Multiple solutions for \( p \)-Laplacian type equations

Alexandru Kristály\textsuperscript{a,}\textsuperscript{*}, Hannelore Lisei\textsuperscript{b}, Csaba Varga\textsuperscript{b}

\textsuperscript{a} Babeş-Bolyai University, Department of Economics, 400591 Cluj-Napoca, Romania
\textsuperscript{b} Babeş-Bolyai University, Faculty of Mathematics and Computer Science, 400084 Cluj-Napoca, Romania

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Abstract

In this paper we establish the existence of three weak solutions of an equation which involves a general elliptic operator in divergence form (in particular, a \( p \)-Laplacian operator), while the nonlinearity has a \((p-1)\)-sublinear growth at infinity. This result completes some recent papers, where mountain pass type solutions were obtained providing the nonlinear term via a \((p-1)\)-superlinear growth at infinity (fulfilling an Ambrosetti–Rabinowitz type condition). In our case, an abstract critical point result is applied, proved by G. Bonanno [G. Bonanno, Some remarks on a three critical points theorem, Nonlinear Analysis 54 (2003) 651–665].

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1. Introduction

Various particular forms of the problem involving elliptic operators in divergence form

\[
\begin{cases}
-\text{div}(a(x, \nabla u)) = f(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]  

(P)

have been studied in the recent years. Here, \( \Omega \subset \mathbb{R}^N \) is a bounded open domain, \( N \geq 2 \), while the nonlinearities \( a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N \) and \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) fulfill certain structural conditions. The simplest case occurs when \( a(x, s) = |s|^{p-2}s, p \geq 2 \); thus (P) reduces to a problem which involves the usual \( p \)-Laplacian operator \( \Delta_p(\cdot) = \text{div}(|\nabla(\cdot)|^{p-2}\nabla(\cdot)) \).

Recently, De Nápoli and Mariani [4] studied problem (P) when the potential \( a \) satisfies a set of assumptions, see H(a) below, which includes the \( p \)-Laplacian and also other important cases, such as the generalized prescribed mean...
curvature operator. Duc and Vu [3] extended the result of [4], considering problem (P) in the ‘nonuniform’ case, for when the potential \(a\) fulfills

\[ |a(x, \xi)| \leq c_0(h_0(x) + h_1(x)|\xi|^{p-1}), \quad \forall (x, \xi) \in \Omega \times \mathbb{R}^N, \]

with \(h_0 \in L^p((\Omega)^{p-1})(\Omega), h_1 \in L^1_{\text{loc}}(\Omega), c_0 > 0.\)

In both papers [3,4], the nonlinear term \(f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}\) verifies the Ambrosetti–Rabinowitz type condition: defining \(F(x,s) = \int_0^s f(x,t)dt\), there exist \(s_0 > 0\) and \(\theta > p\) such that

\[ 0 < \theta F(x,s) \leq sf(x,s), \quad \forall x \in \Omega, s \in \mathbb{R}, |s| \geq s_0. \]  

(AR)

By (AR), one can deduce that

\[ |f(x,s)| \geq c|s|^{\theta-1}, \quad \forall x \in \Omega, s \in \mathbb{R}, |s| \geq s_0, \]

i.e., \(f\) is \((p-1)\)-superlinear at infinity.

The purpose of this paper is to handle the counterpart of the above case, i.e., when \(f\) is \((p-1)\)-sublinear at infinity. For the sake of simplicity, we assume in the sequel that \(f\) is autonomous, i.e., \(f(x,s) = f(s)\). We consider the condition

\[ (f_1) \quad \lim_{|s| \rightarrow +\infty} \frac{f(s)}{|s|^{p-1}} = 0. \]

If \(f(s) = \lambda (\arctan(s))^2\), with \(\lambda \in \mathbb{R}\) fixed (thus \(f\) clearly fulfills \((f_1)\)), while \(a(x,s) = s\) (thus in (P) there appears the standard Laplacian operator \(\Delta u = \nabla \nabla u\)), an easy computation shows that (P) possesses only the zero solution, whenever \(|\lambda| < \pi^{-1}c_2^{-2}\), where \(c_2 > 0\) is the best Sobolev constant of the embedding \(H^1_0(\Omega) \hookrightarrow L^2(\Omega)\). Therefore, it is more appropriate to investigate, instead of (P), the following eigenvalue problem:

\[ \begin{cases} -\text{div}(a(x, \nabla u)) = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega. \end{cases} \]

