Available online at www.sciencedirect.com
ScienceDirect
J. Differential Equations 245 (2008) 3849–3868
Journal of Differential Equations
www.elsevier.com/locate/jde

Detection of arbitrarily many solutions for perturbed elliptic problems involving oscillatory terms ☆

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Received 21 December 2007; revised 11 March 2008
Available online 16 June 2008

To Professor Vicenţiu Rădulescu, with esteem, on his fiftieth birthday

Abstract

We propose a direct approach for detecting arbitrarily many solutions for perturbed elliptic problems involving oscillatory terms. Although the method works in various frameworks, we illustrate it on the problem

$$\begin{cases} -\Delta u + u = Q(x)\big[f(u) + \varepsilon g(u)\big], & x \in \mathbb{R}^N,\ N \geqslant 2, \\ u \geqslant 0, \\ u(x) \to 0 \quad \text{as } |x| \to \infty, \end{cases} \tag{P_ε}$$

where $Q : \mathbb{R}^N \to \mathbb{R}$ is a radial, positive potential, $f : [0, \infty) \to \mathbb{R}$ is a continuous nonlinearity which oscillates near the origin or at infinity and $g : [0, \infty) \to \mathbb{R}$ is any arbitrarily continuous function with $g(0) = 0$. Our aim is to prove that: (a) the unperturbed problem (P_0), i.e. $\varepsilon = 0$ in (P_ε), has infinitely many distinct solutions; (b) the number of distinct solutions for (P_ε) becomes greater and greater whenever $|\varepsilon|$ is smaller and smaller. In fact, our method surprisingly shows that (a) and (b) are *equivalent* in the sense that they are deducible from each other. Various properties of the solutions are also described in L^∞- and H^1-norms. Our method is variational and a specific construction enforces the use of the principle of symmetric criticality for *non*-smooth Szulkin-type functionals.

Keywords: Perturbed elliptic problem; Arbitrarily many solutions; Szulkin-type functional; Symmetric criticality

☆ Research for this article was supported by the Grant PN II, ID_527/2007.
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0022-0396/$ – see front matter
doi:10.1016/j.jde.2008.05.014

1. Introduction and main results

Having infinitely many solutions for a given equation, after a 'small' perturbation of it, one expects to find still many solutions for the perturbed equation; moreover, once the perturbation tends to zero, the number of solutions for the perturbed equation should tend to infinity. Such phenomenon is well known in the case of the equation $\sin s = c$ with $c \in (-1, 1)$ fixed, and its perturbation $\sin s = c + \varepsilon s$, $s \in \mathbb{R}$; the perturbed equation has more and more solutions as $|\varepsilon|$ decreases to 0. To the best of our knowledge, this natural phenomenon has been first exploited in an abstract framework by Krasnosel'skii [6]. More precisely, by using topological methods, Krasnosel'skii asserts the existence of more and more critical points of an even C^1-class functional perturbed by a non-even term tending to zero, the critical points of the perturbed functional being the solutions for the studied equation.

Later on, Krasnosel'skii's idea served for further developments; in order to describe them, we consider the equation

$$-\Delta u + V(x)u = f(x, u) + \varepsilon g(x, u) \quad \text{in } \Omega, \tag{E_ε}$$

where $\Omega \subseteq \mathbb{R}^N$ is an open domain, $V : \Omega \to \mathbb{R}$ is a measurable function, while $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions. Subject to certain boundary condition, we assume the unperturbed equation

$$-\Delta u + V(x)u = f(x, u) \quad \text{in } \Omega, \tag{E_0}$$

has *infinitely many* distinct solutions. Then, the main question is:

(q) *Fixing $k \in \mathbb{N}$, can one find a number $\varepsilon_k > 0$ such that the perturbed equation* (E_ε) *has at least k distinct solutions whenever $\varepsilon \in [-\varepsilon_k, \varepsilon_k]$?*

Two different classes of results are available in the literature answering affirmatively question (q), both for *bounded* domains subjected to zero Dirichlet boundary condition, and $V \equiv 0$:

A. *Perturbation of symmetric problems.* Assume $f(x, s) = -f(x, -s)$ for every $(x, s) \in \Omega \times \mathbb{R}$. It is well known that if the energy functional has the Mountain Pass Geometry, problem (E_0) has infinitely many solutions, due to the symmetric version of the Mountain Pass theorem, see Ambrosetti and Rabinowitz [1]. Furthermore, question (q) was fully answered by Li and Liu [9] for arbitrarily continuous nonlinearity g, following the topological approach developed by Degiovanni and Lancelotti [3] and Degiovanni and Rădulescu [4].
B. *Perturbation of oscillatory problems.* Assume $f(x, \cdot)$ oscillates near the origin or at infinity, uniformly with respect to $x \in \Omega$. Special kinds of oscillations produce infinitely many solutions for (E_0), as shown by Omari and Zanolin [10], and Saint Raymond [13]. Concerning the perturbed problem, Anello and Cordaro [2] answered question (q), by using the abstract variational principle of Ricceri [12].

