inequalities.* The hemivariational inequalities appear as a generalization of the variational inequalities, but actually they are much more general than the last ones, because they are not equivalent to minimum prob-

☆ This article was supported by the Research Center of Sapientia Foundation.
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[1] The research of A. Kristály is partially supported by Central European University (CEU) Special and Extension Programs, and an Istituto Nazionale di Alta Matematica (INDAM) research grant.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2006.02.062

lems. Various approaches from Nonlinear Analysis are applied in order to establish existence of solutions for hemivariational inequalities, such as the theory of monotone operators or variational/topological methods. The interested reader is referred to the monographs of D. Motreanu and P.D. Panagiotopoulos [18], D. Motreanu and V. Rădulescu [19], Z. Naniewicz and P.D. Panagiotopoulos [20], P.D. Panagiotopoulos [21].

In [18, Chapter 3], D. Motreanu and P.D. Panagiotopoulos unified two nonsmooth critical points theories. To be more precise, let X be a Banach space, $\mathcal{I}: X \to (-\infty, +\infty]$ be a *Motreanu–Panagiotopoulos type functional*, i.e., $\mathcal{I} = h + \psi$, where $h : X \to \mathbb{R}$ is locally Lipschitz and $\psi : X \to (-\infty, +\infty]$ convex, proper, and lower semicontinuous. One says that an element $u \in X$ is a *critical point of* $\mathcal{I} = h + \psi$, if

$$h^0(u; v-u) + \psi(v) - \psi(u) \geqslant 0, \quad \forall v \in X.$$

Here, $h^0(u; z)$ is the generalized directional derivative of h at the point $u \in X$ in the direction $z \in X$, see Section 2. Note that, if $\psi \equiv 0$, then the above notion coincides with that introduced by K.-C. Chang [5], while if h is of class $\mathcal{C}^1$, the above critical point theory reduces to that of A. Szulkin [25]. The critical point theory elaborated in [18] is applied in solving certain variational–hemivariational inequalities on *bounded domain*s, originated from nonsmooth mechanical problems.

We emphasize that most of the treated problems in the aforementioned monographs are formulated on *bounded domain*s of $\mathbb{R}^n$. The first existence result for hemivariational inequalities on *unbounded domain*s is due to F. Gazolla and V. Rădulescu [10]. In this paper, the basic Sobolev space (where the solution of the problem is sought) involves such a nonlinearity which makes possible the compactness of its embedding into an adequate Lebesgue space; in this way, standard variational methods can be applied to establish nontrivial solutions of the studied problem. However, one can meet several concrete cases when the Sobolev space associated to the studied problem cannot be compactly embedded into any Lebesgue space. A large class of problems of this kind can be handled by constructing the subspace of radially symmetric functions of the original Sobolev space, and applying the appropriate version of the so-called *Principle of Symmetric Criticality* (PSC, shortly).

The aim of this paper is to prove the PSC for Motreanu–Panagiotopoulos type functionals. Briefly speaking, if we consider a compact group G which acts linearly on the reflexive Banach space X, h and ψ are G-invariant, $\mathcal{I} = h + \psi$, we will prove the following principle: *Every critical point* $u \in \Sigma = \{u \in X\colon\ gu = u,\ \forall g \in G\}$ *of* $\mathcal{I}|_\Sigma$ *will be also a critical point of* $\mathcal{I}$ *in the whole space* X. Here, $\mathcal{I}|_\Sigma$ denotes the restriction of $\mathcal{I}$ to Σ. To the best of our knowledge, this PSC is the most general principle of this kind. Indeed, specializing the form of the functional $\mathcal{I} = h + \psi$, the above principle formulated for Motreanu–Panagiotopoulos type functionals includes the following three well-known versions of the PSC stated by:

- R.S. Palais [22] for $\mathcal{C}^1$ functionals (i.e., h is of class $\mathcal{C}^1$, and $\psi \equiv 0$);
- W. Krawcewicz and W. Marzantowicz [12] for locally Lipschitz functions (i.e., $\psi \equiv 0$);
- J. Kobayashi and M. Ôtani [16] for Szulkin type functionals (i.e., h is of class $\mathcal{C}^1$).

The power of the PSC relies on the fact that it is enough to study the existence of critical points of a given function on a carefully chosen subspace of X (Σ from above) and not on the whole space X. Studying various elliptic problems on unbounded domains (e.g., on an unbounded strip $\Omega = \omega \times \mathbb{R}^l$, where $\omega \subset \mathbb{R}^m$ is bounded and open, $l, m \geqslant 1$), usually the space X (e.g., $H_0^1(\omega \times \mathbb{R}^l)$) cannot be compactly embedded into any 'reasonable' space (e.g., $L^p(\omega \times \mathbb{R}^l)$), as we asserted before. However, if we consider for instance the space of radially symmetric functions of $H_0^1(\omega \times \mathbb{R}^l)$ (which is nothing but $\Sigma = \{u \in H_0^1(\omega \times \mathbb{R}^l)\colon\ gu = u$ for every $g \in \mathrm{id}_{\mathbb{R}^m} \times O(l)\}$), this space can be compactly embedded into $L^p(\omega \times \mathbb{R}^l)$ whenever $l \geqslant 2$ and $2 < p < 2^*$, where 2^* denotes the Sobolev critical exponent, that is, $2^* = 2(m+l)(m+l-2)^{-1}$. Thus, by means of standard variational methods (e.g., Mountain Pass theorem, Fountain theorem) one can guarantee critical points of the restricted energy functional associated to the studied problem, see for instance T. Bartsch and Z.-Q. Wang [3], T. Bartsch and M. Willem [4], W.A. Strauss [24] ($\mathcal{C}^1$-case); Zs. Dályai and Cs. Varga [7], A. Kristály [13–15], Cs. Varga [26] (locally Lipschitz case); J. Kobayashi and M. Ôtani [16] (Szulkin type functionals).

