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Multiple solutions for p-Laplacian type equations

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Abstract

In this paper we establish the existence of three weak solutions of an equation which involves a general elliptic operator in divergence form (in particular, a p-Laplacian operator), while the nonlinearity has a $(p-1)$-sublinear growth at infinity. This result completes some recent papers, where mountain pass type solutions were obtained providing the nonlinear term via a $(p-1)$-superlinear growth at infinity (fulfilling an Ambrosetti–Rabinowitz type condition). In our case, an abstract critical point result is applied, proved by G. Bonanno [G. Bonanno, Some remarks on a three critical points theorem, Nonlinear Analysis 54 (2003) 651–665].

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1. Introduction

Various particular forms of the problem involving elliptic operators in divergence form

$$\begin{cases} -\mathrm{div}(a(x,\nabla u)) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{P}$$

have been studied in the recent years. Here, $\Omega \subset \mathbb{R}^N$ is a bounded open domain, $N \geq 2$, while the nonlinearities $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ fulfill certain structural conditions. The simplest case occurs when $a(x,s) = |s|^{p-2}s$, $p \geq 2$; thus (P) reduces to a problem which involves the usual p-Laplacian operator $\Delta_p(\cdot) = \mathrm{div}(|\nabla(\cdot)|^{p-2}\nabla(\cdot))$.

Recently, De Nápoli and Mariani [4] studied problem (P) when the potential a satisfies a set of assumptions, see H(a) below, which includes the p-Laplacian and also other important cases, such as the generalized prescribed mean

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curvature operator. Duc and Vu [3] extended the result of [4], considering problem (P) in the 'nonuniform' case, for when the potential a fulfills

$$|a(x,\xi)| \le c_0(h_0(x) + h_1(x)|\xi|^{p-1}), \quad \forall (x,\xi) \in \Omega \times \mathbb{R}^N,$$

with $h_0 \in L^{p/(p-1)}(\Omega)$, $h_1 \in L^1_{\mathrm{loc}}(\Omega)$, $c_0 > 0$.

In both papers [3,4], the nonlinear term $f : \Omega \times \mathbb{R} \to \mathbb{R}$ verifies the Ambrosetti–Rabinowitz type condition: defining $F(x,s) = \int_0^s f(x,t)\mathrm{d}t$, there exist $s_0 > 0$ and $\theta > p$ such that

$$0 < \theta F(x,s) \le s f(x,s), \quad \forall x \in \Omega, s \in \mathbb{R}, |s| \ge s_0. \tag{AR}$$

By (AR), one can deduce that

$$|f(x,s)| \ge c|s|^{\theta-1}, \quad \forall x \in \Omega, s \in \mathbb{R}, |s| \ge s_0, \tag{AR$'$}$$

i.e., f is $(p-1)$-*superlinear* at infinity.

The purpose of this paper is to handle the counterpart of the above case, i.e., when f is $(p-1)$-*sublinear* at infinity. For the sake of simplicity, we assume in the sequel that f is autonomous, i.e., $f(x,s) = f(s)$. We consider the condition

(f_1) $\lim_{|s|\to+\infty} \frac{f(s)}{|s|^{p-1}} = 0$.

If $f(s) = \lambda(\arctan s)^2$, with $\lambda \in \mathbb{R}$ fixed (thus f clearly fulfills (f_1)), while $a(x,s) = s$ (thus in (P) there appears the standard Laplacian operator $\Delta u = \mathrm{div}\nabla u$), an easy computation shows that (P) possesses only the zero solution, whenever $|\lambda| < \pi^{-1}c_2^{-2}$, where $c_2 > 0$ is the best Sobolev constant of the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$. Therefore, it is more appropriate to investigate, instead of (P), the following eigenvalue problem:

$$\begin{cases} -\mathrm{div}(a(x,\nabla u)) = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P$_\lambda$}$$