The main purpose of the present paper is to propose a third, *direct method* for answering question (q) whenever the nonlinear term $f(x, \cdot)$ belongs to a wide class of oscillatory functions. Although our method works in various frameworks (for instance, the domain Ω is bounded, and the studied problem is subject to Dirichlet, Neumann or more general boundary conditions), we illustrate this new approach by treating the problem

$$\begin{cases} -\Delta u + u = Q(x)\big[f(u) + \varepsilon g(u)\big], & x \in \mathbb{R}^N,\ N \geqslant 2, \\ u \geqslant 0, \\ u(x) \to 0 \quad \text{as } |x| \to \infty, \end{cases} \tag{P$_\varepsilon$}$$

where $f:[0,\infty) \to \mathbb{R}$ is a continuous nonlinearity which oscillates near the origin or at infinity, see hypotheses (f_1^0) and (f_2^0), or (f_1^∞) and (f_2^∞), respectively. On the nonlinear term $g:[0,\infty) \to \mathbb{R}$ we assume only its *continuity* and that $g(0)=0$.

Throughout the paper we assume

(Q) $Q:\mathbb{R}^N \to \mathbb{R}$ is a positive, continuous, radially symmetric potential such that $Q \in L^p(\mathbb{R}^N)$ for every $p \in [1,2]$.

In order to formulate our results, we recall some notations. The Hilbert space $H^1(\mathbb{R}^N)$ is endowed with its usual inner product and norm,

$$\langle u, v\rangle_{H^1} = \int_{\mathbb{R}^N} (\nabla u \nabla v + uv)\,dx, \qquad \|u\|_{H^1} = \sqrt{\langle u,u\rangle_{H^1}}, \quad u, v \in H^1\big(\mathbb{R}^N\big).$$

The space $L^q(\mathbb{R}^N)$ is endowed with its usual norm $\|\cdot\|_{L^q}$, $q \in [1,\infty]$.

Let $f \in C([0,\infty),\mathbb{R})$ and $F(s) = \int_0^s f(t)\,dt$, $s \geqslant 0$. We assume:

(f_1^0) $-\infty < \liminf_{s\to 0^+} \frac{F(s)}{s^2} \leqslant \limsup_{s\to 0^+} \frac{F(s)}{s^2} = +\infty$.
(f_2^0) There exists a sequence $\{s_i\}_i \subset (0,\infty)$ converging to 0 such that $f(s_i) < 0$ for every $i \in \mathbb{N}$.

Remark 1.1. (a) Hypotheses (f_1^0) and (f_2^0) imply an oscillatory behaviour of f near the origin.

(b) Let $\alpha, \beta, \gamma \in \mathbb{R}$ such that $0 < \alpha < 1 < \alpha + \beta$, and $\gamma \in (0,1)$. Then, the function $f:[0,\infty) \to \mathbb{R}$ defined by $f(0)=0$ and $f(s) = s^\alpha(\gamma + \sin s^{-\beta})$, $s > 0$, verifies (f_1^0) and (f_2^0), respectively.

The first result deals with the unperturbed problem (P_0):

Theorem 1.1. *Assume* (Q) *and let $f \in C([0,\infty),\mathbb{R})$ satisfying (f_1^0) and (f_2^0). Then there exists a sequence $\{u_i^0\}_i \subset H^1(\mathbb{R}^N)$ of distinct, radially symmetric weak solutions of* (P_0) *such that*

$$\lim_{i\to\infty} \big\|u_i^0\big\|_{L^\infty} = \lim_{i\to\infty} \big\|u_i^0\big\|_{H^1} = 0. \tag{1}$$

Keeping in mind Theorem 1.1, we expect an affirmative answer to question (q) for the perturbed problem (P_ε). This is indeed the case:

Theorem 1.2. *Assume* (Q), *let $f \in C([0,\infty),\mathbb{R})$ satisfying (f_1^0) and (f_2^0), and let $g \in C([0,\infty),\mathbb{R})$ with $g(0)=0$. Then, for every $k \in \mathbb{N}$, there exists $\varepsilon_k^0 > 0$ such that* (P_ε) *has at* *least k distinct, radially symmetric weak solutions in* $H^1(\mathbb{R}^N)$ *whenever* $\varepsilon \in [-\varepsilon_k^0, \varepsilon_k^0]$. *Moreover, if the* (*first k*) *solutions are denoted by* $u_{i,\varepsilon}^0 \in H^1(\mathbb{R}^N)$, $i = \overline{1,k}$, *then*

$$\|u_{i,\varepsilon}^0\|_{L^\infty} < \frac{1}{i} \quad \text{and} \quad \|u_{i,\varepsilon}^0\|_{H^1} < \frac{1}{i} \quad \text{for any } i = \overline{1,k};\ \varepsilon \in \left[-\varepsilon_k^0, \varepsilon_k^0\right]. \tag{2}$$

Remark 1.2. Note that (1) and (2) are in a perfect concordance. Furthermore, an unexpected situation occurs: the perturbed and unperturbed problems are *equivalent* in the sense that they are deducible from each other. Clearly, the perturbed problem contains the unperturbed problem by choosing $g \equiv 0$. Conversely, exploiting the behaviour of certain sequences which appear in the proof of Theorem 1.1, we are able to answer affirmatively question (q) for problem (P_ε); this construction represents actually the core of our method. For details, see Section 3.