The proof of the PSC for Motreanu–Panagiotopoulos functionals is based on [16] and [11]. Indeed, J. Kobayashi and M. Ôtani [16] proved recently the PSC not only for a highly nonsmooth class of functions (i.e., for Szulkin functionals) but for functionals which do not require full variational structure. Due to this latter fact and exploiting the basic properties of the generalized gradient of a locally Lipschitz function, we will be able to include the PSC for Motreanu–Panagiotopoulos functionals within Kobayashi and Ôtani's framework (see Theorem 2.1). Here, a key result is used which characterizes the critical points as solutions of certain differential inclusion, as it was pointed out by N.C. Kourogenis, J. Papadrianos and N.S. Papageorgiou [11].

In Section 2 we will prove the PSC. Although in the last section we apply this principle to establish the existence of multiple solutions for a variational–hemivariational inequality which is defined on an unbounded strip, we believe that this principle will be helpful in future works in order to study further important problems: variational–hemivariational inequalities of Schrödinger type involving singularities (in the spirit of [2]), Klein–Gordon and Born–Infeld problems (see [1] and [9]), and problems formulated on Sobolev spaces with variable exponent. We conclude our paper with a simple example which fulfils our hypotheses.

2. PSC for Motreanu–Panagiotopoulos functionals

Let $(X, \|\cdot\|)$ be a real Banach space and X^* its topological dual. A function $h : X \to \mathbb{R}$ is called *locally Lipschitz* if each point $u \in X$ possesses a neighborhood $\mathcal{N}_u$ such that $|h(u_1) - h(u_2)| \leqslant L\|u_1 - u_2\|$ for all $u_1, u_2 \in \mathcal{N}_u$, for a constant $L > 0$ depending on $\mathcal{N}_u$. The *generalized directional derivative* of h at the point $u \in X$ in the direction $z \in X$ is

$$h^0(u; z) = \limsup_{w \to u,\, t \to 0^+} \frac{h(w + tz) - h(w)}{t}.$$

The *generalized gradient* of h at $u \in X$ is defined by

$$\partial h(u) = \bigl\{x^* \in X^*\colon\ \langle x^*, z\rangle_X \leqslant h^0(u; z),\ \forall z \in X\bigr\},$$

[6], where $\langle\cdot,\cdot\rangle_X$ is the duality pairing between X^* and X.

Let $\mathcal{I} = h + \psi$, with $h : X \to \mathbb{R}$ locally Lipschitz and $\psi : X \to (-\infty, +\infty]$ convex, proper (i.e., $\psi \not\equiv +\infty$), and lower semicontinuous. $\mathcal{I}$ is called a *Motreanu–Panagiotopoulos type* functional, see [18, Chapter 3].

Definition 2.1. [18, Definition 3.1] An element $u \in X$ is said to be a critical point of $\mathcal{I} = h + \psi$, if

$$h^0(u; v - u) + \psi(v) - \psi(u) \geqslant 0, \quad \forall v \in X.$$

Kourogenis, Papadrianos and Papageorgiou [11] observed that one can characterize the critical points (in the sense of Definition 2.1) by means of differential inclusions using convex analysis. Namely, one has

Proposition 2.1. [11] *An element $u \in X$ is a critical point of $\mathcal{I} = h + \psi$, if and only if $0 \in \partial h(u) + \partial\psi(u)$, where $\partial\psi(u)$ denotes the subdifferential of the convex function ψ at u, i.e.,*

$$\partial\psi(u) = \left\{x^* \in X^*:\ \psi(v) - \psi(u) \geqslant \langle x^*, v-u\rangle_X \text{ for every } v \in X\right\}.$$

Let G be a topological group which acts *linearly* on X, i.e., the action $G \times X \to X$: $[g,u] \mapsto gu$ is continuous and for every $g \in G$, the map $u \mapsto gu$ is linear. The group G induces an action of the same type on the dual space X^* defined by $\langle gx^*, u\rangle_X = \langle x^*, g^{-1}u\rangle_X$ for every $g \in G$, $u \in X$ and $x^* \in X^*$. A function $h : X \to \mathbb{R} \cup \{+\infty\}$ is *G-invariant* if $h(gu) = h(u)$ for every $g \in G$ and $u \in X$. A set $K \subseteq X$ (or $K \subseteq X^*$) is *G-invariant* if $gK = \{gu:\ u \in K\} \subseteq K$ for every $g \in G$. Let

$$\Sigma = \{u \in X:\ gu = u \text{ for every } g \in G\}$$

the *fixed point set of X under G*. Now, we are in the position to state the PSC for Motreanu–Panagiotopoulos functionals.