In the next section we will state our main result (Theorem 2.1) which guarantees the existence of at least *three* weak solutions of (P$_\lambda$) for certain $\lambda > 0$. Our result completes in a natural way not only the papers of Duc and Vu [3], and De Nápoli and Mariani [4] (superlinear nonlinearities), but also some earlier works in the sublinear context (where $a(x,s) = s$). For instance, Brézis and Oswald [2] studied problem (P$_\lambda$) when the behaviour of $f(s)/s$ is suitably controlled at infinity, obtaining an existence and uniqueness result via the minimization technique and maximum principle. Lin [5] exploited a sub–super-solution argument, applying the sweeping principle of Serrin in order to obtain existence, uniqueness, and asymptotical properties of the solutions of (P$_\lambda$) when $f(s)$ behaves like s^q $(0 < q < 1)$ with s large.

The proof of our main result (Theorem 2.1) is based on a recent abstract critical point theorem proved by Bonanno [1] which is an extension of the famous result of Ricceri [6,7]. In the next section we give the precise statement of Theorem 2.1, Section 3 contains auxiliary results, while in Section 4 we will give the proof of Theorem 2.1.

2. Main result

In the sequel, let $p > 1$ and $\Omega \subset \mathbb{R}^N$ be a bounded open domain, where $N \ge 2$. Now, we recall the same assumptions as in [4], concerning the potential $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$.

H(a): Let $A : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$, $A = A(x,\xi)$ be a continuous function in $\overline{\Omega} \times \mathbb{R}^N$, with continuous derivative with respect to ξ, $a = DA = A'$, and suppose that the following conditions hold:

(a) $A(x,0) = 0$, $\forall x \in \Omega$.

(b) a satisfies the growth condition

$$|a(x,\xi)| \le c_1(1 + |\xi|^{p-1}), \quad \forall x \in \Omega, \xi \in \mathbb{R}^N \tag{1}$$

for some constant $c_1 > 0$.

(c) A is p-uniformly convex: There exists a constant $k > 0$, such that

$$A\left(x, \frac{\xi+\eta}{2}\right) \le \frac{1}{2}A(x,\xi) + \frac{1}{2}A(x,\eta) - k|\xi - \eta|^p, \quad \forall x \in \Omega, \xi, \eta \in \mathbb{R}^N.$$

(d) A is p-subhomogeneous:

$$0 \le a(x,\xi)\cdot\xi \le pA(x,\xi), \quad \forall x \in \Omega, \xi \in \mathbb{R}^N. \tag{2}$$

(e) A satisfies the ellipticity condition: There exists a constant $C > 0$ such that

$$A(x,\xi) \ge C|\xi|^p, \quad \forall x \in \Omega, \xi \in \mathbb{R}^N. \tag{3}$$

Remark 2.1. Let $p \ge 2$. If $A(x,s) = \frac{1}{p}|s|^p$, then $a(x,s) = |s|^{p-2}s$ and one obtains the usual p-Laplacian. If $A(x,s) = \frac{1}{p}[(1+s^2)^{\frac{p}{2}} - 1]$, then we obtain the generalized mean curvature operator $\mathrm{div}((1+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u)$. For another specific choice of a, see [4].

Let $f : \mathbb{R} \to \mathbb{R}$ be a function which, besides (f_1), satisfies the following conditions:

(f_2) $\lim_{s\to 0} \frac{f(s)}{|s|^{p-1}} = 0$.

(f_3) There exists $s_0 \in \mathbb{R}$ such that $F(s_0) > 0$, where $F(s) = \int_0^s f(t)\mathrm{d}t$.

Our main result is the following:

Theorem 2.1. *Let $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a potential which fulfills the hypothesis* H(a), *and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies* (f_1), (f_2) *and* (f_3). *Then, there exist an open interval $\Lambda \subset (0, +\infty)$ and a constant $\mu > 0$ such that for every $\lambda \in \Lambda$ problem* (P_λ) *has at least three distinct weak solutions in $W_0^{1,p}(\Omega)$, whose $W_0^{1,p}(\Omega)$-norms are less than μ.*