In the sequel, we will state the counterparts of Theorems 1.1 and 1.2 whenever f oscillates at infinity. We assume:

(f_1^∞) $-\infty < \liminf_{s\to\infty} \frac{F(s)}{s^2} \leqslant \limsup_{s\to\infty} \frac{F(s)}{s^2} = +\infty$.
(f_2^∞) There exists a sequence $\{s_i\}_i \subset (0,\infty)$ converging to $+\infty$ such that $f(s_i) < 0$ for every $i \in \mathbb{N}$.

Remark 1.3. (a) Hypotheses (f_1^∞) and (f_2^∞) imply an oscillatory behaviour of f at infinity.

(b) Let $\alpha, \beta, \gamma \in \mathbb{R}$ such that $1 < \alpha$, $|\alpha - \beta| < 1$, and $\gamma \in (0,1)$. Then, the function $f : [0,\infty) \to \mathbb{R}$ defined by $f(s) = s^\alpha(\gamma + \sin s^\beta)$ verifies the hypotheses (f_1^∞) and (f_2^∞), respectively.

Concerning problem (P_0), we have the counterpart of Theorem 1.1:

Theorem 1.3. *Assume* (Q) *and let* $f \in C([0,\infty),\mathbb{R})$ *satisfying* (f_1^∞), (f_2^∞) *and* $f(0) = 0$. *Then there exists a sequence* $\{u_i^\infty\}_i \subset H^1(\mathbb{R}^N)$ *of radially symmetric weak solutions of* (P_0) *such that*

$$\lim_{i\to\infty} \|u_i^\infty\|_{L^\infty} = \infty. \tag{3}$$

Remark 1.4. Note that beside of (f_1^∞) and (f_2^∞), no further growth condition is assumed on the nonlinear term at infinity. Actually, this is the reason why we are not able to give H^1-estimates for the solutions obtained in Theorem 1.3. However, if we assume that f has a *half-subcritical* growth at infinity, i.e., there exist $q \in (1, 2^*/2)$ and $c > 0$ such that

$$|f(s)| \leqslant c\left(1 + s^{q-1}\right) \quad \text{for all } s \in [0,\infty), \tag{4}$$

then we have

$$\lim_{i\to\infty} \|u_i^\infty\|_{H^1} = \infty, \tag{3'}$$

see Section 4. Here, the number 2^* is the usual critical exponent. Let us observe that relation (4) and the right side of (f_1^∞) imply $2 < q$. Thus, (3′) is possible for the lower dimensions $N = 2, 3$, since $2 < 2^*/2$ must hold. Another way to guarantee (3′) is to complete hypothesis (Q) by allowing for instance $Q \in L^\infty(\mathbb{R}^N)$ and (4) with $q \in (2, 2^*)$.

Throughout Theorem 1.3, another affirmative answer to (q) can be done:

Theorem 1.4. *Assume* (Q), *let* $f \in C([0,\infty),\mathbb{R})$ *satisfying* (f_1^∞), (f_2^∞) *with* $f(0)=0$, *and let* $g \in C([0,\infty),\mathbb{R})$ *with* $g(0)=0$. *Then, for every* $k \in \mathbb{N}$, *there exists* $\varepsilon_k^\infty > 0$ *such that* (P_ε) *has at least k distinct, radially symmetric weak solutions in* $H^1(\mathbb{R}^N)$ *whenever* $\varepsilon \in [-\varepsilon_k^\infty, \varepsilon_k^\infty]$. *Moreover, if the* (*first k*) *solutions are denoted by* $u_{i,\varepsilon}^\infty \in H^1(\mathbb{R}^N)$, $i=\overline{1,k}$, *then*

$$\left\|u_{i,\varepsilon}^\infty\right\|_{L^\infty} > i-1 \quad \textit{for any } i=\overline{1,k};\ \varepsilon \in \left[-\varepsilon_k^\infty, \varepsilon_k^\infty\right]. \tag{5}$$

Remark 1.5. Relations (3) and (5) are also in concordance. Moreover, if both functions f and g verify (4) with $q \in (2, 2^*/2)$, then beside of (5), we also have

$$\left\|u_{i,\varepsilon}^\infty\right\|_{H^1} > i-1 \quad \text{for any } i=\overline{1,k};\ \varepsilon \in \left[-\varepsilon_k^\infty, \varepsilon_k^\infty\right]. \tag{5$'$}$$

For details, see Section 4.