Theorem 2.1. *Let X be a reflexive Banach space and $\mathcal{I} = h + \psi : X \to \mathbb{R} \cup \{+\infty\}$ be a Motreanu–Panagiotopoulos type functional. If a compact group G acts linearly on X, and the functionals h and ψ are G-invariant, then every critical point of $\mathcal{I}|_\Sigma$ is also a critical point of $\mathcal{I}$.*

In order to give the proof of Theorem 2.1, we recall first some facts from [16]. Let

$$\Phi(X) = \left\{\psi : X \to \mathbb{R} \cup \{\infty\}:\ \psi \text{ is convex, proper, lower semicontinuous}\right\};$$

$$\Phi_G(X) = \left\{\psi \in \Phi(X):\ \psi \text{ is } G\text{-invariant}\right\};$$

$$\Gamma_G(X^*) = \{K \subseteq X^*:\ K \text{ is } G\text{-invariant, weak}^*\text{-closed, convex}\}.$$

Proposition 2.2. [16, Theorem 3.16] *Assume that a compact group G acts linearly on a reflexive Banach space X. Then for every $K \in \Gamma_G(X^*)$ and $\psi \in \Phi_G(X)$ one has*

$$K|_\Sigma \cap \partial(\psi|_\Sigma)(u) \neq \emptyset \quad \Rightarrow \quad K \cap \partial\psi(u) \neq \emptyset, \quad u \in \Sigma, \tag{2.1}$$

where $K|_\Sigma = \{x^*|_\Sigma:\ x^* \in K\}$ with $\langle x^*|_\Sigma, u\rangle_\Sigma = \langle x^*, u\rangle_X$, $u \in \Sigma$.

Let $A : X \to X$ be the *averaging operator* over G, defined by

$$Au = \int_G gu\, d\mu(g), \quad u \in X, \tag{2.2}$$

where μ is the normalized Haar measure on G. Relation (2.2) can read as follows:

$$\langle x^*, Au\rangle_X = \int_G \langle x^*, gu\rangle_X\, d\mu(g), \quad u \in X,\ x^* \in X^*. \tag{2.3}$$

It is easy to verify that A is a continuous linear projection from X to Σ and for every G-invariant closed convex set $K \subseteq X$ we have $A(K) \subseteq K$. The adjoint operator $A^* : \Sigma^* \to X^*$ of $A : X \to \Sigma$ is defined by

$$\langle A^*w^*, z\rangle_X = \langle w^*, Az\rangle_\Sigma, \quad z \in X,\ w^* \in \Sigma^*. \tag{2.4}$$

Lemma 2.1. *Let $h : X \to \mathbb{R}$ be a G-invariant locally Lipschitz function and $u \in \Sigma$. Then*

(a) $\partial(h|_\Sigma)(u) \subseteq \partial h(u)|_\Sigma$.
(b) $\partial h(u) \in \Gamma_G(X^*)$.

Proof. (a) Let us fix $w^* \in \partial(h|_\Sigma)(u)$. Then by definition, one has

$$\langle w^*, v\rangle_\Sigma \leqslant (h|_\Sigma)^0(u; v) \quad \text{for every } v \in \Sigma.$$

First, a simple estimation shows that $(h|_\Sigma)^0(u; v) \leqslant h^0(u; v)$ for every $v \in \Sigma$. Thus, applying the above inequality for $v = Az \in \Sigma$ with $z \in X$ arbitrarily fixed, by (2.4) one has

$$\langle A^*w^*, z\rangle_X = \langle w^*, Az\rangle_\Sigma \leqslant h^0(u; Az). \tag{2.5}$$

Using [6, Proposition 2.1.2(b)] and (2.3), we get

$$\begin{aligned} h^0(u; Az) &= \max\bigl\{\langle x^*, Az\rangle_X\colon\ x^* \in \partial h(u)\bigr\} \\ &= \max\Biggl\{\int_G \langle x^*, gz\rangle_X \, d\mu(g)\colon\ x^* \in \partial h(u)\Biggr\} \\ &\leqslant \int_G h^0(u; gz)\, d\mu(g) = \int_G h^0\bigl(g^{-1}u; z\bigr)\, d\mu(g) = \int_G h^0(u; z)\, d\mu(g) \\ &= h^0(u; z). \end{aligned}$$

Combining this relation with (2.5), we conclude that $A^*w^* \in \partial h(u)$. Since $w^* = A^*w^*|_\Sigma$, we obtain that $w^* \in \partial h(u)|_\Sigma$, completing the proof of (a).