(P_\lambda)

In the next section we will state our main result (Theorem 2.1) which guarantees the existence of at least three weak solutions of (P_\lambda) for certain \(\lambda > 0\). Our result completes in a natural way not only the papers of Duc and Vu [3], and De Nápoli and Mariani [4] (superlinear nonlinearities), but also some earlier works in the sublinear context (where \(a(x,s) = s\)). For instance, Brézis and Oswald [2] studied problem (P_\lambda) when the behaviour of \(f(s)/s\) is suitably controlled at infinity, obtaining an existence and uniqueness result via the minimization technique and maximum principle. Lin [5] exploited a sub–super-solution argument, applying the sweeping principle of Serrin in order to obtain existence, uniqueness, and asymptotical properties of the solutions of (P_\lambda) when \(f(s)\) behaves like \(s^q\) \((0 < q < 1)\) with \(s\) large.

The proof of our main result (Theorem 2.1) is based on a recent abstract critical point theorem proved by Bonanno [1] which is an extension of the famous result of Ricceri [6,7]. In the next section we give the precise statement of Theorem 2.1, Section 3 contains auxiliary results, while in Section 4 we will give the proof of Theorem 2.1.

2. Main result

In the sequel, let \(p > 1\) and \(\Omega \subset \mathbb{R}^N\) be a bounded open domain, where \(N \geq 2\). Now, we recall the same assumptions as in [4], concerning the potential \(a : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N\).

H(a): Let \(A : \overline{\Omega} \times \mathbb{R}^N \rightarrow \mathbb{R}, A = A(x, \xi)\) be a continuous function in \(\overline{\Omega} \times \mathbb{R}^N\), with continuous derivative with respect to \(\xi, a = DA = A'\), and suppose that the following conditions hold:

(a) \(A(x,0) = 0, \forall x \in \Omega\).

(b) \(a\) satisfies the growth condition

\[ |a(x, \xi)| \leq c_1(1 + |\xi|^{p-1}), \quad \forall x \in \Omega, \xi \in \mathbb{R}^N \]

(1)

for some constant \(c_1 > 0\).

(c) \(A\) is \(p\)-uniformly convex: There exists a constant \(k > 0\), such that

\[ A\left(x, \frac{\xi + \eta}{2}\right) \leq \frac{1}{2} A(x, \xi) + \frac{1}{2} A(x, \eta) - k|\xi - \eta|^p, \quad \forall x \in \Omega, \xi, \eta \in \mathbb{R}^N. \]
(d) $A$ is $p$-subhomogeneous:
\[
0 \leq a(x, \xi) : \xi \leq p A(x, \xi), \quad \forall x \in \Omega, \xi \in \mathbb{R}^N.
\]
(e) $A$ satisfies the ellipticity condition: There exists a constant $C > 0$ such that
\[
A(x, \xi) \geq C|\xi|^p, \quad \forall x \in \Omega, \xi \in \mathbb{R}^N.
\]

**Remark 2.1.** Let $p \geq 2$. If $A(x, s) = \frac{1}{p}|s|^p$, then $a(x, s) = |s|^{p-2}s$ and one obtains the usual $p$-Laplacian. If $A(x, s) = \frac{1}{p}[(1 + s^2)^{\frac{p}{2}} - 1]$, then we obtain the generalized mean curvature operator $\text{div}((1 + |\nabla u|^2)^{\frac{p-2}{2}}\nabla u)$. For another specific choice of $A$, see [4].

Let $f : \mathbb{R} \to \mathbb{R}$ be a function which, besides $(f_1)$, satisfies the following conditions:

(f2) $\lim_{s \to 0} \frac{f(s)}{|s|^{p-1}} = 0$.

(f3) There exists $s_0 \in \mathbb{R}$ such that $F(s_0) > 0$, where $F(s) = \int_0^s f(t)\,dt$.

