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PERTURBED NEUMANN PROBLEMS WITH MANY SOLUTIONS

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□ *Given $f, g : [0,\infty) \to \mathbb{R}$ two continuous nonlinearities with $f(0) = g(0) = 0$ and f having a suitable oscillatory behavior at zero or at infinity, we prove by a direct method that for every $k \in \mathbb{N}$, there exists $\varepsilon_k > 0$ such that the problem*

$$\begin{cases} -\triangle_p u + \alpha(x)u^{p-1} = f(u) + \varepsilon g(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

has at least k distinct nonnegative weak solutions in $W^{1,p}(\Omega)$ whenever $|\varepsilon| \leq \varepsilon_k$. We also give various $W^{1,p}$- and L^∞-estimates of the solutions. No growth assumption on g is needed, and $\alpha \in L^\infty(\Omega)$ may be sign-changing or even negative depending on the rate of the oscillation of f.

Keywords Arbitrarily many solutions; Oscillatory nonlinearity; Perturbed Neumann problem.

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1. INTRODUCTION AND MAIN RESULTS

Very recently, in [3] the authors studied the Neumann problem

$$\begin{cases} -\triangle_p u + \alpha(x)|u|^{p-2}u = \beta(x)f(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{P$_0$}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open domain with C^2-boundary $\partial\Omega$, $1 < p < \infty$, $\triangle_p(\cdot) = \text{div}(|\nabla(\cdot)|^{p-2}\nabla(\cdot))$ is the p-Laplacian operator, ν is the

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outer unit normal to $\partial\Omega$, $f \in L^\infty_{\text{loc}}([0,\infty))$ with $f(0) = 0$, and $\alpha, \beta \in L^\infty(\Omega)$ with $\text{essinf}_\Omega \beta > 0$. Because f is *not* necessarily continuous, problem (P_0) has been reformulated into a hemivariational inequality, and the existence of *infinitely* many nonnegative solutions for (P_0) are guaranteed whenever f has a suitable oscillatory behavior at the origin or at infinity (see hypotheses (H_0^f) and (H_∞^f) below).

The goal of the current paper is to treat the *perturbed* problem

$$\begin{cases} -\triangle_p u + \alpha(x)|u|^{p-2}u = f(u) + \varepsilon g(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_ε}$$

where f is continuous and verifies the same conditions as in [3], and $g : [0,\infty) \to \mathbb{R}$ is an *arbitrarily* continuous function with $g(0) = 0$. Having infinitely many solutions for problem (P_0) cf. [3], we expect to find still *many* solutions for the perturbed problem (P_ε) whenever $|\varepsilon|$ is small enough. The purpose of the current paper is to show that this is indeed the case. Here, a solution for (P_ε) is meant as a weak solution in $W^{1,p}(\Omega)$ in the usual sense.

In the sequel, we state our results, recalling simultaneously the hypotheses and results from [3] in the smooth context (and taking $\beta = 1$, see (P_0)). If we denote by $F(s) = \int_0^s f(t)dt$, $s \geq 0$, we assume

$\left(H_0^f\right)$ $\limsup_{s\to 0^+} \frac{pF(s)}{s^p} > \frac{\int_\Omega \alpha(x)dx}{\text{meas}(\Omega)} \geq \text{essinf}_\Omega \alpha > \liminf_{s\to 0^+} \frac{f(s)}{s^{p-1}}$.

Note that (H_0^f) implies an oscillatory behavior of f at zero.

Theorem A [3, Theorem 1.2]. *Let $\alpha \in L^\infty(\Omega)$ and a continuous function $f : [0,\infty) \to \mathbb{R}$ with $f(0) = 0$, fulfilling (H_0^f). Then (P_0) admits a sequence of distinct nonnegative solutions $\{u_i^0\}$ in $W^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that*

$$\lim_{i\to\infty} \|u_i^0\|_{W^{1,p}} = \lim_{i\to\infty} \|u_i^0\|_\infty = 0. \tag{1.1}$$

Here, the norms $\|\cdot\|_{W^{1,p}}$ and $\|\cdot\|_{L^\infty}$ are the usual ones on the spaces $W^{1,p}(\Omega)$ and $L^\infty(\Omega)$, respectively. The first main result of the current paper reads as follows.

Theorem 1.1. *Let $\alpha \in L^\infty(\Omega)$ and two continuous functions $f, g : [0,\infty) \to \mathbb{R}$ with $f(0) = g(0) = 0$. Assume that (H_0^f) holds.*

Then, for every $k \in \mathbb{N}$, there exists $\varepsilon_k^0 > 0$ such that (P_ε) has at least k distinct nonnegative solutions in $W^{1,p}(\Omega) \cap L^\infty(\Omega)$ whenever $\varepsilon \in [-\varepsilon_k^0, \varepsilon_k^0]$. Moreover, *if the (first k) solutions are denoted by* $u^0_{i,\varepsilon}$, $i=\overline{1,k}$, *then*

$$\|u^0_{i,\varepsilon}\|_{L^\infty} < \frac{1}{i} \quad \textit{and} \quad \|u^0_{i,\varepsilon}\|_{W^{1,p}} < \frac{1}{i} \quad \textit{for any } i=\overline{1,k};\;\; \varepsilon \in [-\varepsilon^0_k, \varepsilon^0_k]. \tag{1.1'}$$

Remark 1.2. It is useful to notice the concordance between relations (1.1) and (1.1′), respectively. Moreover, no growth assumption is required on g.

Dealing with the case when f oscillates at infinity, in [3] is required a *subcritical* growth condition at infinity for f; namely

(f_{p^*}) $\limsup_{s\to\infty} \frac{|f(s)|}{s^{q-1}} < \infty$ for some $q \in (p, p^*)$.