(b) Since $\partial h(u)$ is a nonempty, convex and weak*-compact subset of X^* (see [6, Proposition 2.1.2(a)]), it is enough to prove that $\partial h(u)$ is G-invariant, i.e., $g\partial h(u) \subseteq \partial h(u)$ for every $g \in G$. To this end, let us fix $g \in G$ and $x^* \in \partial h(u)$. Then, for every $z \in X$ we have

$$\langle gx^*, z\rangle_X = \bigl\langle x^*, g^{-1}z\bigr\rangle_X \leqslant h^0\bigl(u; g^{-1}z\bigr) = h^0(gu; z) = h^0(u; z),$$

i.e., $gx^* \in \partial h(u)$. □

Proof of Theorem 2.1. Let $u \in \Sigma$ be a critical point of $\mathcal{I}|_\Sigma$. Thanks to Proposition 2.1 one has $0 \in \partial(h|_\Sigma)(u) + \partial(\psi|_\Sigma)(u)$. Moreover, due to Lemma 2.1(a) we have

$$\emptyset \neq -\partial(h|_\Sigma)(u) \cap \partial(\psi|_\Sigma)(u) \subseteq -\partial h(u)|_\Sigma \cap \partial(\psi|_\Sigma)(u).$$

By choosing $K = \partial h(u)$ in Proposition 2.2 and taking into account Lemma 2.1(b), relation (2.1) implies that $\emptyset \neq -\partial h(u) \cap \partial\psi(u)$. Thus, in particular $0 \in \partial h(u) + \partial\psi(u)$, i.e., u is indeed a critical point of $\mathcal{I}$. □

3. A variational–hemivariational inequality on unbounded strips

Let $\Omega = \omega \times \mathbb{R}^l$ be an unbounded strip (or, in other words, a strip-like domain), where $\omega \subset \mathbb{R}^m$ is open bounded, and $l \geqslant 2$, $m \geqslant 1$. Let $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be a continuous function, which is locally Lipschitz in the second variable and satisfies the following condition:

(F1) $F(x, 0) = 0$, and there exist $c_1 > 0$ and $p \in (2, 2^\star)$ such that

$$|\xi| \leqslant c_1\bigl(|s| + |s|^{p-1}\bigr), \quad \text{for every } \xi \in \partial F(x, s),\ (x, s) \in \Omega \times \mathbb{R}.$$

Here, we denoted by $\partial F(x,s)$ the generalized gradient of $F(x,\cdot)$ at the point $s \in \mathbb{R}$ and $2^* = 2(m+l)(m+l-2)^{-1}$.

As usual, $H_0^1(\Omega)$ is the Sobolev space endowed with the inner product

$$\langle u, v\rangle_0 = \int_\Omega \nabla u \nabla v\, dx$$

and norm $\|\cdot\|_0 = \sqrt{\langle\cdot,\cdot\rangle_0}$, while the norm of $L^\alpha(\Omega)$ will be denoted by $\|\cdot\|_\alpha$. It is well known that the embedding $H_0^1(\Omega) \hookrightarrow L^\alpha(\Omega)$, $\alpha \in [2, 2^*]$, is continuous, that is, there exists $k_\alpha > 0$ such that $\|u\|_\alpha \leqslant k_\alpha \|u\|_0$ for every $u \in H_0^1(\Omega)$.

Consider finally the closed convex cone

$$\mathcal{K} = \left\{u \in H_0^1(\Omega)\colon\ u \geqslant 0 \text{ a.e. in } \Omega\right\}.$$

The aim of this section is to study the following (eigenvalue) problem for variational–hemivariational inequality (denoted by (P)):

Find $(u,\lambda) \in \mathcal{K} \times (0,\infty)$ *such that*

$$\int_\Omega \nabla u(x)\big(\nabla v(x) - \nabla u(x)\big)\, dx + \lambda \int_\Omega F^\circ\big(x, u(x); -v(x) + u(x)\big)\, dx \geqslant 0, \quad \forall v \in \mathcal{K}.$$

We say that a function $h : \Omega \to \mathbb{R}$ is *axially symmetric,* if $h(x,y) = h(x, gy)$ for all $x \in \omega$, $y \in \mathbb{R}^l$ and $g \in O(l)$, where $O(l)$ is the orthogonal group in $\mathbb{R}^l$. In particular, we denote by $H_{0,s}^1(\Omega)$ the closed subspace of axially symmetric functions of $H_0^1(\Omega)$.

Beside of (F1), we require on the nonlinearity F the following three assumptions:

(F2) $$\lim_{s\to 0} \frac{\max\{|\xi|\colon\ \xi \in \partial F(x,s)\}}{s} = 0$$

uniformly for every $x \in \Omega$.

(F3) There exist $q \in\,]0, 2[$, $\nu \in [2, 2^\star]$, $\alpha \in L^{\nu/(\nu-q)}(\Omega)$, $\beta \in L^1(\Omega)$ such that

$$F(x,s) \leqslant \alpha(x)|s|^q + \beta(x).$$

(F4) There exists $u_0 \in H_{0,s}^1(\Omega) \cap \mathcal{K}$ such that $\int_\Omega F(x, u_0(x))\, dx > 0$.

The main result of this section can be formulated as follows:

Theorem 3.1. *Let* $F : \Omega : \mathbb{R} \to \mathbb{R}$ *be a continuous function which satisfies* (F1)–(F4) *and* $F(\cdot, s)$ *is axially symmetric for every* $s \in \mathbb{R}$. *Then there is an open interval* $\Lambda_0 \subset (0,\infty)$ *such that for every* $\lambda \in \Lambda_0$ *there are at least three distinct elements* $u_i^\lambda \in \mathcal{K}$ $(i \in \{1,2,3\})$ *which are axially symmetric, having the property that* (u_i^λ, λ) *are solutions of* (P) *for every* $i \in \{1,2,3\}$.