Our main result is the following:

**Theorem 2.1.** Let $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a potential which fulfills the hypothesis $H(a)$, and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies $(f_1), (f_2)$ and $(f_3)$. Then, there exist an open interval $\Lambda \subset (0, +\infty)$ and a constant $\mu > 0$ such that for every $\lambda \in \Lambda$ problem (P$\lambda$) has at least three distinct weak solutions in $W^{1,p}_0(\Omega)$, whose $W^{1,p}_0(\Omega)$-norms are less than $\mu$.

### 3. Preliminaries

We assume that the assumptions of Theorem 2.1 are verified. The norm of the space $L^p(\Omega)$ will be denoted by $\| \cdot \|_p$. The Sobolev space $W^{1,p}_0(\Omega)$ is endowed with the usual norm $\|u\| = (\int_\Omega |\nabla u|^p\,dx)^{1/p}$. Since the embedding $W^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega)$ ($q \in [1, p^*)$) is compact, let $c_q > 0$ be the best Sobolev constant, i.e. $\|u\|_q \leq c_q \|u\|$ for every $u \in W^{1,p}_0(\Omega)$, and $c_q = \gamma_q^{-1}$, with $\gamma_q = \inf\{\|u\| : \|u\|_q = 1\}$. Above, $p^*$ denotes the usual Sobolev critical exponent.

We introduce the energy functional $\mathcal{E}_\lambda : W^{1,p}_0(\Omega) \to \mathbb{R}$ given by $\mathcal{E}_\lambda(u) = A(u) - \lambda F(u)$, where
\[
A(u) = \int_\Omega A(x, \nabla u(x))\,dx \quad \text{and} \quad F(u) = \int_\Omega F(u(x))\,dx.
\]

It is easy to see that the functional $\mathcal{E}_\lambda$ is of class $C^1$ and its derivative is given by
\[
\langle \mathcal{E}_\lambda'(u), \varphi \rangle = \int_\Omega a(x, \nabla u(x)) \nabla \varphi(x)\,dx - \lambda \int_\Omega f(u(x))\varphi(x)\,dx.
\]

Here, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $W^{1,p}_0(\Omega)$ and its dual $W^{-1,p'}(\Omega)$, $1/p + 1/p' = 1$. Moreover, the critical points of the functional $\mathcal{E}_\lambda$ are exactly the weak solutions of problem (P$\lambda$).

**Remark 3.1.** Due to hypothesis $H(a)$, a simple calculation shows that the functional $A$ is locally uniformly convex. Moreover, $A' : W^{1,p}_0(\Omega) \to W^{-1,p'}(\Omega)$ verifies the $(S_\lambda)$ condition, i.e., for every sequence $\{u_n\} \subset W^{1,p}_0(\Omega)$ such that $u_n \rightharpoonup u$ weakly and $\limsup_{n \to \infty} (A'(u_n), u_n - u) \leq 0$, we have $u_n \to u$ strongly; see Proposition 2.1 in [4].

**Lemma 3.1.** For every $\lambda \in \mathbb{R}$, the functional $\mathcal{E}_\lambda : W^{1,p}_0(\Omega) \to \mathbb{R}$ is sequentially weakly lower semicontinuous.

**Proof.** The functional $A$ being locally uniformly convex is weakly lower semicontinuous. On the other hand, condition $(f_1)$ implies the existence of a constant $c > 0$ such that $|f(s)| \leq c(1 + |s|^{p-1})$ for every $s \in \mathbb{R}$. Since the embedding $W^{1,p}_0(\Omega) \hookrightarrow L^p(\Omega)$ is compact, one can deduce in a standard way that $F$ is sequentially weakly continuous. \hfill $\square$

**Lemma 3.2.** For every $\lambda \in \mathbb{R}$, the functional $\mathcal{E}_\lambda$ is coercive and satisfies the Palais–Smale condition.
Proof. Let us fix \( \lambda \in \mathbb{R} \), arbitrary. By (f₁) there exists \( \delta = \delta(\lambda) \) such that
\[
|f(s)| \leq pCc_p^{-p}(1 + |\lambda|)^{-1}|s|^{p-1} \quad \text{for every } |s| \geq \delta.
\]
(Here, \( C \) is from \( H(a)(e) \)).