As we already pointed out, the method developed in the present paper is applicable in more general settings; not only the type of the domain Ω can vary with various boundary conditions, but also equations involving the p-Laplacian can be considered. We emphasize that existence of infinitely many solutions for elliptic problems in $\mathbb{R}^N$ involving the p-Laplacian and an oscillatory term has been already studied by Kristály [7] and Kristály, Moroşanu and Tersian [8]. However, in those papers the assumption $p > N \geqslant 2$ was essential, due to a Morrey-type embedding, and only the 'unperturbed' case was considered. Consequently, the unperturbed problem (P_0) in the present paper may be considered as a natural completion of [7] and [8] from the point of view of the parameter p and the space dimension N. Finally, we mention that elliptic problems involving decaying or unbounded terms can be also treated by this method, exploiting recent embedding results of Su, Wang and Willem [14].

The paper is divided as follows. First, we prove a key result which is based on the principle of symmetric criticality for (non-differentiable) Szulkin-type functionals. We emphasize that although our problems (P_0) and (P_ε) are smooth ones, we are forced to use a typically non-smooth principle; this is due to a specific construction performed in Section 2. Then, in Section 3 we prove Theorems 1.1 and 1.2, while in Section 4 we are dealing with Theorems 1.3 and 1.4. Finally, in Appendix A, we recall the principle of symmetric criticality for Szulkin-type functionals, following the paper of Kobayashi and Ôtani [5].

2. Preliminaries and a key result

Due to the fact that problems (P_0) and (P_ε) will be treated simultaneously, in this section we consider the generic problem

$$\begin{cases} -\Delta u + u = Q(x)h(u), & x \in \mathbb{R}^N,\ N \geqslant 2, \\ u(x) \to 0 \quad \text{as } |x| \to \infty, \end{cases} \tag{P_h}$$

and beside of hypothesis (Q), we assume that

(h_1) $h:[0,\infty) \to \mathbb{R}$ is a continuous, bounded function such that $h(0)=0$;
(h_2) there are $0 < a < b$ such that $h(s) \leqslant 0$ for all $s \in [a,b]$.

Due to (h_1), we may extend h continuously to the whole $\mathbb{R}$, putting $h(s) = 0$ for all $s \leqslant 0$.

We introduce the energy functional $E_h : H^1(\mathbb{R}^N) \to \mathbb{R}$ associated with problem (P_h), defined by

$$E_h(u) = \frac{1}{2}\|u\|_{H^1}^2 - \int_{\mathbb{R}^N} Q(x)H\big(u(x)\big)\,dx, \quad u \in H^1\big(\mathbb{R}^N\big),$$

where $H(s) = \int_0^s h(t)\,dt$, $s \in \mathbb{R}$. One can easily show that E_h is well defined; indeed, by the mean value theorem, we have

$$\int_{\mathbb{R}^N} Q(x)\big|H\big(u(x)\big)\big|\,dx \leqslant M_h\|Q\|_{L^2}\|u\|_{L^2} < \infty, \quad u \in H^1\big(\mathbb{R}^N\big), \tag{6}$$

where $M_h = \sup_{s\in\mathbb{R}} |h(s)|$. Moreover, standard arguments show that E_h is of class C^1 on $H^1(\mathbb{R}^N)$.

Now, we denote by $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ the radial functions in $H^1(\mathbb{R}^N)$, and let

$$R_h = E_h|_{H^1_{\mathrm{rad}}(\mathbb{R}^N)},$$

i.e., the restriction of E_h to $H^1_{\mathrm{rad}}(\mathbb{R}^N)$. Finally, considering the number $b \in \mathbb{R}$ from (h_2), we introduce

$$W^b = \big\{u \in H^1\big(\mathbb{R}^N\big)\colon \|u\|_{L^\infty} \leqslant b\big\} \quad \text{and} \quad W^b_{\mathrm{rad}} = W^b \cap H^1_{\mathrm{rad}}\big(\mathbb{R}^N\big).$$

The main result of this section can be stated as follows.

Theorem 2.1. *Assume that* (h_1), (h_2) *and* (Q) *hold. Then*

(i) *the functional R_h is bounded from below on W^b_{rad} and its infimum is attained at $u_h \in W^b_{\mathrm{rad}}$;*
(ii) *$u_h(x) \in [0, a]$ for a.e. $x \in \mathbb{R}^N$;*
(iii) *u_h is a weak solution of* (P_h).