From now on, we assume that the hypotheses of Theorem 3.1 are fulfilled. Before to prove Theorem 3.1, some preliminary results will be given.

Lemma 3.1. [14] *For every* $\varepsilon > 0$ *there exists* $c(\varepsilon) > 0$ *such that*

(i) $|\xi| \leqslant \varepsilon|s| + c(\varepsilon)|s|^{p-1}$ *for every* $\xi \in \partial F(x,s)$, $(x,s) \in \Omega \times \mathbb{R}$.
(ii) $|F(x,s)| \leqslant \varepsilon s^2 + c(\varepsilon)|s|^p$ *for every* $\xi \in \partial F(x,s)$, $(x,s) \in \Omega \times \mathbb{R}$.

Let us define $\mathcal{F}: H_0^1(\Omega) \to \mathbb{R}$ by

$$\mathcal{F}(u) = \int_\Omega F\big(x, u(x)\big)\, dx.$$

Lemma 3.2. [14] *The function $\mathcal{F}: H_0^1(\Omega) \to \mathbb{R}$ is locally Lipschitz, and for every closed subspace Y of $H_0^1(\Omega)$ one has*

$$(\mathcal{F}|_Y)^\circ(u, v) \leqslant \int_\Omega F^\circ\big(x, u(x); v(x)\big),$$

for every $u, v \in Y$. (Here, $\mathcal{F}|_Y$ denotes the restriction of $\mathcal{F}$ to Y.)

We consider the indicator function of the set $\mathcal{K}$, i.e., $\psi_\mathcal{K}: H_0^1(\Omega) \to\ ]-\infty, \infty]$,

$$\psi_\mathcal{K}(u) = \begin{cases} 0, & \text{if } u \in \mathcal{K}, \\ +\infty, & \text{if } u \notin \mathcal{K}, \end{cases}$$

which is clearly convex, proper and lower semicontinuous. Moreover, define for $\lambda > 0$ the function $\mathcal{I}_\lambda: H_0^1(\Omega) \to\ ]-\infty, \infty]$ by

$$\mathcal{I}_\lambda(u) = \frac{1}{2}\|u\|_0^2 - \lambda \mathcal{F}(u) + \psi_\mathcal{K}(u). \tag{3.1}$$

It is easily seen that $\mathcal{I}_\lambda$ is a Motreanu–Panagiotopoulos type functional. Furthermore, one has

Proposition 3.1. *If $u \in H_0^1(\Omega)$ is a critical point of $\mathcal{I}_\lambda$, then (u, λ) is a solution of* (P).

Proof. By assumption, one has

$$\langle u, v - u\rangle_0 + \lambda(-\mathcal{F})^0(u; v - u) + \psi_\mathcal{K}(v) - \psi_\mathcal{K}(u) \geqslant 0, \quad \forall v \in H_0^1(\Omega).$$

In particular, u should belong to $\mathcal{K}$. If we fix arbitrarily $v \in \mathcal{K}$, and we take into account Lemma 3.2 (with $Y = H_0^1(\Omega)$) the assertion yields. □

Let $G = \mathrm{id}_{\mathbb{R}^m} \times O(l) \subset O(m + l)$. Define the action of G on $H_0^1(\Omega)$ by $gu(x) = u(g^{-1}x)$ for every $g \in G$, $u \in H_0^1(\Omega)$ and $x \in \Omega$. Since $\mathcal{K}$ is a G-invariant set, then $\psi_\mathcal{K}$ is a G-invariant function. In this way, $\psi_\mathcal{K} \in \Phi_G(H_0^1(\Omega))$. Since $F(\cdot, s)$ is axially symmetric for every $s \in \mathbb{R}$, then $\mathcal{F}$ is also a G-invariant function. The norm $\|\cdot\|_0$ is a G-invariant function as well. In conclusion, if we consider the set

$$\Sigma = H_{0,s}^1(\Omega) = \big\{u \in H_0^1(\Omega):\ gu = u \text{ for every } g \in G\big\},$$

then, in view of Theorem 3.1, every critical point of $\mathcal{I}_\lambda|_\Sigma$ becomes as well critical point of $\mathcal{I}_\lambda$. In order to find critical points of $\mathcal{I}_\lambda|_\Sigma$ we recall a nonsmooth critical point result in its full generality, proved by Marano and Motreanu [17]. Note that the smooth version of this result is due to Ricceri [23].

Let $(X, \|\cdot\|)$ be a real Banach space, $\mathcal{I} = h + \psi$ a Motreanu–Panagiotopoulos type functional. First, we need

Definition 3.1. [18, Definition 3.2] $\mathcal{I} = h + \psi$ is said to *satisfy the Palais–Smale condition at level* $c \in \mathbb{R}$ (shortly, $(PS)_c$) if every sequence $\{u_n\}$ in X satisfying $\mathcal{I}(u_n) \to c$ and

$$h^0(u_n; v - u_n) + \psi(v) - \psi(u_n) \geqslant -\varepsilon_n \|v - u_n\|, \quad \forall v \in X,$$

for a sequence $\{\varepsilon_n\}$ in $[0, \infty)$ with $\varepsilon_n \to 0$, contains a convergent subsequence. If $(PS)_c$ is verified for all $c \in \mathbb{R}$, $\mathcal{I}$ is said to *satisfy the Palais–Smale condition* (shortly, (PS)).