Integrating the above inequality we have
\[
|F(s)| \leq Cc_p^{-p}(1 + |\lambda|)^{-1}|s|^p + \max_{|t| \leq \delta} |f(t)||s|, \quad \forall s \in \mathbb{R}.
\]

Thus, for every \( u \in W_0^{1,p}(\Omega) \) we obtain
\[
\mathcal{E}_\lambda(u) \geq A(u) - |\lambda|\|\mathcal{F}(u)\|
\geq C\|u\|^p - C|\lambda|\|u\|^p - c_p|\lambda|(\nu(\Omega))^\frac{1}{p}\|u\|\max_{|t| \leq \delta} |f(t)|,
\]
where \( \nu(\Omega) \) denotes the Lebesgue measure of \( \Omega \). Since \( p > 1 \), \( \mathcal{E}_\lambda(u) \to +\infty \) whenever \( \|u\| \to +\infty \). Hence \( \mathcal{E}_\lambda \) is coercive.

Now, let \( \{u_n\} \subset W_0^{1,p}(\Omega) \) be a sequence such that \( \{\mathcal{E}_\lambda(u_n)\} \) is bounded and \( \|\mathcal{E}_\lambda'(u_n)\|_{W^{-1,p'}} \to 0 \). Since \( \mathcal{E}_\lambda \) is coercive, it follows that the sequence \( \{u_n\} \) is bounded in \( W_0^{1,p}(\Omega) \). Up to a subsequence, \( u_n \rightharpoonup u \) weakly in \( W_0^{1,p}(\Omega) \) and \( u_n \to u \) strongly in \( L^p(\Omega) \). From \( \mathcal{E}_\lambda = A - \lambda\mathcal{F} \) we get
\[
\langle A'(u_n), u_n - u \rangle = \langle \mathcal{E}_\lambda'(u_n), u_n - u \rangle + \lambda \int_\Omega f(u_n(x))(u_n(x) - u(x))dx.
\]
Since \( \|\mathcal{E}_\lambda'(u_n)\|_{W^{-1,p'}} \to 0 \) and \( \{u_n - u\} \) is bounded in \( W_0^{1,p}(\Omega) \), by the inequality \( |\langle \mathcal{E}_\lambda'(u_n), u_n - u \rangle| \leq \|\mathcal{E}_\lambda'(u_n)\|_{W^{-1,p'}}\|u_n - u\| \) it follows that
\[
\langle \mathcal{E}_\lambda'(u_n), u_n - u \rangle \to 0.
\]
As before, (f₁) implies the existence of a constant \( c > 0 \) such that \( |f(s)| \leq c(1 + |s|^{p-1}) \) for every \( s \in \mathbb{R} \). Therefore
\[
\int_\Omega |f(u_n(x))||u_n(x) - u(x)|dx \leq c \int_\Omega |u_n(x) - u(x)|dx + c \int_\Omega |(u_n(x))|^{p-1}|u_n(x) - u(x)|dx
\leq c((\nu(\Omega))^\frac{1}{p} + \|u_n\|_p^{p-1})\|u_n - u\|_p.
\]
Since \( u_n \to u \) strongly in \( L^p(\Omega) \), we get
\[
\lim_{n \to \infty} \int_\Omega |f(u_n(x))||u_n(x) - u(x)|dx = 0.
\]
In conclusion, relation (4) implies
\[
\limsup_{n \to \infty} \langle A'(u_n), u_n - u \rangle \leq 0.
\]
But the operator \( A' \) has the \((S_+)\) property; therefore we have \( u_n \to u \) strongly in \( W_0^{1,p}(\Omega) \). □

Lemma 3.3. The following property holds:
\[
\lim_{\rho \to 0^+} \frac{\sup\{\mathcal{F}(u) : A(u) < \rho\}}{\rho} = 0.
\]

Proof. Due to (f₂), for an arbitrary small \( \epsilon > 0 \), there exists \( \delta > 0 \) such that
\[
|f(s)| \leq \epsilon pC_p^{-p}|s|^{p-1} \quad \text{for every } |s| \leq \delta.
\]
Combining the above inequality with
\[
|f(s)| \leq c(1 + |s|^{p-1}) \quad \text{for every } s \in \mathbb{R},
\]
we obtain
\[
|F(s)| \leq \epsilon c_p^{-p}p|s|^p + K(\delta)|s|^q \quad \text{for every } s \in \mathbb{R},
\]
where \( p \in (p, p^*) \) is fixed, and \( K(\delta) > 0 \) does not depend on \( s \). For \( \rho > 0 \), define the sets
\[
S_\rho^1 = \{ u \in W_0^{1, p}(\Omega) : A(u) < \rho \}
\]
and
\[
S_\rho^2 = \{ u \in W_0^{1, p}(\Omega) : C\|u\|^p < \rho \}.
\]
By H(a)(e) it follows that \( S_\rho^1 \subset S_\rho^2 \).