Proof. (i) Actually, R_h is bounded from below on the whole $H^1_{\mathrm{rad}}(\mathbb{R}^N)$. Indeed, due to (6), for all $u \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$ we have

$$\begin{aligned} R_h(u) &= \frac{1}{2}\|u\|_{H^1}^2 - \int_{\mathbb{R}^N} Q(x)H(u)\,dx \geqslant \frac{1}{2}\|u\|_{H^1}^2 - M_h\|Q\|_{L^2}\|u\|_{H^1} \\ &\geqslant -\frac{1}{2}M_h^2\|Q\|_{L^2}^2. \end{aligned}$$

Now, we prove that R_h attains its infimum on W^b_{rad}. Note that W^b_{rad} is convex and closed in $H^1_{\mathrm{rad}}(\mathbb{R}^N)$, thus weakly closed. Due to the boundedness from below of R_h on W^b_{rad}, it is enough to prove that R_h is sequentially weakly lower semicontinuous. The latter fact follows at once if we prove that $u \mapsto \int_{\mathbb{R}^N} Q(x)H(u)\,dx$, $u \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$, is sequentially weakly continuous. We argue by contradiction; let $\{u_i\}_i \subset H^1_{\mathrm{rad}}(\mathbb{R}^N)$ be a sequence which converges weakly to $u \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$ but, up to a subsequence, one can find a number $\varepsilon_0 > 0$ such that

$$0 < \varepsilon_0 \leqslant \left| \int_{\mathbb{R}^N} Q(x)H(u_i)\,dx - \int_{\mathbb{R}^N} Q(x)H(u)\,dx \right| \quad \text{for all } i \in \mathbb{N},$$

and u_i converges strongly to u in $L^q(\mathbb{R}^N)$, for some $q \in (2, 2^*)$. Here, we employed the fact that $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ is compactly embedded into $L^q(\mathbb{R}^N)$ for all $q \in (2, 2^*)$. Using the mean value theorem and Hölder inequality, from the above inequality we deduce that

$$0 < \varepsilon_0 \leqslant M_h \int_{\mathbb{R}^N} Q(x)|u_i - u| \leqslant M_h \|Q\|_{L^{q/(q-1)}} \|u_i - u\|_{L^q}.$$

But the right-hand side tends to 0 as $i \to \infty$, contradicting $\varepsilon_0 > 0$. This proves (i); let $u_h \in W^b_{\mathrm{rad}}$ be a minimum point of R_h over W^b_{rad}.

(ii) Let $A = \{x \in \mathbb{R}^N\colon u_h(x) \notin [0, a]\}$ and suppose that $\mathrm{meas}(A) > 0$. Define the function $\gamma : \mathbb{R} \to \mathbb{R}$ by $\gamma(s) = \min(s_+, a)$, where $s_+ = \max(s, 0)$. Now, set $w = \gamma \circ u_h$. Since γ is a Lipschitz function and $\gamma(0) = 0$, the theorem of Marcus and Mizel [11] shows that $w \in H^1(\mathbb{R}^N)$. In addition, w is radial, since $u_h \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$. Thus $w \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$. Moreover, by definition, $0 \leqslant w(x) \leqslant a$ for a.e. $\mathbb{R}^N$.

We introduce the sets

$$A_1 = \{x \in A\colon u_h(x) < 0\} \quad \text{and} \quad A_2 = \{x \in A\colon u_h(x) > a\}.$$

Thus, $A = A_1 \cup A_2$, and we have that $w(x) = u_h(x)$ for all $x \in \mathbb{R}^N \setminus A$, $w(x) = 0$ for all $x \in A_1$, and $w(x) = a$ for all $x \in A_2$. Moreover, we have

$$\begin{aligned} R_h(w) - R_h(u_h) &= \frac{1}{2}\left[\|w\|^2_{H^1} - \|u_h\|^2_{H^1}\right] - \int_{\mathbb{R}^N} Q(x)\left[H(w) - H(u_h)\right] \\ &= -\frac{1}{2}\int_A |\nabla u_h|^2 + \frac{1}{2}\int_A \left[w^2 - u_h^2\right] - \int_A Q(x)\left[H(w) - H(u_h)\right]. \end{aligned}$$

Note that

$$\int_A \left[w^2 - u_h^2\right] = -\int_{A_1} u_h^2 + \int_{A_2} \left[a^2 - u_h^2\right] \leqslant 0.$$

Due to the fact that $h(s) = 0$ for all $s \leqslant 0$, one has

$$\int_{A_1} Q(x)\left[H(w) - H(u_h)\right] = 0.$$

By the mean value theorem, for a.e. $x \in A_2$, there exists $\theta(x) \in [a, u_h(x)] \subseteq [a, b]$ such that

$$H\big(w(x)\big) - H\big(u_h(x)\big) = H(a) - H\big(u_h(x)\big) = h\big(\theta(x)\big)\big(a - u_h(x)\big).$$

Thus, on account of (h_2), one has

$$\int_{A_2} Q(x)\big[H(w) - H(u_h)\big] \geqslant 0.$$

Consequently, every term of the expression $R_h(w) - R_h(u_h)$ is non-positive. On the other hand, since $w \in W^b_{\mathrm{rad}}$, then $R_h(w) \geqslant R_h(u_h) = \inf_{W^b_{\mathrm{rad}}} R_h$. So, every term in $R_h(w) - R_h(u_h)$ should be zero. In particular,

$$\int_{A_1} u_h^2 = \int_{A_2} \big[a^2 - u_h^2\big] = 0,$$

which imply that $\mathrm{meas}(A)$ should be 0, contradicting our assumption.