Here, $p^* = pN/(N-p)$ if $N > p$ and $p^* = \infty$ if $p \geq N$. The counterpart of the hypothesis (H^f_0) at infinity is

(H^f_∞) $\limsup_{s\to\infty} \frac{pF(s)}{s^p} > \frac{\int_\Omega \alpha(x)dx}{\text{meas}(\Omega)} \geq \text{essinf}_\Omega \alpha > \liminf_{s\to\infty} \frac{f(s)}{s^{p-1}}$.

Theorem B [3, Theorem 1.3]. *Let* $\alpha \in L^\infty(\Omega)$ *and a continuous function* $f:[0,\infty)\to \mathbb{R}$ *with* $f(0)=0$, *fulfilling* (f_{p^*}) *and* (H^f_∞). *Then* (P_0) *admits a sequence of distinct nonnegative solutions* $\{u^\infty_i\}$ *in* $W^{1,p}(\Omega)\cap L^\infty(\Omega)$ *such that*

$$\lim_{i\to\infty} \|u^\infty_i\|_{W^{1,p}} = \lim_{i\to\infty} \|u^\infty_i\|_\infty = \infty. \tag{1.2}$$

In our second result, we can avoid the subcritical growth condition (f_{p^*}) as follows.

Theorem 1.3. *Let* $\alpha \in L^\infty(\Omega)$ *and two continuous functions* $f, g : [0,\infty)\to\mathbb{R}$ *with* $f(0)=g(0)=0$. *Assume that* (H^f_∞) *holds.*

Then, for every $k\in\mathbb{N}$, *there exists* $\varepsilon^\infty_k > 0$ *such that* (P_ε) *has at least* k *distinct nonnegative solutions in* $W^{1,p}(\Omega)\cap L^\infty(\Omega)$ *whenever* $\varepsilon \in [-\varepsilon^\infty_k, \varepsilon^\infty_k]$. *Moreover, if the (first k) solutions are denoted by* $u^\infty_{i,\varepsilon}$, $i=\overline{1,k}$, *then*

$$\|u^\infty_{i,\varepsilon}\|_{L^\infty} > i-1 \quad \textit{for any } i=\overline{1,k};\;\; \varepsilon\in[-\varepsilon^\infty_k, \varepsilon^\infty_k]. \tag{1.2'}$$

The proofs of Theorems A and B play crucial roles in Theorems 1.1 and 1.3, respectively; in fact, the proofs are based on a careful analysis of two special sequences involving the energy functional associated to (P_ε). For details, see Sections 3 and 4.

We give two simple functions for f fulfilling the hypotheses of Theorems 1.1 and 1.3, respectively.

(a) Let $\alpha,\beta,\gamma\in\mathbb{R}$ such that $0<\alpha<1<\alpha+\beta$, and $\gamma\in(0,1)$. Then, the function $f:[0,\infty)\to\mathbb{R}$ defined by $f(0)=0$ and $f(s)=s^\alpha(\gamma+\sin s^{-\beta})$,

$s > 0$, verifies (H_0^f) with $p = 2$. Note that α may be any negative or sign-changing function that belongs to $L^\infty(\Omega)$.

(b) Let $\alpha, \beta, \gamma \in \mathbb{R}$ such that $1 < \alpha$, $|\alpha - \beta| < 1$, and $\gamma \in (0, 1)$. Then, the function $f : [0, \infty) \to \mathbb{R}$ defined by $f(s) = s^\alpha(\gamma + \sin s^\beta)$ verifies the hypotheses (H_∞^f) with $p = 2$. The same remark is valid for α as before.

Equations involving oscillatory terms usually produce infinitely many solutions. This phenomenon has been exploited by several authors in various contexts: for Neumann boundary problems, see Ricceri [7], Faraci and Kristály [2], Kristály and Motreanu [3], for Dirichlet boundary problems, see Anello and Cordaro [1], Omari and Zanolin [5], and Saint Raymond [8].

2. AN AUXILIARY RESULT

In this section, we consider the problem

$$\begin{cases} -\triangle_p u + \lambda(x)|u|^{p-2}u = h(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

assuming that $\lambda \in L^\infty(\Omega)$ with $\mathrm{essinf}_\Omega \lambda > 0$ and

(h_1) $h : [0, \infty) \to \mathbb{R}$ is a continuous, bounded function such that $h(0) = 0$;
(h_2) there are $0 < a < b$ such that $h(s) \le 0$ for all $s \in [a, b]$.

Because of (h_1), we may extend h continuously to the whole $\mathbb{R}$, taking $h(s) = 0$ for all $s \le 0$.

We may introduce the energy functional $\mathscr{E} : W^{1,p}(\Omega) \to \mathbb{R}$ associated with problem (P), which is defined by

$$\mathscr{E}(u) = \frac{1}{p}\|u\|_\lambda^p - \int_\Omega H(u(x))dx, \quad u \in W^{1,p}(\Omega),$$

where

$$\|u\|_\lambda = \left(\int_\Omega |\nabla u(x)|^p \, dx + \int_\Omega \lambda(x)|u(x)|^p \, dx \right)^{1/p}$$

and $H(s) = \int_0^s h(t)dt$, $s \in \mathbb{R}$. Note that the norms $\|\cdot\|_\lambda$ and $\|\cdot\|_{W^{1,p}}$ are equivalent, as $\mathrm{essinf}_\Omega \lambda > 0$. Standard arguments show that $\mathscr{E}$ is well-defined and is of class C^1 on $W^{1,p}(\Omega)$. Moreover, its critical points are weak solutions for problem (P).

We consider the number $b \in \mathbb{R}$ from (h_2), and we introduce the level-set

$$W^b = \{u \in W^{1,p}(\Omega) : \|u\|_{L^\infty} \leq b\}.$$

Now, we are ready to state the main result of this section.