Now, let $h_1, h_2 : X \to \mathbb{R}$ be two locally Lipschitz functions, and let $\psi_1 : X \to]-\infty, +\infty]$ be a convex, proper, lower semicontinuous function. Then $h_1 + \psi_1 + \lambda h_2$ is a Motreanu–Panagiotopoulos type functional for every $\lambda \in \mathbb{R}$. Furthermore, one has

Theorem 3.2. [17, Theorem B] *Let X be a separable and reflexive Banach space, let $I_1 = h_1 + \psi_1$ and $I_2 = h_2$, and let $\Lambda \subseteq \mathbb{R}$ be an interval. Suppose that*

(a_1) *h_1 is weakly sequentially lower semicontinuous and h_2 is weakly sequentially continuous*;
(a_2) *for every $\lambda \in \Lambda$ the function $I_1 + \lambda I_2$ fulfils $(PS)_c$, $c \in \mathbb{R}$, with*

$$\lim_{\|u\| \to +\infty} \big(I_1(u) + \lambda I_2(u)\big) = +\infty;$$

(a_3) *there exists a continuous concave function $h : \Lambda \to \mathbb{R}$ satisfying*

$$\sup_{\lambda \in \Lambda} \inf_{u \in X} \big(I_1(u) + \lambda I_2(u) + h(\lambda)\big) < \inf_{u \in X} \sup_{\lambda \in \Lambda} \big(I_1(u) + \lambda I_2(u) + h(\lambda)\big).$$

Then there is an open interval $\Lambda_0 \subseteq \Lambda$, such that for each $\lambda \in \Lambda_0$ the function $I_1 + \lambda I_2$ has at least three critical points in X.

We will apply Theorem 3.2 by choosing

$$X = \Sigma = H^1_{0,s}(\Omega), \qquad h_1 = \frac{1}{2}\|\cdot\|^2_\Sigma, \qquad \psi_1 = \psi_{\mathcal{K}}|_\Sigma, \qquad h_2 = -\mathcal{F}|_\Sigma,$$
$$\Lambda = [0, \infty[.$$

As usual, $\|\cdot\|_\Sigma$, $\psi_{\mathcal{K}}|_\Sigma$ and $\mathcal{F}|_\Sigma$ denote the restrictions of $\|\cdot\|_0$, $\psi_{\mathcal{K}}$ and $\mathcal{F}$ to Σ, respectively. We will use also the notation $\langle\cdot,\cdot\rangle_\Sigma$ for the restriction of $\langle\cdot,\cdot\rangle_0$ to Σ.

Now, we are going to verify the hypotheses (a_1)–(a_3) of Theorem 3.2.

Step 1 *(Verification of (a_1)).* The weakly sequentially lower semicontinuity of h_1 is standard. We prove that h_2 is weakly sequentially continuous.

Let $\{u_n\}$ be a sequence from Σ which converges weakly to some $u \in \Sigma$. In particular, $\{u_n\}$ is bounded in Σ and by virtue of Lemma 3.1, $F(x, s) = o(s^2)$ as $s \to 0$, and $F(x, s) = o(s^{2^*})$ as $s \to +\infty$, uniformly for every $x \in \Omega$. But, from [8, Lemma 4, p. 368] it follows that $h_2(u_n) \to h_2(u)$ as $n \to \infty$, i.e., h_2 is weakly sequentially continuous.

Step 2 *(Verification of (a_2)).* Fix $\lambda \in \Lambda$. First, we will prove that $I_1 + \lambda I_2 \equiv h_1 + \psi_1 + \lambda h_2$ is coercive. Indeed, due to (F3), by Hölder's inequality we have for every $u \in \Sigma$ that

$$I_1(u) + \lambda I_2(u) \geqslant \frac{1}{2}\|u\|^2_\Sigma - \lambda \int_\Omega \alpha(x)\big|u(x)\big|^q\,dx - \lambda \int_\Omega \beta(x)\,dx$$
$$\geqslant \frac{1}{2}\|u\|^2_\Sigma - \lambda\|\alpha\|_{\nu/(\nu-q)} k^q_\nu \|u\|^q_\Sigma - \lambda\|\beta\|_1.$$

Since $q < 2$, it is clear that $\|u\|_\Sigma \to +\infty$ implies $I_1(u) + \lambda I_2(u) \to +\infty$, as claimed.