3. Preliminaries

We assume that the assumptions of Theorem 2.1 are verified. The norm of the space $L^p(\Omega)$ will be denoted by $\|\cdot\|_p$. The Sobolev space $W_0^{1,p}(\Omega)$ is endowed with the usual norm $\|u\| = (\int_\Omega |\nabla u|^p \mathrm{d}x)^{1/p}$. Since the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ ($q \in [1, p^*)$) is compact, let $c_q > 0$ be the best Sobolev constant, i.e. $\|u\|_q \le c_q\|u\|$ for every $u \in W_0^{1,p}(\Omega)$, and $c_q = \gamma_q^{-1}$, with $\gamma_q = \inf\{\|u\| : \|u\|_q = 1\}$. Above, p^* denotes the usual Sobolev critical exponent.

We introduce the energy functional $\mathcal{E}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ given by $\mathcal{E}_\lambda(u) = \mathcal{A}(u) - \lambda\mathcal{F}(u)$, where

$$\mathcal{A}(u) = \int_\Omega A(x, \nabla u(x))\mathrm{d}x \quad \text{and} \quad \mathcal{F}(u) = \int_\Omega F(u(x))\mathrm{d}x.$$

It is easy to see that the functional $\mathcal{E}_\lambda$ is of class $\mathcal{C}^1$ and its derivative is given by

$$\langle \mathcal{E}'_\lambda(u), \varphi\rangle = \int_\Omega a(x, \nabla u(x))\nabla\varphi(x)\mathrm{d}x - \lambda\int_\Omega f(u(x))\varphi(x)\mathrm{d}x.$$

Here, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $W_0^{1,p}(\Omega)$ and its dual $W^{-1,p'}(\Omega)$, $1/p + 1/p' = 1$. Moreover, the critical points of the functional $\mathcal{E}_\lambda$ are exactly the weak solutions of problem (P_λ).

Remark 3.1. Due to hypothesis H(a), a simple calculation shows that the functional $\mathcal{A}$ is locally uniformly convex. Moreover, $\mathcal{A}' : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ verifies the (S_+) condition, i.e., for every sequence $\{u_n\} \subset W_0^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ weakly and $\limsup_{n\to\infty}\langle\mathcal{A}'(u_n), u_n - u\rangle \le 0$, we have $u_n \to u$ strongly; see Proposition 2.1 in [4].

Lemma 3.1. *For every $\lambda \in \mathbb{R}$, the functional $\mathcal{E}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ is sequentially weakly lower semicontinuous.*

Proof. The functional $\mathcal{A}$ being locally uniformly convex is weakly lower semicontinuous. On the other hand, condition (f_1) implies the existence of a constant $c > 0$ such that $|f(s)| \le c(1 + |s|^{p-1})$ for every $s \in \mathbb{R}$. Since the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, one can deduce in a standard way that $\mathcal{F}$ is sequentially weakly continuous. □

Lemma 3.2. *For every $\lambda \in \mathbb{R}$, the functional $\mathcal{E}_\lambda$ is coercive and satisfies the Palais–Smale condition.*

Proof. Let us fix $\lambda \in \mathbb{R}$, arbitrary. By (f_1) there exists $\delta = \delta(\lambda)$ such that

$$|f(s)| \leq pCc_p^{-p}(1+|\lambda|)^{-1}|s|^{p-1} \quad \text{for every } |s| \geq \delta.$$

(Here, C is from H(a)(e).) Integrating the above inequality we have

$$|F(s)| \leq Cc_p^{-p}(1+|\lambda|)^{-1}|s|^p + \max_{|t|\leq\delta}|f(t)||s|, \quad \forall s \in \mathbb{R}.$$

Thus, for every $u \in W_0^{1,p}(\Omega)$ we obtain

$$\begin{aligned}\mathcal{E}_\lambda(u) &\geq \mathcal{A}(u) - |\lambda||\mathcal{F}(u)| \\ &\geq C\|u\|^p - C\frac{|\lambda|}{(1+|\lambda|)}\|u\|^p - c_p|\lambda|(\nu(\Omega))^{\frac{1}{p'}}\|u\|\max_{|t|\leq\delta}|f(t)|,\end{aligned}$$

where $\nu(\Omega)$ denotes the Lebesgue measure of Ω. Since $p > 1$, $\mathcal{E}_\lambda(u) \to +\infty$ whenever $\|u\| \to +\infty$. Hence $\mathcal{E}_\lambda$ is coercive.