Theorem 2.1. *Assume that* (h_1), (h_2) *hold. Then*

(i) *the functional* $\mathscr{E}$ *is bounded from below on* W^b *and its infimum is attained at* $\tilde{u} \in W^b$*;*
(ii) $\tilde{u}(x) \in [0, a]$ *for a.e.* $x \in \Omega$*;*
(iii) $\tilde{u}$ *is a weak solution of* (P).

Proof. (i) For every $u \in W^b$, we have

$$\mathscr{E}(u) = \frac{1}{p}\|u\|_\lambda^p - \int_\Omega H(u(x))dx \geq -\mathrm{meas}(\Omega)\max_{[-b,b]} H > -\infty.$$

Thus, $\mathscr{E}$ is bounded from below on W^b. On the other hand, due to the theorem of Rellich–Kondrachov, $\mathscr{E}$ is sequentially weakly continuous. Because W^b is convex and closed, thus weakly closed in $W^{1,p}(\Omega)$, the infimum of $\mathscr{E}$ on W^b is attained at an element $\tilde{u} \in W^b$.

(ii) Let $W = \{x \in \Omega : \tilde{u}(x) \notin [0, a]\}$ and suppose that $\mathrm{meas}(W) > 0$. Define the function $\gamma(s) = \min(s_+, a)$ where $s_+ = \max(s, 0)$, and set $\tilde{w} = \gamma \circ \tilde{u}$. Due to Marcus and Mizel [6], $\tilde{w}$ belongs to $W^{1,p}(\Omega)$ (as γ is Lipschitz continuous). Moreover, $\tilde{w} \in W^b$. We introduce the following two sets

$$W_1 = \{x \in W : \tilde{u}(x) < 0\} \quad \text{and} \quad W_2 = \{x \in W : \tilde{u}(x) > a\}.$$

Then, $W = W_1 \cup W_2$, and we have that $\tilde{w}(x) = \tilde{u}(x)$ for all $x \in \Omega \setminus W$, $\tilde{w}(x) = 0$ for all $x \in W_1$, and $\tilde{w}(x) = a$ for all $x \in W_2$. Furthermore,

$$\begin{aligned}
&\mathscr{E}(\tilde{w}) - \mathscr{E}(\tilde{u}) \\
&\quad = -\frac{1}{p}\int_W |\nabla \tilde{u}|^p\, dx + \frac{1}{p}\int_W \lambda(x)[|\tilde{w}|^p - |\tilde{u}|^p]dx - \int_W [H(\tilde{w}) - H(\tilde{u})]dx \\
&\quad = -\frac{1}{p}\int_W |\nabla \tilde{u}|^p\, dx - \frac{1}{p}\int_{W_1} \lambda(x)|\tilde{u}|^p\, dx + \frac{1}{p}\int_{W_2} \lambda(x)[a^p - \tilde{u}^p]dx \\
&\qquad - \int_{W_1} [H(0) - H(\tilde{u}(x))]dx - \int_{W_2} [H(a) - H(\tilde{u}(x))]dx.
\end{aligned}$$

First, $\int_{W_1}[H(0) - H(\tilde{u}(x))]dx = 0$. Then, by using the mean value theorem and hypotheses (h_2), we obtain

$$\int_{W_2}[H(a) - H(\tilde{u}(x))]dx \geq 0.$$

Therefore, every term of the above expression is nonpositive. But, taking into account that $\mathscr{E}(\tilde{w}) \geq \mathscr{E}(\tilde{u}) = \inf_{W^b}\mathscr{E}$, every term should be zero. In particular,

$$\int_{W_1}\lambda(x)|\tilde{u}|^p = \int_{W_2}\lambda(x)[a^p - \tilde{u}^p] = 0.$$

Because $\mathrm{essinf}_\Omega\lambda > 0$, the above relations imply that $\mathrm{meas}(W_1) = \mathrm{meas}(W_2) = 0$, so $\mathrm{meas}(W) = 0$, contradicting the initial assumption.

(iii) A direct consequence of (i) is that

$$\mathscr{E}'(\tilde{u})(w - \tilde{u}) \geq 0, \quad \forall w \in W^b,$$

that is,

$$\int_\Omega \left[|\nabla\tilde{u}|^{p-2}\nabla\tilde{u}\nabla(w - \tilde{u}) + \lambda(x)\tilde{u}^{p-1}(w - \tilde{u})\right] - \int_\Omega h(\tilde{u})(w - \tilde{u}) \geq 0,$$
$$\forall w \in W^b.$$

Let us define the function $\gamma(s) = \mathrm{sgn}(s)\min(|s|, b)$, and fix $\varepsilon > 0$ and $v \in W^{1,p}(\Omega)$ arbitrarily. Because γ is Lipschitz continuous, $w = \gamma \circ (\tilde{u} + \varepsilon v)$ belongs to $W^{1,p}(\Omega)$, see Marcus and Mizel [6]. The explicit expression of w is

$$w(x) = \begin{cases} -b, & \text{if } x \in \{\tilde{u} + \varepsilon v < -b\} \\ \tilde{u}(x) + \varepsilon v(x), & \text{if } x \in \{-b \leq \tilde{u} + \varepsilon v < b\} \\ b, & \text{if } x \in \{b \leq \tilde{u} + \varepsilon v\}. \end{cases}$$

Consequently, $w \in W^b$. Considering w as a test function in the above inequality, we obtain

$$\begin{aligned} 0 \leq &- \int_{\{\tilde{u}+\varepsilon v<-b\}} |\nabla\tilde{u}|^p - \int_{\{\tilde{u}+\varepsilon v<-b\}} \lambda(x)\tilde{u}^{p-1}(b + \tilde{u}) + \int_{\{\tilde{u}+\varepsilon v<-b\}} h(\tilde{u})(b + \tilde{u}) \\ &+ \varepsilon\int_{\{-b\leq\tilde{u}+\varepsilon v<b\}} |\nabla\tilde{u}|^{p-2}\nabla\tilde{u}\nabla v + \varepsilon\int_{\{-b\leq\tilde{u}+\varepsilon v<b\}} \lambda(x)\tilde{u}^{p-1}v - \varepsilon\int_{\{-b\leq\tilde{u}+\varepsilon v<b\}} h(\tilde{u})v \\ &- \int_{\{b\leq\tilde{u}+\varepsilon v\}} |\nabla\tilde{u}|^p + \int_{\{b\leq\tilde{u}+\varepsilon v\}} \lambda(x)\tilde{u}^{p-1}(b - \tilde{u}) - \int_{\{b\leq\tilde{u}+\varepsilon v\}} h(\tilde{u})(b - \tilde{u}). \end{aligned}$$