Now, we will prove that $I_1 + \lambda I_2$ verifies $(PS)_c$, $c \in \mathbb{R}$. Let $\{u_n\} \subset \Sigma$ be a sequence such that

$$I_1(u_n) + \lambda I_2(u_n) \to c \tag{3.2}$$

and for every $v \in \Sigma$ we have

$$\langle u_n, v - u_n\rangle_\Sigma + \lambda h_2^0(u_n; v - u_n) + \psi_1(v) - \psi_1(u_n) \geqslant -\varepsilon_n \|v - u_n\|_\Sigma, \tag{3.3}$$

for a sequence $\{\varepsilon_n\}$ in $[0, +\infty[$ with $\varepsilon_n \to 0$. In particular, (3.2) shows that $\{u_n\} \subset \mathcal{K}$. Moreover, the coerciveness of the function $I_1 + \lambda I_2$ implies that the sequence $\{u_n\}$ is bounded in $\Sigma \cap \mathcal{K}$. Therefore, there exists an element $u \in \mathcal{K} \cap \Sigma$ such that $\{u_n\}$ converges weakly to u in Σ. (Note that $\mathcal{K}$ is convex and closed, thus, weakly closed.) Moreover, since the embedding $\Sigma \hookrightarrow L^p(\Omega)$ is compact (see [8]), up to a subsequence, $\{u_n\}$ converges strongly to u in $L^p(\Omega)$. Choosing in particular $v = u$ in (3.3), we have

$$\|u_n - u\|_\Sigma^2 \leqslant \lambda h_2^0(u_n; u - u_n) + \langle u, u - u_n\rangle_\Sigma + \varepsilon_n \|u - u_n\|_\Sigma.$$

The last two terms tend to zero as $n \to \infty$. Thus, in order to prove $\|u_n - u\|_\Sigma \to 0$, it is enough to show that the first term in the right-hand side tends to zero as well. To do this, we use Lemma 3.2 (with $Y = \Sigma$) and Lemma 3.1(a), obtaining

$$\begin{aligned} h_2^0(u_n; u - u_n) &\leqslant \int_\Omega F^0\big(x, u_n(x); -u(x) + u_n(x)\big)\,dx \\ &= \int_\Omega \max\big\{\xi_n(x)\big(-u(x) + u_n(x)\big):\ \xi_n(x) \in \partial F\big(x, u_n(x)\big)\big\}\,dx \\ &\leqslant \int_\Omega \big[\varepsilon\big|u_n(x)\big| + c(\varepsilon)\big|u_n(x)\big|^{p-1}\big]\big|u_n(x) - u(x)\big|\,dx \\ &\leqslant \varepsilon k_2^2 \|u_n\|_\Sigma \|u_n - u\|_\Sigma + c(\varepsilon)\|u_n\|_p^{p-1}\|u_n - u\|_p. \end{aligned}$$

Due to the arbitrariness of $\varepsilon > 0$, the last term tends to zero, therefore, $\|u_n - u\|_\Sigma \to 0$ as $n \to \infty$.

Step 3 *(Verification of (a_3)).* Let us define the function $\gamma : [0, \infty[\to \mathbb{R}$ by

$$\gamma(t) = \sup\big\{-h_2(u):\ \|u\|_\Sigma^2 \leqslant 2t\big\}.$$

Due to Lemma 3.1(b), one has

$$\gamma(t) \leqslant 2\varepsilon k_2^2 t + 2^p c(\varepsilon) k_p^p t^{\frac{p}{2}}.$$

On the other hand, we know that $\gamma(t) \geqslant 0$ for $t \geqslant 0$. Due to the arbitrariness of $\varepsilon > 0$, we deduce

$$\lim_{t \to 0^+} \frac{\gamma(t)}{t} = 0.$$

By (F4) it is clear that $u_0 \neq 0$ ($h_2(0) = 0$). Therefore it is possible to choose a number η such that

$$0 < \eta < -2h_2(u_0)\|u_0\|_\Sigma^{-2}.$$

From $\lim_{t\to 0^+}\gamma(t)/t = 0$ it follows the existence of a number $t_0 \in\,]0, \|u_0\|_\Sigma^2/2[$ such that $\gamma(t_0) < \eta t_0$. Therefore,

$$\gamma(t_0) < -2h_2(u_0)\|u_0\|_\Sigma^{-2} t_0.$$

Choose $\rho_0 > 0$ such that

$$\gamma(t_0) < \rho_0 < -2h_2(u_0)\|u_0\|_\Sigma^{-2} t_0. \tag{3.4}$$

Due to the choice of t_0 and (3.4) we have

$$\rho_0 < -h_2(u_0). \tag{3.5}$$

Define $h : \Lambda = [0, +\infty[\, \to \mathbb{R}$ by $h(\lambda) = \rho_0\lambda$. We prove that the function h satisfies the inequality

$$\sup_{\lambda\in\Lambda}\inf_{u\in\Sigma}\big(I_1(u) + \lambda I_2(u) + \rho_0\lambda\big) < \inf_{u\in\Sigma}\sup_{\lambda\in\Lambda}\big(I_1(u) + \lambda I_2(u) + \rho_0\lambda\big).$$

Note, that in the previous inequality we can put $\Sigma \cap \mathcal{K}$ instead of Σ; indeed, if $u \in \Sigma \setminus \mathcal{K}$, then $I_1(u) = +\infty$.