From (5) we obtain
\[
\mathcal{F}(u) \leq \varepsilon \|u\|^p + K(\delta)c_{\mathcal{Q}}^p \|u\|^p.
\] (6)

Since \( 0 \in S_\rho^1 \) (due to H(a)(b)), and \( \mathcal{F}(0) = 0 \), one has \( 0 \leq \sup_{u \in S_\rho^1} \mathcal{F}(u) \). On the other hand, if \( u \in S_\rho^2 \), then
\[ \|u\| \leq C^{-\frac{1}{p}} \rho^{\frac{1}{p}} \]
and using (6) we get
\[
\frac{\sup_{u \in S_\rho^1} \mathcal{F}(u)}{\rho} \leq \frac{\sup_{u \in S_\rho^2} \mathcal{F}(u)}{\rho} \leq \varepsilon C^{-1} + K(\delta)c_{\mathcal{Q}}^p C^{-\frac{q}{p}} \rho^{\frac{q}{p} - 1}.
\]
Because \( \varepsilon > 0 \) is arbitrary and \( \rho \to 0^+ \), we get the desired result. \( \square \)

4. Proof of Theorem 2.1

The main ingredient for the proof of Theorem 2.1 is a recent critical point result due to Bonanno [1] which it is actually a refinement of a result of Ricceri [6,7]. For completeness, we recall the result from [1].

**Theorem B** ([1, Theorem 2.1]). Let \( X \) be a separable and reflexive real Banach space, and let \( A, F : X \to \mathbb{R} \) be two continuously Gâteaux differentiable functionals. Assume that there exists \( x_0 \in X \) such that \( A(x_0) = F(x_0) = 0 \) and \( A(x) \geq 0 \) for every \( x \in X \) and that there exists \( x_1 \in X, \rho > 0 \) such that

(i) \( \rho < A(x_1) \);

(ii) \( \sup_{A(x) < \rho} F(x) < \rho \frac{F(x_1)}{A(x_1)} \).

Further, put
\[
\overline{a} = \frac{\zeta \rho}{\rho F(x_1) - \sup_{A(x) < \rho} F(x)},
\]
with \( \zeta > 1 \), and assume that the functional \( A - \lambda F \) is sequentially weakly lower semicontinuous, satisfies the Palais–Smale condition and

(iii) \( \lim_{\|x\| \to +\infty} (A(x) - \lambda F(x)) = +\infty \), for every \( \lambda \in [0, \overline{a}] \).

Then, there exist an open interval \( \Lambda \subset [0, \overline{a}] \) and a number \( \mu > 0 \) such that for each \( \lambda \in \Lambda \), the equation \( A'(x) - \lambda F'(x) = 0 \) admits at least three solutions in \( X \) having norm less than \( \mu \).

**Proof of Theorem 2.1.** Let \( s_0 \in \mathbb{R} \) be from \( \text{(f)} \), i.e., \( F(s_0) > 0 \). Fix an element \( x_0 \in \Omega \). Choose \( R_0 > 0 \) in such a way that
\[
\{ x \in \mathbb{R}^N : |x - x_0| \leq R_0 \} \subseteq \Omega,
\]
where \( |\cdot| \) denotes the usual euclidean norm in \( \mathbb{R}^N \). Let us denote by \( B_N(x_0, r) \) the \( N \)-dimensional closed euclidean ball with center \( x_0 \in \mathbb{R}^N \) and radius \( r > 0 \).