After a suitable rearrangement of the terms in the above inequality, we obtain

$$
\begin{aligned}
0 \le\; & \varepsilon \int_\Omega |\nabla \tilde{u}|^{p-2}\nabla \tilde{u}\nabla v + \varepsilon \int_\Omega \lambda(x)\tilde{u}^{p-1}v - \varepsilon \int_\Omega h(\tilde{u})v - \int_{\{\tilde{u}+\varepsilon v<-b\}} |\nabla \tilde{u}|^p \\
& - \int_{\{b\le \tilde{u}+\varepsilon v\}} |\nabla \tilde{u}|^p + \int_{\{\tilde{u}+\varepsilon v<-b\}} [h(\tilde{u}) - \lambda(x)\tilde{u}^{p-1}](b+\tilde{u}+\varepsilon v) \\
& + \int_{\{b\le \tilde{u}+\varepsilon v\}} [h(\tilde{u}) - \lambda(x)\tilde{u}^{p-1}](-b+\tilde{u}+\varepsilon v) \\
& - \varepsilon \int_{\{\tilde{u}+\varepsilon v<-b\}} |\nabla \tilde{u}|^{p-2}\nabla \tilde{u}\nabla v - \varepsilon \int_{\{b\le \tilde{u}+\varepsilon v\}} |\nabla \tilde{u}|^{p-2}\nabla \tilde{u}\nabla v.
\end{aligned}
$$

First, due to (ii), we have

$$
\begin{aligned}
& \int_{\{\tilde{u}+\varepsilon v<-b\}} [h(\tilde{u}) - \lambda(x)\tilde{u}^{p-1}](b+\tilde{u}+\varepsilon v) \\
& \quad \le -\varepsilon \int_{\{\tilde{u}+\varepsilon v<-b\}} \Big[\max_{s\in[0,a]} |h(s)| + a^{p-1}\lambda(x)\Big] v.
\end{aligned}
$$

A similar estimation shows that

$$
\begin{aligned}
& \int_{\{b\le \tilde{u}+\varepsilon v\}} [h(\tilde{u}) - \lambda(x)\tilde{u}^{p-1}](-b+\tilde{u}+\varepsilon v) \\
& \quad \le \varepsilon \int_{\{b\le \tilde{u}+\varepsilon v\}} \Big[\max_{s\in[0,a]} |h(s)| + a^{p-1}\lambda(x)\Big] v.
\end{aligned}
$$

Taking into account the above estimates and dividing by $\varepsilon > 0$, we obtain that

$$
\begin{aligned}
0 \le\; & \int_\Omega |\nabla \tilde{u}|^{p-2}\nabla \tilde{u}\nabla v + \int_\Omega \lambda(x)\tilde{u}^{p-1}v - \int_\Omega h(\tilde{u})v \\
& - \int_{\{\tilde{u}+\varepsilon v<-b\}} \Big(\max_{s\in[0,a]} |h(s)|v + a^{p-1}\lambda(x)v + |\nabla \tilde{u}|^{p-2}\nabla \tilde{u}\nabla v\Big) \\
& - \int_{\{b\le \tilde{u}+\varepsilon v\}} \Big(\max_{s\in[0,a]} |h(s)|v + a^{p-1}\lambda(x)v + |\nabla \tilde{u}|^{p-2}\nabla \tilde{u}\nabla v\Big).
\end{aligned}
$$

Now, letting $\varepsilon \to 0^+$, and taking into account that $0 \le \tilde{u}(x) \le a$ a.e. $x \in \Omega$, we have $\text{meas}(\{\tilde{u}+\varepsilon v < -b\}) \to 0$ and $\text{meas}(\{b \le \tilde{u}+\varepsilon v\}) \to 0$, respectively. Consequently, the above inequality reduces to

$$
0 \le \int_\Omega |\nabla \tilde{u}|^{p-2}\nabla \tilde{u}\nabla v + \int_\Omega \lambda(x)\tilde{u}^{p-1}v - \int_\Omega h(\tilde{u})v.
$$

Because $v \in W^{1,p}(\Omega)$ was arbitrarily chosen, $\tilde{u}$ is a nonnegative solution for (P).

3. PROOF OF THEOREM 1.1

Because of (H_0^f), one can fix $c_0 \in \mathbb{R}$ such that

$$\text{essinf}_\Omega \alpha > c_0 > \liminf_{s\to 0^+} \frac{f(s)}{s^{p-1}}. \tag{3.1}$$

In particular, there is a sequence $\{s_i\} \subset (0,1)$ converging (decreasingly) to 0, such that

$$f(s_i) < c_0 s_i^{p-1}. \tag{3.2}$$

Let us define the functions

$$j(s) = f(s) - c_0 s_+^{p-1} \quad \text{and} \quad J(s) = \int_0^s j(t)\,dt, \quad s \in \mathbb{R} \tag{3.3}$$

and $\lambda_0(x) = \alpha(x) - c_0$, $x \in \Omega$.