(iii) We divide this part into two steps.

Step 1. $E'_h(u_h)(w - u_h) \geqslant 0$ *for every* $w \in W^b$.

Standard argument shows that W^b is closed and convex in $H^1(\mathbb{R}^N)$. Let ζ_{W^b} be the indicator function of the set W^b (i.e., $\zeta_{W^b}(u) = 0$ if $u \in W^b$, and $\zeta_{W^b}(u) = +\infty$, otherwise). Since E_h is of class C^1 on $H^1(\mathbb{R}^N)$, and ζ_{W^b} is convex, lower semicontinuous and proper (i.e., $\not\equiv +\infty$), we may define the Szulkin-type functional $I : H^1(\mathbb{R}^N) \to \mathbb{R} \cup \{+\infty\}$ by $I = E_h + \zeta_{W^b}$, see Appendix A. Since $W^b_{\mathrm{rad}} = W^b \cap H^1_{\mathrm{rad}}(\mathbb{R}^N)$, the restriction of ζ_{W^b} to $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ is precisely the indicator function $\zeta_{W^b_{\mathrm{rad}}}$ of the set W^b_{rad}. Recall that u_h is a local minimum point of R_h relative to W^b_{rad} (see (i)), thus a local minimum point of the Szulkin-type functional $\tilde{I} : H^1_{\mathrm{rad}}(\mathbb{R}^N) \to \mathbb{R} \cup \{+\infty\}$, defined by $\tilde{I} = R_h + \zeta_{W^b_{\mathrm{rad}}}$. Due to Proposition A.1 (see Appendix A), u_h is a critical point of $\tilde{I}$, i.e.,

$$0 \in R'_h(u_h) + \partial\zeta_{W^b_{\mathrm{rad}}}(u_h) \quad \text{in } \big(H^1_{\mathrm{rad}}\big(\mathbb{R}^N\big)\big)^*. \tag{7}$$

On the other hand, we introduce the action of the orthogonal group $G = O(N)$ on $H^1(\mathbb{R}^N)$ by

$$(gu)(x) = u\big(g^{-1}x\big) \quad \text{for all } g \in O(N),\ u \in H^1\big(\mathbb{R}^N\big),\ x \in \mathbb{R}^N.$$

Clearly, this action is linear and continuous on $H^1(\mathbb{R}^N)$. Since the potential $Q : \mathbb{R}^N \to \mathbb{R}$ is radial, one can easily check that the functional E_h is $O(N)$-invariant. Moreover, due to the fact that the set W^b is $O(N)$-invariant, the functional ζ_{W^b} is $O(N)$-invariant as well. The set $H^1_{\mathrm{rad}}(\mathbb{R}^N)$ is exactly the subspace of $O(N)$-symmetric points of $H^1(\mathbb{R}^N)$. Therefore, on account of (7) and Theorem A.1 from Appendix A, we obtain

$$0 \in E'_h(u_h) + \partial\zeta_{W^b}(u_h) \quad \text{in } \big(H^1\big(\mathbb{R}^N\big)\big)^*.$$

Consequently, for every $w \in H^1(\mathbb{R}^N)$, we have

$$E_h'(u_h)(w-u_h)+\zeta_{W^b}(w)-\zeta_{W^b}(u_h)\geqslant 0,$$

which implies our claim.

Step 2. (Proof concluded) u_h *is a weak solution of* (P_h).

First of all, by a Strauss-type inequality (see for instance Willem [16, p. 76]), we have that $u_h(x)\to 0$ as $|x|\to\infty$. It remains to prove that

$$\int_{\mathbb{R}^N}\nabla u_h\nabla v+\int_{\mathbb{R}^N}u_h v-\int_{\mathbb{R}^N}Q(x)h(u_h)v=0 \quad \text{for all } v\in H^1\big(\mathbb{R}^N\big). \tag{8}$$

By Step 1, we have

$$\int_{\mathbb{R}^N}\nabla u_h\nabla(w-u_h)+\int_{\mathbb{R}^N}u_h(w-u_h)-\int_{\mathbb{R}^N}Q(x)h(u_h)(w-u_h)\geqslant 0, \quad \forall w\in W^b.$$