Now, let $\{u_n\} \subset W_0^{1,p}(\Omega)$ be a sequence such that $\{\mathcal{E}_\lambda(u_n)\}$ is bounded and $\|\mathcal{E}'_\lambda(u_n)\|_{W^{-1,p'}} \to 0$. Since $\mathcal{E}_\lambda$ is coercive, it follows that the sequence $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Up to a subsequence, $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega)$ and $u_n \to u$ strongly in $L^p(\Omega)$. From $\mathcal{E}_\lambda = \mathcal{A} - \lambda\mathcal{F}$ we get

$$\langle \mathcal{A}'(u_n), u_n - u\rangle = \langle \mathcal{E}'_\lambda(u_n), u_n - u\rangle + \lambda\int_\Omega f(u_n(x))(u_n(x)-u(x))\mathrm{d}x. \tag{4}$$

Since $\|\mathcal{E}'_\lambda(u_n)\|_{W^{-1,p'}} \to 0$ and $\{u_n - u\}$ is bounded in $W_0^{1,p}(\Omega)$, by the inequality $|\langle \mathcal{E}'_\lambda(u_n), u_n - u\rangle| \leq \|\mathcal{E}'_\lambda(u_n)\|_{W^{-1,p'}}\|u_n - u\|$ it follows that

$$\langle \mathcal{E}'_\lambda(u_n), u_n - u\rangle \to 0.$$

As before, (f_1) implies the existence of a constant $c > 0$ such that $|f(s)| \leq c(1+|s|^{p-1})$ for every $s \in \mathbb{R}$. Therefore

$$\begin{aligned}\int_\Omega |f(u_n(x))||u_n(x)-u(x)|\mathrm{d}x &\leq c\int_\Omega |u_n(x)-u(x)|\mathrm{d}x + c\int_\Omega |(u_n(x))|^{p-1}|u_n(x)-u(x)|\mathrm{d}x \\ &\leq c((\nu(\Omega))^{\frac{1}{p'}} + \|u_n\|_p^{p-1})\|u_n - u\|_p.\end{aligned}$$

Since $u_n \to u$ strongly in $L^p(\Omega)$, we get

$$\lim_{n\to\infty}\int_\Omega |f(u_n(x))||u_n(x)-u(x)|\mathrm{d}x = 0.$$

In conclusion, relation (4) implies

$$\limsup_{n\to\infty}\langle \mathcal{A}'(u_n), u_n - u\rangle \leq 0.$$

But the operator $\mathcal{A}'$ has the (S_+) property; therefore we have $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$. □

Lemma 3.3. *The following property holds:*

$$\lim_{\rho\to 0^+}\frac{\sup\{\mathcal{F}(u) : \mathcal{A}(u) < \rho\}}{\rho} = 0.$$

Proof. Due to (f_2), for an arbitrary small $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(s)| \leq \varepsilon pc_p^{-p}|s|^{p-1} \quad \text{for every } |s| \leq \delta.$$

Combining the above inequality with

$$|f(s)| \leq c(1+|s|^{p-1}) \quad \text{for every } s \in \mathbb{R},$$

we obtain

$$|F(s)| \leq \varepsilon c_p^{-p}|s|^p + K(\delta)|s|^q \quad \text{for every } s \in \mathbb{R}, \tag{5}$$

where $q \in (p, p^\star)$ is fixed, and $K(\delta) > 0$ does not depend on s. For $\rho > 0$, define the sets

$$S^1_\rho = \{u \in W^{1,p}_0(\Omega) : \mathcal{A}(u) < \rho\}$$

and

$$S^2_\rho = \{u \in W^{1,p}_0(\Omega) : C\|u\|^p < \rho\}.$$

By H(a)(e) it follows that $S^1_\rho \subset S^2_\rho$.