Because $j(s_i) < 0$ (see (3.2)), and using the continuity of j and g as well as hypothesis (H_0^f), we may fix the positive sequences $\{a_i\}_i$, $\{b_i\}_i$, $\{\tilde{s}_i\}_i$, and $\{\varepsilon_i\}_i$ such that for all $i \in \mathbb{N}$,

$$b_{i+1} < a_i < s_i < b_i < 1; \tag{3.4}$$

$$\tilde{s}_i \leq b_i \leq \left\{ \frac{1}{i}, \frac{\min(1, \text{essinf}_\Omega \lambda_0)}{p i^p \text{meas}(\Omega)[\max_{[0,1]}|f| + \max_{[0,1]}|g| + |c_0| + 1]} \right\}; \tag{3.5}$$

$$j(s) + \varepsilon g(s) \leq 0 \quad \text{for all } s \in [a_i, b_i] \text{ and } \varepsilon \in [-\varepsilon_i, \varepsilon_i]; \tag{3.6}$$

$$\frac{pJ(\tilde{s}_i)}{\tilde{s}_i^p} > \frac{\int_\Omega \alpha(x)\,dx}{\text{meas}(\Omega)} - c_0. \tag{3.7}$$

In particular, we have $\lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i = 0$. For every $i \in \mathbb{N}$, we define the truncation functions $j_i, g_i : [0,\infty) \to \mathbb{R}$ by

$$j_i(s) = j(\min(s, b_i)) \quad \text{and} \quad g_i(s) = g(\min(s, b_i)). \tag{3.8}$$

Because $j(0) = g(0) = 0$, we may extend continuously the functions j_i and g_i to the whole real line, taking 0 for negative values. For every $s \in \mathbb{R}$ and $i \in \mathbb{N}$, let $J_i(s) = \int_0^s j_i(t)\,dt$ and $G_i(s) = \int_0^s g_i(t)\,dt$.

For every $i \in \mathbb{N}$ and $\varepsilon \in [-\varepsilon_i, \varepsilon_i]$, the function $h_{i,\varepsilon}^0 : [0,\infty) \to \mathbb{R}$ defined by $h_{i,\varepsilon}^0 = j_i + \varepsilon g_i$ is continuous, bounded, and $h_{i,\varepsilon}^0(0) = 0$. On account of relations (3.6) and (3.8), we have $h_{i,\varepsilon}^0(s) \leq 0$ for all $s \in [a_i, b_i]$. Moreover, $\text{essinf}_\Omega \lambda_0 = \text{essinf}_\Omega \alpha - c_0 > 0$, see (3.1). Thus, we may apply

Theorem 2.1 to the function $h^0_{i,\varepsilon}$ obtaining that for every $i \in \mathbb{N}$ and $\varepsilon \in [-\varepsilon_i, \varepsilon_i]$, the problem

$$\begin{cases} -\Delta_p u + \lambda_0(x)|u|^{p-2}u = h^0_{i,\varepsilon}(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathrm{P}^0_{i,\varepsilon}$}$$

has a weak solution $u^0_{i,\varepsilon} \in W^{1,p}(\Omega)$ such that

$$u^0_{i,\varepsilon} \in [0, a_i] \quad \text{for a.e. } x \in \Omega; \tag{3.9}$$

$$u^0_{i,\varepsilon} \quad \text{is the infimum of the functional } \mathscr{E}^\varepsilon_i \text{ on } W^{b_i}, \tag{3.10}$$

where

$$\mathscr{E}^\varepsilon_i(u) = \frac{1}{p}\|u\|^p_{\lambda_0} - \int_{\mathbb{R}^N} [J_i(u) + \varepsilon G_i(u)], \quad u \in W^{1,p}(\Omega). \tag{3.11}$$

Because of (3.3), (3.8), (3.9) and the definition of the function λ_0, the element $u^0_{i,\varepsilon}$ is a weak solution not only for $(\mathrm{P}^0_{i,\varepsilon})$ but also for our problem (P_ε). Consequently, it remains to prove that for every $k \in \mathbb{N}$, there are at least k distinct elements $u^0_{i,\varepsilon}$ verifying the required properties.

As we pointed out in the Introduction, the proof of the above fact is based on Theorem A (i.e., on the unperturbed case); consequently, we recall some partial results from [3]. To do this, take for abbreviation $u^0_i = u^0_{i,0}$ and let $w_{\tilde{s}_i} \in W^{1,p}(\Omega)$, $w_{\tilde{s}_i}(x) = \tilde{s}_i$ $(x \in \Omega)$ for every $i \in \mathbb{N}$. The core of Theorem A, which is based on (3.7), is to prove the relations

$$\mathscr{E}^0_i(u^0_i) \le \mathscr{E}^0_i(w_{\tilde{s}_i}) < 0 \quad \text{for all } i \in \mathbb{N}; \tag{3.12}$$

$$\lim_{i\to\infty} \mathscr{E}^0_i(u^0_i) = \lim_{i\to\infty} \mathscr{E}^0_i(w_{\tilde{s}_i}) = 0, \tag{3.13}$$

see Propositions 3.1 and 3.3 from [3], respectively. In particular, because of (3.8) and (3.9), we observe that $\mathscr{E}^0_i(u^0_i) = \mathscr{E}^0_1(u^0_i)$ for all $i \in \mathbb{N}$. Combining this relation with (3.12) and (3.13), we see that the sequence $\{u^0_i\}_i$ contains infinitely many distinct elements.