Let us define the function $\gamma(s)=\operatorname{sgn}(s)\min(|s|,b)$, and fix $\varepsilon>0$ and $v\in H^1(\mathbb{R}^N)$ arbitrarily. Since γ is Lipschitz continuous and $\gamma(0)=0$, the element $w_\gamma=\gamma\circ(u_h+\varepsilon v)$ belongs to $H^1(\mathbb{R}^N)$, see Marcus and Mizel [11]. The explicit expression of the truncation function w_γ is

$$w_\gamma(x)=\begin{cases}-b, & \text{if } x\in\{u_h+\varepsilon v<-b\},\\ u_h(x)+\varepsilon v(x), & \text{if } x\in\{-b\leqslant u_h+\varepsilon v<b\},\\ b, & \text{if } x\in\{b\leqslant u_h+\varepsilon v\}.\end{cases}$$

Therefore, $w_\gamma\in W^b$. Taking $w=w_\gamma$ as a test function in the previous inequality, we obtain

$$\begin{aligned}0\leqslant{}&-\int_{\{u_h+\varepsilon v<-b\}}\big[|\nabla u_h|^2+u_h(b+u_h)-Q(x)h(u_h)(b+u_h)\big]\\&+\varepsilon\int_{\{-b\leqslant u_h+\varepsilon v<b\}}\big[\nabla u_h\nabla v+u_h v-Q(x)h(u_h)v\big]\\&-\int_{\{b\leqslant u_h+\varepsilon v\}}\big[|\nabla u_h|^2-u_h(b-u_h)+Q(x)h(u_h)(b-u_h)\big].\end{aligned}$$

After a suitable rearrangement of the terms in this inequality, we obtain that

$$\begin{aligned}0\leqslant{}&\varepsilon\int_{\mathbb{R}^N}\nabla u_h\nabla v+\varepsilon\int_{\mathbb{R}^N}u_h v-\varepsilon\int_{\mathbb{R}^N}Q(x)h(u_h)v\\&-\int_{\{u_h+\varepsilon v<-b\}}|\nabla u_h|^2-\int_{\{b\leqslant u_h+\varepsilon v\}}|\nabla u_h|^2\end{aligned}$$

$$+ \int\limits_{\{u_h+\varepsilon v<-b\}} \big[Q(x)h(u_h) - u_h\big](b + u_h + \varepsilon v)$$
$$+ \int\limits_{\{b\leqslant u_h+\varepsilon v\}} \big[Q(x)h(u_h) - u_h\big](-b + u_h + \varepsilon v)$$
$$- \varepsilon \int\limits_{\{u_h+\varepsilon v<-b\}} \nabla u_h \nabla v - \varepsilon \int\limits_{\{b\leqslant u_h+\varepsilon v\}} \nabla u_h \nabla v.$$

Recalling the notation $M_h = \sup_{s\in\mathbb{R}} |h(s)| < \infty$, and taking into account that $u_h(x) \in [0, a] \subset [-b, b]$ for a.e. $x \in \mathbb{R}^N$, we have

$$\int\limits_{\{u_h+\varepsilon v<-b\}} \big[Q(x)h(u_h) - u_h\big](b + u_h + \varepsilon v) \leqslant -\varepsilon \int\limits_{\{u_h+\varepsilon v<-b\}} \big[M_h Q(x) + u_h(x)\big]v(x)\,dx$$

and

$$\int\limits_{\{b\leqslant u_h+\varepsilon v\}} \big[Q(x)h(u_h) - u_h\big](-b + u_h + \varepsilon v) \leqslant \varepsilon M_h \int\limits_{\{b\leqslant u_h+\varepsilon v\}} Q(x)v(x)\,dx.$$

Using the above estimates and dividing by $\varepsilon > 0$, we obtain

$$0 \leqslant \int\limits_{\mathbb{R}^N} \nabla u_h \nabla v + \int\limits_{\mathbb{R}^N} u_h v - \int\limits_{\mathbb{R}^N} Q(x)h(u_h)v - \int\limits_{\{u_h+\varepsilon v<-b\}} \big[\nabla u_h \nabla v + u_h v + M_h Q(x)v\big]$$
$$- \int\limits_{\{b\leqslant u_h+\varepsilon v\}} \big[\nabla u_h \nabla v - M_h Q(x)v\big].$$

Now, letting $\varepsilon \to 0^+$, and taking into account (ii), that is, $0 \leqslant u_h(x) \leqslant a$ for a.e. $x \in \mathbb{R}^N$, we have

$$\text{meas}\big(\{u_h + \varepsilon v < -b\}\big) \to 0 \quad \text{and} \quad \text{meas}\big(\{b \leqslant u_h + \varepsilon v\}\big) \to 0,$$

respectively. Consequently, the above inequality reduces to

$$0 \leqslant \int\limits_{\mathbb{R}^N} \nabla u_h \nabla v + \int\limits_{\mathbb{R}^N} u_h v - \int\limits_{\mathbb{R}^N} Q(x)h(u_h)v.$$

Putting $(-v)$ instead of v, we arrive to (8), i.e., u_h is a weak solution of (P_h). This ends the proof. □

We conclude this section by constructing a special function which will be useful in the proof of our theorems. In the sequel, B_c denotes the closed N-dimensional ball with radius $c > 0$ and center 0.