From (5) we obtain

$$\mathcal{F}(u) \leq \varepsilon\|u\|^p + K(\delta)c_q^q\|u\|^q. \tag{6}$$

Since $0 \in S^1_\rho$ (due to H(a)(a)), and $\mathcal{F}(0) = 0$, one has $0 \leq \sup_{u\in S^1_\rho} \mathcal{F}(u)$. On the other hand, if $u \in S^2_\rho$, then $\|u\| \leq C^{-\frac{1}{p}}\rho^{\frac{1}{p}}$, and using (6) we get

$$0 \leq \frac{\sup_{u\in S^1_\rho} \mathcal{F}(u)}{\rho} \leq \frac{\sup_{u\in S^2_\rho} \mathcal{F}(u)}{\rho} \leq \varepsilon C^{-1} + K(\delta)c_q^q C^{-\frac{q}{p}}\rho^{\frac{q}{p}-1}.$$

Because $\varepsilon > 0$ is arbitrary and $\rho \to 0^+$, we get the desired result. □

4. Proof of Theorem 2.1

The main ingredient for the proof of Theorem 2.1 is a recent critical point result due to Bonanno [1] which it is actually a refinement of a result of Ricceri [6,7]. For completeness, we recall the result from [1].

Theorem B (*[1, Theorem 2.1]*). *Let X be a separable and reflexive real Banach space, and let $\mathcal{A}, \mathcal{F} : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals. Assume that there exists $x_0 \in X$ such that $\mathcal{A}(x_0) = \mathcal{F}(x_0) = 0$ and $\mathcal{A}(x) \geq 0$ for every $x \in X$ and that there exists $x_1 \in X$, $\rho > 0$ such that*

(i) $\rho < \mathcal{A}(x_1)$;
(ii) $\sup_{\mathcal{A}(x)<\rho} \mathcal{F}(x) < \rho\frac{\mathcal{F}(x_1)}{\mathcal{A}(x_1)}$.

Further, put

$$\overline{a} = \frac{\zeta\rho}{\rho\frac{\mathcal{F}(x_1)}{\mathcal{A}(x_1)} - \sup\limits_{\mathcal{A}(x)<\rho} \mathcal{F}(x)},$$

with $\zeta > 1$, and assume that the functional $\mathcal{A} - \lambda\mathcal{F}$ is sequentially weakly lower semicontinuous, satisfies the Palais–Smale condition and

(iii) $\lim_{\|x\|\to+\infty}(\mathcal{A}(x) - \lambda\mathcal{F}(x)) = +\infty$, *for every $\lambda \in [0, \overline{a}]$.*

Then, there exist an open interval $\Lambda \subset [0, \overline{a}]$ and a number $\mu > 0$ such that for each $\lambda \in \Lambda$, the equation $\mathcal{A}'(x) - \lambda\mathcal{F}'(x) = 0$ admits at least three solutions in X having norm less than μ.

Proof of Theorem 2.1. Let $s_0 \in \mathbb{R}$ be from (f_3), i.e., $F(s_0) > 0$. Fix an element $x_0 \in \Omega$. Choose $R_0 > 0$ in such a way that

$$\{x \in \mathbb{R}^N : |x - x_0| \leq R_0\} \subseteq \Omega,$$

where $|\cdot|$ denotes the usual euclidean norm in $\mathbb{R}^N$. Let us denote by $B_N(x_0, r)$ the N-dimensional closed euclidean ball with center $x_0 \in \mathbb{R}^N$ and radius $r > 0$.