Up to a subsequence, we may consider a sequence $\{\gamma_i\}_i$ with negative terms such that

$$\gamma_i < \mathscr{E}^0_i(u^0_i) \le \mathscr{E}^0_i(w_{\tilde{s}_i}) < \gamma_{i+1}. \tag{3.14}$$

Let us denote

$$\varepsilon'_i = \frac{\gamma_{i+1} - \mathscr{E}^0_i(w_{\tilde{s}_i})}{|G_i(\tilde{s}_i)|\mathrm{meas}(\Omega) + 1} \quad \text{and} \quad \varepsilon''_i = \frac{\mathscr{E}^0_i(u^0_i) - \gamma_i}{\max_{s\in[0,a_i]}|G_i(s)|\mathrm{meas}(\Omega) + 1}, \ \ i \in \mathbb{N}.$$

Fix $k \in \mathbb{N}$. Because of (3.14),

$$\varepsilon_k^0 = \min(1, \varepsilon_1, \ldots, \varepsilon_k, \varepsilon_1', \ldots, \varepsilon_k', \varepsilon_1'', \ldots, \varepsilon_k'') > 0.$$

Then, for every $i \in \{1, \ldots, k\}$ and $\varepsilon \in [-\varepsilon_k^0, \varepsilon_k^0]$, we have

$$\begin{aligned}
\mathcal{E}_i^\varepsilon(u_{i,\varepsilon}^0) &\leq \mathcal{E}_i^\varepsilon(w_{\tilde{s}_i}) \quad \text{(see (3.10) and (3.5))} \\
&= \mathcal{E}_i^0(w_{\tilde{s}_i}) - \varepsilon \int_\Omega G_i(w_{\tilde{s}_i}) \\
&< \gamma_{i+1}, \quad \text{(see the choice of } \varepsilon_i')
\end{aligned}$$

and taking into account that $u_{i,\varepsilon}^0$ belongs to W^{b_i}, and u_i^0 is the minimum point of $\mathcal{E}_i^0$ over the set W^{b_i}, see relation (3.10) for $\varepsilon = 0$, we have

$$\begin{aligned}
\mathcal{E}_i^\varepsilon(u_{i,\varepsilon}^0) &= \mathcal{E}_i^0(u_{i,\varepsilon}^0) - \varepsilon \int_\Omega G_i(u_{i,\varepsilon}^0) \\
&\geq \mathcal{E}_i^0(u_i^0) - \varepsilon \int_\Omega G_i(u_{i,\varepsilon}^0) \\
&> \gamma_i. \quad \text{(see the choice of } \varepsilon_i'' \text{ and (3.9))}
\end{aligned}$$

In conclusion, for every $i \in \{1, \ldots, k\}$ and $\varepsilon \in [-\varepsilon_k^0, \varepsilon_k^0]$, we have

$$\gamma_i < \mathcal{E}_i^\varepsilon(u_{i,\varepsilon}^0) < \gamma_{i+1},$$

thus

$$\mathcal{E}_1^\varepsilon(u_{1,\varepsilon}^0) < \cdots < \mathcal{E}_k^\varepsilon(u_{k,\varepsilon}^0).$$

Let us observe that $u_{i,\varepsilon}^0 \in W^{b_1}$ for every $i \in \{1, \ldots, k\}$, so $\mathcal{E}_i^\varepsilon(u_{i,\varepsilon}^0) = \mathcal{E}_1^\varepsilon(u_{i,\varepsilon}^0)$, see relation (3.8). From above, we obtain that for every $\varepsilon \in [-\varepsilon_k^0, \varepsilon_k^0]$,

$$\mathcal{E}_1^\varepsilon(u_{1,\varepsilon}^0) < \cdots < \mathcal{E}_1^\varepsilon(u_{k,\varepsilon}^0).$$

In particular, this fact shows that the elements $u_{1,\varepsilon}^0, \ldots, u_{k,\varepsilon}^0$ are distinct whenever $\varepsilon \in [-\varepsilon_k^0, \varepsilon_k^0]$.

Now, we prove (1.1'). The first relation easily follows by (3.9) and (3.5). To check the second relation, we observe that for every $i \in \{1, \ldots, k\}$ and $\varepsilon \in [-\varepsilon_k^0, \varepsilon_k^0]$,

$$\mathcal{E}_1^\varepsilon(u_{i,\varepsilon}^0) = \mathcal{E}_i^\varepsilon(u_{i,\varepsilon}^0) < \gamma_{i+1} < 0.$$

Consequently, for every $i \in \{1, \dots, k\}$ and $\varepsilon \in [-\varepsilon_k^0, \varepsilon_k^0]$, by using a mean value theorem, we obtain

$$\begin{aligned}
&\frac{1}{p}\|u_{i,\varepsilon}^0\|_{W^{1,p}}^p \\
&\quad \le \frac{1}{p}[\min(1, \mathrm{essinf}_\Omega \lambda_0)]^{-1}\,\|u_{i,\varepsilon}^0\|_{\lambda_0}^p \\
&\quad < [\min(1, \mathrm{essinf}_\Omega \lambda_0)]^{-1}\int_\Omega [J_i(u_{i,\varepsilon}^0) + \varepsilon G_i(u_{i,\varepsilon}^0)] \\
&\quad \le [\min(1, \mathrm{essinf}_\Omega \lambda_0)]^{-1}\,\mathrm{meas}(\Omega)\Big[\max_{[0,1]}|f| + \max_{[0,1]}|g| + |c_0| a_i^{p-1}\Big] a_i \\
&\qquad \text{(see (3.3), (3.4), (3.9) and } \varepsilon_k^0 \le 1) \\
&\quad < \frac{1}{p i^p}, \quad \text{(see (3.4) and (3.5))}
\end{aligned}$$

which concludes the proof.