For \( \sigma \in (0, 1) \) define
\[
\begin{align*}
u_\sigma(x) &= \begin{cases} 0, & \text{if } x \in \mathbb{R}^N \setminus B_N(x_0, R_0); \\ s_0, & \text{if } x \in B_N(x_0, \sigma R_0); \\ \frac{R_0 - |x - x_0|}{R_0(1 - \sigma)}, & \text{if } x \in B_N(x_0, R_0) \setminus B_N(x_0, \sigma R_0). \end{cases}
\end{align*}
\] (7)
It is clear that \( u_\sigma \in W^{1,p}_0(\Omega) \). Moreover, we have
\[
|u_\sigma(x)| \leq |s_0| \quad \text{for each } x \in \mathbb{R}^N,
\]
and
\[
\|u_\sigma\|^p = \int_{\Omega} |\nabla u_\sigma|^p = \frac{|s_0|^p(1 - \sigma^N)}{(1 - \sigma)^p} R_0^{N-p} \omega_N > 0,
\]  
(8)
where \( \omega_N \) is the volume of \( B_N(0, 1) \). Using the definition of \( u_\sigma \) we obtain
\[
\mathcal{F}(u_\sigma) = [F(s_0)\sigma^N - \max_{|t| \leq |s_0|} |F(t)|(1 - \sigma^N)] R_0^N \omega_N.
\]  
(9)
For \( \sigma \) close enough to 1, the right-hand side of the last inequality becomes strictly positive; let \( s_0 \) be such a number.

On account of Lemma 3.3, we may choose \( \rho_0 \in (0, 1) \) such that
\[
\rho_0 < C \| u_{s_0} \|^p \quad (\leq A(u_{s_0}))
\]
and
\[
\sup\{\mathcal{F}(u) : A(u) < \rho_0\} < \frac{[F(s_0)\sigma_0^N - \max_{|t| \leq |s_0|} |F(t)|(1 - \sigma_0^N)] R_0^N \omega_N}{2A(u_{s_0})}.
\]  
(10)
In Theorem B we choose \( x_1 = u_{s_0} \) and \( x_0 = 0 \) and observe that the hypotheses (i) and (ii) are satisfied. We define
\[
\bar{\alpha} = \frac{\mathcal{F}(u_{s_0})}{A(u_{s_0})} - \frac{1 + \rho_0}{\sup\{\mathcal{F}(u) : A(u) < \rho_0\}}.
\]  
(11)
Taking into account Lemmas 3.1 and 3.2, all the assumptions of Theorem B are verified.

Thus there exist an open interval \( \Lambda \subset [0, \bar{\alpha}] \) and a number \( \mu > 0 \) such that for each \( \lambda \in \Lambda \), the equation \( E'_{\lambda}(u) = A'(u) - \lambda F'(u) = 0 \) admits at least three solutions in \( W^{1,p}_0(\Omega) \) having \( W^{1,p}_0(\Omega) \)-norms less than \( \mu \). This concludes the proof. □

Remark 4.1. A natural question arises when the interval \( \Lambda \) is obtained in Theorem 2.1: can we estimate it? In order to give such an estimation, let us fix \( s_0, R_0, \) and \( \sigma_0 \) as before. Due to (9) and (10), we have
\[
\frac{\sup\{\mathcal{F}(u) : A(u) < \rho_0\}}{\rho_0} < \frac{\mathcal{F}(u_{s_0})}{2A(u_{s_0})}.
\]
Thus, according to (11) and \( \rho_0 < 1 \), one has \( \bar{\alpha} < \frac{4A(u_{s_0})}{\mathcal{F}(u_{s_0})} \). Using H(a) (a), (b), we have
\[
A(u_{s_0}) \leq c_1 (\text{meas}(\Omega))^{1-1/p} \| u_{s_0} \| + \| u_{s_0} \|^p.
\]
In conclusion, invoking now (8) and (9), we have
\[
\Lambda \subset [0, \bar{\alpha}] \subset \left[ \frac{4c_1 (\text{meas}(\Omega))^{1-1/p} C(s_0, \sigma_0) R_0^{N/p-N-1} \omega_N^{1/p-1} + C(s_0, \sigma_0) R_0^{-p}}{F(s_0)\sigma_0^N - \max_{|t| \leq |s_0|} |F(t)|(1 - \sigma_0^N)} \right],
\]
where
\[
C(s_0, \sigma_0) = \frac{|s_0|(1 - \sigma_0^N)^{1/p}}{1 - \sigma_0}.
\]
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