The function

$$\Lambda \ni \lambda \mapsto \inf_{u\in\Sigma\cap\mathcal{K}}\big[\|u\|_\Sigma^2/2 + \lambda\big(\rho_0 + h_2(u)\big)\big]$$

is upper semicontinuous on Λ. Relation (3.5) implies that

$$\lim_{\lambda\to+\infty}\inf_{u\in\Sigma\cap\mathcal{K}}\big[I_1(u) + \lambda I_2(u) + \rho_0\lambda\big] \leqslant \lim_{\lambda\to+\infty}\big[\|u_0\|_\Sigma^2/2 + \lambda\big(\rho_0 + h_2(u_0)\big)\big] = -\infty.$$

Thus we find an element $\bar\lambda \in \Lambda$ such that

$$\sup_{\lambda\in\Lambda}\inf_{u\in\Sigma\cap\mathcal{K}}\big(I_1(u) + \lambda I_2(u) + \rho_0\lambda\big) = \inf_{u\in\Sigma\cap\mathcal{K}}\big[\|u\|_\Sigma^2/2 + \bar\lambda\big(\rho_0 + h_2(u)\big)\big]. \tag{3.6}$$

Since $\gamma(t_0) < \rho_0$, for all $u \in \Sigma$ such that $\|u\|_\Sigma^2 \leqslant 2t_0$, we have $h_2(u) > -\rho_0$. Thus, we have

$$t_0 \leqslant \inf\big\{\|u\|_\Sigma^2/2\colon\ h_2(u) \leqslant -\rho_0\big\} \leqslant \inf\big\{\|u\|_\Sigma^2/2\colon\ u \in \mathcal{K},\ h_2(u) \leqslant -\rho_0\big\}. \tag{3.7}$$

On the other hand,

$$\begin{aligned}\inf_{u\in\Sigma\cap\mathcal{K}}\sup_{\lambda\in\Lambda}\big(I_1(u) + \lambda I_2(u) + \rho_0\lambda\big) &= \inf_{u\in\Sigma\cap\mathcal{K}}\Big[\|u\|_\Sigma^2/2 + \sup_{\lambda\in\Lambda}\big(\lambda\big(\rho_0 + h_2(u)\big)\big)\Big] \\ &= \inf_{u\in\Sigma\cap\mathcal{K}}\big\{\|u\|_\Sigma^2/2\colon\ h_2(u) \leqslant -\rho_0\big\}.\end{aligned}$$

Therefore, relation (3.7) can be written as

$$t_0 \leqslant \inf_{u\in\Sigma\cap\mathcal{K}}\sup_{\lambda\in\Lambda}\big(I_1(u) + \lambda I_2(u) + \rho_0\lambda\big). \tag{3.8}$$

There are two distinct cases:

(A) If $0 \leqslant \bar\lambda < t_0/\rho_0$, we have

$$\inf_{u\in\Sigma\cap\mathcal{K}}\big[\|u\|_\Sigma^2/2 + \bar\lambda\big(\rho_0 + h_2(u)\big)\big] \leqslant \bar\lambda\big(\rho_0 + h_2(0)\big) = \bar\lambda\rho_0 < t_0.$$

Combining this inequality with (3.6) and (3.8) we obtain the desired inequality.

(B) If $t_0/\rho_0 \leqslant \bar{\lambda}$, from $\rho_0 < -h_2(u_0)$ and (3.4) it follows

$$\inf_{u \in \Sigma \cap \mathcal{K}} \left[\|u\|_\Sigma^2/2 + \bar{\lambda}\big(\rho_0 + h_2(u)\big) \right] \leqslant \|u_0\|_\Sigma^2/2 + \bar{\lambda}\big(\rho_0 + h_2(u_0)\big)$$
$$\leqslant \|u_0\|_\Sigma^2/2 + t_0\big(\rho_0 + h_2(u_0)\big)/\rho_0 < t_0.$$

Now, we will repeat the last part of (A), which concludes Step 3.

Proof of Theorem 3.1. Due to the above three steps, Theorem 3.2 implies the existence of an open interval $\Lambda_0 \subset [0, \infty[$, such that for each $\lambda \in \Lambda_0$, the function $I_\lambda|_\Sigma \equiv I_1 + \lambda I_2$ has at least three critical points in $\Sigma \cap \mathcal{K}$. It remains to apply Theorem 2.1 and Proposition 3.1. □

Example 3.1. Let $\Omega = (0,1) \times \mathbb{R}^2$ and define the function $F : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$F(x,s) = F\big((x_1,x_2),s\big) = \big(1 + |x_2|^2\big)^{-2} \min\big\{|s|^3, |s|^{3/2}\big\},$$

for every $x_1 \in (0,1)$, $x_2 \in \mathbb{R}^2$ and $s \in \mathbb{R}$. Here, $|x_2|$ denotes the Euclidean norm of x_2 in $\mathbb{R}^2$.

The assumptions of Theorem 3.1 are fulfilled with the following choice: $c_1 = 3$; $p = 3$; $q = 3/2$; ν any number between 2 and 6; $\alpha(x_1,x_2) = (1 + |x_2|^2)^{-2}$; $\beta \equiv 0$; and $u_0(x_1,x_2) = 1$ if $|x_2| \leqslant 1$, $u_0(x_1,x_2) = 2 - |x_2|$ if $1 \leqslant |x_2| \leqslant 2$, $u_0(x_1,x_2) = 0$ if $|x_2| \geqslant 2$.

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