4. PROOF OF THEOREM 1.3

The proof of this part is similar to that of Theorem 1.1. Because of (H_∞^f), one can fix $c_\infty \in \mathbb{R}$ such that

$$\mathrm{essinf}_\Omega \alpha > c_\infty > \liminf_{s\to\infty}\frac{f(s)}{s^{p-1}}. \tag{4.1}$$

So, there is a sequence $\{s_i\} \subset (0, \infty)$ converging increasingly to $+\infty$, such that

$$f(s_i) < c_\infty s_i^{p-1}. \tag{4.2}$$

We define the functions

$$j(s) = f(s) - c_\infty s_+^{p-1} \quad \text{and} \quad J(s) = \int_0^s j(t)dt, \quad s \in \mathbb{R} \tag{4.3}$$

and $\lambda_\infty(x) = \alpha(x) - c_\infty$, $x \in \Omega$. Because $j(s_i) < 0$ (see (4.2)), and using the continuity of j and g as well as hypothesis (H_∞^f), we may fix a subsequence $\{s_{m_i}\}_i$ of $\{s_i\}_i$ and the positive sequences $\{a_i\}_i$, $\{b_i\}_i$, $\{\tilde{s}_i\}_i$, and $\{\varepsilon_i\}_i$ such that for all $i \in \mathbb{N}$,

$$i \le a_i < s_{m_i} < b_i < a_{i+1}; \tag{4.4}$$

$$\tilde{s}_i \le b_i; \tag{4.5}$$

$$j(s) + \varepsilon g(s) \le 0 \quad \text{for all } s \in [a_i, b_i] \text{ and } \varepsilon \in [-\varepsilon_i, \varepsilon_i]; \tag{4.6}$$

$$\frac{pJ(\tilde{s}_i)}{\tilde{s}_i^p} > \frac{\int_\Omega \alpha(x)\,dx}{\mathrm{meas}(\Omega)} - c_\infty, \tag{4.7}$$

and $\lim_{i\to\infty} \tilde{s}_i = \infty$.

In the same way as we did in (3.8), let us define the truncation functions $j_i, g_i : [0,\infty) \to \mathbb{R}$ by

$$j_i(s) = j(\min(s, b_i)) \quad \text{and} \quad g_i(s) = g(\min(s, b_i)). \tag{4.8}$$

Because $j_i(0) = g_i(0) = 0$, we may extend continuously the functions j_i and g_i to the whole real line, taking 0 for negative values. For every $s \in \mathbb{R}$ and $i \in \mathbb{N}$, let $J_i(s) = \int_0^s j_i(t)\,dt$ and $G_i(s) = \int_0^s g_i(t)\,dt$.

For every $i \in \mathbb{N}$ fixed and $\varepsilon \in [-\varepsilon_i, \varepsilon_i]$, the function $h_{i,\varepsilon}^\infty : [0,\infty) \to \mathbb{R}$ defined by $h_{i,\varepsilon}^\infty = j_i + \varepsilon g_i$ is continuous, bounded, and $h_{i,\varepsilon}^\infty(0) = 0$. On account of relations (4.5) and (4.8), one has $h_{i,\varepsilon}^\infty(s) \le 0$ for all $s \in [a_i, b_i]$. Consequently, we may apply Theorem 2.1 to the function $h_{i,\varepsilon}^\infty$ obtaining that for every $i \in \mathbb{N}$ and $\varepsilon \in [-\varepsilon_i, \varepsilon_i]$, the problem

$$\begin{cases} -\Delta_p u + \lambda_\infty(x)|u|^{p-2}u = h_{i,\varepsilon}^\infty(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathrm{P}_{i,\varepsilon}^\infty$}$$

has a weak solution $u_{i,\varepsilon}^\infty \in W^{1,p}(\Omega)$ such that

$$u_{i,\varepsilon}^\infty \in [0, a_i] \quad \text{for a.e. } x \in \Omega; \tag{4.9}$$

$$u_{i,\varepsilon}^\infty \text{ is the infimum of the functional } \mathscr{E}_i^\varepsilon \text{ on } W^{b_i}, \tag{4.10}$$

where $\mathscr{E}_i^\varepsilon$ is defined exactly as in (3.11). Because of (4.8) and (4.9), $u_{i,\varepsilon}^\infty$ is a weak solution not only for $(\mathrm{P}_{i,\varepsilon}^\infty)$ but also for the initial problem (P_ε). Consequently, we have to prove that for every $k \in \mathbb{N}$, there are at least k distinct elements $u_{i,\varepsilon}^\infty$ verifying (1.2′) when ε belongs to a certain interval around the origin.

Let $u_i^\infty = u_{i,0}^\infty$. The crucial step of Theorem B in [3], see also (4.5) and (4.7), is

$$\lim_{i\to\infty} \mathscr{E}_i^0(u_i^\infty) = \lim_{i\to\infty} \mathscr{E}_i^0(w_{\tilde{s}_i}) = -\infty, \tag{4.11}$$

where $w_{\tilde{s}_i}$ denotes the constant function with value $\tilde{s}_i$. In particular, it follows that the sequence $\{u_i^\infty\}_i$ contains infinitely many distinct elements. So, up to a subsequence, we can fix a sequence $\{\gamma_i\}_i$ with negative terms such that

$$\gamma_{i+1} < \mathscr{E}_i^0(u_i^\infty) \le \mathscr{E}_i^0(w_{\tilde{s}_i}) < \gamma_i. \tag{4.12}$$

Let us denote

$$\varepsilon_i' = \frac{\gamma_i - \mathscr{E}_i^0(w_{\tilde{s}_i})}{|G_i(\tilde{s}_i)|\text{meas}(\Omega) + 1} \quad \text{and} \quad \varepsilon_i'' = \frac{\mathscr{E}_i^0(u_i^\infty) - \gamma_{i+1}}{\max_{s\in[0,a_i]}|G_i(s)|\text{meas}(\Omega) + 1}, \quad i \in \mathbb{N}.$$

Fix $k \in \mathbb{N}$. Because of (4.12), we have

$$\varepsilon_k^\infty = \min(1, \varepsilon_1, \ldots, \varepsilon_k, \varepsilon_1', \ldots, \varepsilon_k', \varepsilon_1'', \ldots, \varepsilon_k'') > 0.$$