For $\sigma \in (0, 1)$ define

$$u_\sigma(x) = \begin{cases} 0, & \text{if } x \in \mathbb{R}^N \setminus B_N(x_0, R_0); \\ s_0, & \text{if } x \in B_N(x_0, \sigma R_0); \\ \dfrac{s_0}{R_0(1-\sigma)}(R_0 - |x - x_0|), & \text{if } x \in B_N(x_0, R_0) \setminus B_N(x_0, \sigma R_0). \end{cases} \tag{7}$$

It is clear that $u_\sigma \in W_0^{1,p}(\Omega)$. Moreover, we have

$$|u_\sigma(x)| \le |s_0| \quad \text{for each } x \in \mathbb{R}^N,$$

and

$$\|u_\sigma\|^p = \int_\Omega |\nabla u_\sigma|^p = \frac{|s_0|^p(1-\sigma^N)}{(1-\sigma)^p} R_0^{N-p}\omega_N > 0, \tag{8}$$

where ω_N is the volume of $B_N(0,1)$. Using the definition of u_σ we obtain

$$\mathcal{F}(u_\sigma) \ge [F(s_0)\sigma^N - \max_{|t|\le|s_0|}|F(t)|(1-\sigma^N)]R_0^N\omega_N. \tag{9}$$

For σ close enough to 1, the right-hand side of the last inequality becomes strictly positive; let σ_0 be such a number.

On account of Lemma 3.3, we may choose $\rho_0 \in (0,1)$ such that

$$\rho_0 < C\|u_{\sigma_0}\|^p \quad (\le \mathcal{A}(u_{\sigma_0}))$$

and

$$\frac{\sup\{\mathcal{F}(u) : \mathcal{A}(u) < \rho_0\}}{\rho_0} < \frac{[F(s_0)\sigma_0^N - \max_{|t|\le|s_0|}|F(t)|(1-\sigma_0^N)]R_0^N\omega_N}{2\mathcal{A}(u_{\sigma_0})}. \tag{10}$$

In Theorem B we choose $x_1 = u_{\sigma_0}$ and $x_0 = 0$ and observe that the hypotheses (i) and (ii) are satisfied. We define

$$\overline{a} = \frac{1+\rho_0}{\frac{\mathcal{F}(u_{\sigma_0})}{\mathcal{A}(u_{\sigma_0})} - \frac{\sup\{\mathcal{F}(u):\mathcal{A}(u)<\rho_0\}}{\rho_0}}. \tag{11}$$

Taking into account Lemmas 3.1 and 3.2, all the assumptions of Theorem B are verified.

Thus there exist an open interval $\Lambda \subset [0,\overline{a}]$ and a number $\mu > 0$ such that for each $\lambda \in \Lambda$, the equation $\mathcal{E}'_\lambda(u) = \mathcal{A}'(u) - \lambda\mathcal{F}'(u) = 0$ admits at least three solutions in $W_0^{1,p}(\Omega)$ having $W_0^{1,p}(\Omega)$-norms less than μ. This concludes the proof. □

Remark 4.1. A natural question arises when the interval Λ is obtained in Theorem 2.1: can we estimate it? In order to give such an estimation, let us fix s_0, R_0, and σ_0 as before. Due to (9) and (10), we have

$$\frac{\sup\{\mathcal{F}(u) : \mathcal{A}(u) < \rho_0\}}{\rho_0} < \frac{\mathcal{F}(u_{\sigma_0})}{2\mathcal{A}(u_{\sigma_0})}.$$

Thus, according to (11) and $\rho_0 < 1$, one has $\overline{a} < \frac{4\mathcal{A}(u_{\sigma_0})}{\mathcal{F}(u_{\sigma_0})}$. Using H(a) (a), (b), we have

$$\mathcal{A}(u_{\sigma_0}) \le c_1(\text{meas}(\Omega)^{1-1/p}\|u_{\sigma_0}\| + \|u_{\sigma_0}\|^p).$$

In conclusion, invoking now (8) and (9), we have

$$\Lambda \subset [0,\overline{a}] \subset \left[0, \frac{4c_1(\text{meas}(\Omega)^{1-1/p}C(s_0,\sigma_0)R_0^{N/p-N-1}\omega_N^{1/p-1} + C(s_0,\sigma_0)^pR_0^{-p})}{F(s_0)\sigma_0^N - \max_{|t|\le|s_0|}|F(t)|(1-\sigma_0^N)}\right],$$

where

$$C(s_0,\sigma_0) = \frac{|s_0|(1-\sigma_0^N)^{1/p}}{1-\sigma_0}.$$

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