Then, for every $i \in \{1, \ldots, k\}$ and $\varepsilon \in [-\varepsilon_k^\infty, \varepsilon_k^\infty]$ we have

$$\begin{aligned} \mathscr{E}_i^\varepsilon(u_{i,\varepsilon}^\infty) &\leq \mathscr{E}_i^\varepsilon(w_{\tilde{s}_i}) \quad (\text{see } (4.10)) \\ &= \mathscr{E}_i^0(w_{\tilde{s}_i}) - \varepsilon \int_\Omega G_i(w_{\tilde{s}_i}) \\ &< \gamma_i, \quad (\text{see the choice of } \varepsilon_i') \end{aligned}$$

and because $u_{i,\varepsilon}^\infty$ belongs to W^{b_i}, and u_i^∞ is the minimum point of $\mathscr{E}_i^0$ on the set W^{b_i}, see relation (4.10) for $\varepsilon = 0$, we have

$$\begin{aligned} \mathscr{E}_i^\varepsilon(u_{i,\varepsilon}^\infty) &= \mathscr{E}_i^0(u_{i,\varepsilon}^\infty) - \varepsilon \int_\Omega G_i(u_{i,\varepsilon}^\infty) \\ &\geq \mathscr{E}_i^0(u_i^\infty) - \varepsilon \int_\Omega G_i(u_{i,\varepsilon}^\infty) \\ &> \gamma_{i+1}. \quad (\text{see the choice of } \varepsilon_i'' \text{ and } (4.9)) \end{aligned}$$

Thus, for every $i \in \{1, \ldots, k\}$ and $\varepsilon \in [-\varepsilon_k^\infty, \varepsilon_k^\infty]$, we have

$$\gamma_{i+1} < \mathscr{E}_i^\varepsilon(u_{i,\varepsilon}^\infty) < \gamma_i.$$

In particular,

$$\mathscr{E}_k^\varepsilon(u_{k,\varepsilon}^\infty) < \cdots < \mathscr{E}_1^\varepsilon(u_{1,\varepsilon}^\infty) < 0. \tag{4.13}$$

By construction, $u_{i,\varepsilon}^\infty \in W^{b_k}$ for every $i \in \{1, \ldots, k\}$, see (4.4); thus, $\mathscr{E}_i^\varepsilon(u_{i,\varepsilon}^\infty) = \mathscr{E}_k^\varepsilon(u_{i,\varepsilon}^\infty)$, see relation (4.8). Therefore, (4.13) implies that for every $\varepsilon \in [-\varepsilon_k^\infty, \varepsilon_k^\infty]$,

$$\mathscr{E}_k^\varepsilon(u_{k,\varepsilon}^\infty) < \cdots < \mathscr{E}_k^\varepsilon(u_{1,\varepsilon}^\infty) < 0.$$

In particular, the elements $u_{1,\varepsilon}^\infty, \ldots, u_{k,\varepsilon}^\infty$ are distinct whenever $\varepsilon \in [-\varepsilon_k^\infty, \varepsilon_k^\infty]$.

Now, we prove relation (1.2′). Fix $\varepsilon \in [-\varepsilon_k^\infty, \varepsilon_k^\infty]$. First of all, because $\mathscr{E}_1^\varepsilon(u_{1,\varepsilon}^\infty) < 0 = \mathscr{E}_1^\varepsilon(0)$, then $\|u_{1,\varepsilon}^\infty\|_{L^\infty} > 0$, which proves relation (1.2′) for $i = 1$. We further prove that

$$\|u_{i,\varepsilon}^\infty\|_{L^\infty} > a_{i-1} \quad \text{for all } i \in \{2, \ldots, k\}. \tag{4.14}$$

Let us assume the contrary, i.e., there exists an element $i_0 \in \{2, \ldots, k\}$ such that $\|u^{\infty}_{i_0,\varepsilon}\|_{L^{\infty}} \leq a_{i_0-1}$. Because $a_{i_0-1} < b_{i_0-1}$, then $u^{\infty}_{i_0,\varepsilon} \in W^{b_{i_0-1}}$. Thus, on account of (4.10) and (4.8), we have

$$\mathscr{E}^{\varepsilon}_{i_0-1}(u^{\infty}_{i_0-1,\varepsilon}) = \min_{W^{b_{i_0-1}}} \mathscr{E}^{\varepsilon}_{i_0-1} \leq \mathscr{E}^{\varepsilon}_{i_0-1}(u^{\infty}_{i_0,\varepsilon}) = \mathscr{E}^{\varepsilon}_{i_0}(u^{\infty}_{i_0,\varepsilon}),$$

which contradicts (4.13). Thus, (4.14) holds true, which can be combined with (4.4), obtaining relation (1.2′). The proof is concluded.

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REFERENCES

1. G. Anello and G. Cordaro (2007). Perturbation from Dirichlet problem involving oscillating nonlinearities. *J. Differential Equations* 234:80–90.
2. F. Faraci and A. Kristály (2007). On an open question of Ricceri concerning a Neumann problem. *Glasg. Math. J.* 49:189–195.
3. A. Kristály and D. Motreanu (2007). Nonsmooth Neumann-type problems involving the *p*-Laplacian. *Numer. Funct. Anal. Optim.* 28(11–12):1309–1326.
4. S. Li and Z. Liu (2002). Perturbations from symmetric elliptic boundary value problems. *J. Differential Equations* 185:271–280.
5. P. Omari and F. Zanolin (1996). Infinitely many solutions of a quasilinear elliptic problem with an oscillatory potential. *Comm. Partial Differential Equations* 21:721–733.
6. M. Marcus and V. J. Mizel (1972). Absolute continuity on tracks and mappings of Sobolev spaces. *Arch. Rational Mech. Anal.* 45:294–320.
7. B. Ricceri (2001). Infinitely many solutions of the Neumann problem for elliptic equations involving the *p*-Laplacian. *Bull. London Math. Soc.* 33:31–340.
8. J. Saint Raymond (2002). On the multiplicity of the solutions of the equations $-\Delta u = \lambda f(u)$. *J. Differential Equations* 180:65–88.