Fix $\rho > 0$. For any $s > 0$ we introduce the function

$$w_s(x) = \begin{cases} 0, & \text{if } x \in \mathbb{R}^N \setminus B_\rho, \\ s, & \text{if } x \in B_{\rho/2}, \\ \frac{2s}{\rho}(\rho - |x|), & \text{if } x \in B_\rho \setminus B_{\rho/2}. \end{cases} \tag{9}$$

It is clear that $w_s \in H^1_{\text{rad}}(\mathbb{R}^N)$ and

$$\|w_s\|^2_{H^1} \leqslant K(\rho)s^2, \tag{10}$$

where $K(\rho) = (4 + \rho^2)\rho^{N-2}\omega_N$, and ω_N denotes the volume of the N-dimensional unit ball.

3. Proof of Theorems 1.1 and 1.2

Due to (f_2^0) and to the continuity of f and g, we may fix the positive sequences $\{a_i\}_i$, $\{b_i\}_i$, and $\{\varepsilon_i\}_i$ such that $\lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i = 0$, and for all $i \in \mathbb{N}$,

$$b_{i+1} < a_i < s_i < b_i < 1; \tag{11}$$

$$f(s) + \varepsilon g(s) \leqslant 0 \quad \text{for all } s \in [a_i, b_i] \text{ and } \varepsilon \in [-\varepsilon_i, \varepsilon_i]. \tag{12}$$

For every $i \in \mathbb{N}$, we define the truncation functions $f_i, g_i : [0, \infty) \to \mathbb{R}$ by

$$f_i(s) = f\big(\min(s, b_i)\big) \quad \text{and} \quad g_i(s) = g\big(\min(s, b_i)\big). \tag{13}$$

By (f_1^0) and (f_2^0) we have $f(0) = 0$. Since $f_i(0) = g_i(0) = 0$, we may extend continuously the functions f_i and g_i to the whole real line, taking 0 for negative arguments. For every $s \in \mathbb{R}$ and $i \in \mathbb{N}$, let $F_i(s) = \int_0^s f_i(t)\,dt$ and $G_i(s) = \int_0^s g_i(t)\,dt$.

For every $i \in \mathbb{N}$ and $\varepsilon \in [-\varepsilon_i, \varepsilon_i]$ the function $h^0_{i,\varepsilon} : [0, \infty) \to \mathbb{R}$ defined by $h^0_{i,\varepsilon} = f_i + \varepsilon g_i$ is continuous, bounded, and $h^0_{i,\varepsilon}(0) = 0$. On account of relations (12) and (13), we have $h^0_{i,\varepsilon}(s) \leqslant 0$ for all $s \in [a_i, b_i]$. Thus, we may apply Theorem 2.1 to the function $h^0_{i,\varepsilon}$ obtaining that for every $i \in \mathbb{N}$ and $\varepsilon \in [-\varepsilon_i, \varepsilon_i]$, the problem

$$\begin{cases} -\Delta u + u = Q(x)h^0_{i,\varepsilon}(u), & x \in \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{$\mathrm{P}^0_{i,\varepsilon}$}$$

has a radially symmetric, weak solution $u^0_{i,\varepsilon} \in H^1(\mathbb{R}^N)$ such that

$$u^0_{i,\varepsilon} \in [0, a_i] \quad \text{for a.e. } x \in \mathbb{R}^N; \tag{14}$$

$$u^0_{i,\varepsilon} \text{ is the infimum of the functional } R_i^\varepsilon \text{ on } W^{b_i}_{\text{rad}}, \tag{15}$$

where

$$R_i^\varepsilon(u) = \frac{1}{2}\|u\|^2_{H^1} - \int_{\mathbb{R}^N} Q(x)\big[F_i(u) + \varepsilon G_i(u)\big], \quad u \in H^1_{\text{rad}}\big(\mathbb{R}^N\big). \tag{16}$$

Due to (13) and (14), $u^0_{i,\varepsilon}$ is a weak solution not only for $(\mathrm{P}^0_{i,\varepsilon})$ but also for our problem (P_ε). Consequently, it remains to prove that

- (I_0) there are infinitely many distinct elements in the sequence $\{u^0_{i,0}\}_i$ verifying (1), see Theorem 1.1;
- (II_0) for every $k \in \mathbb{N}$, there are at least k distinct elements $u^0_{i,\varepsilon}$ verifying (2) when ε belongs to a certain interval around the origin, see Theorem 1.2.

Proof of ($\mathbf{I_0}$); Theorem 1.1 concluded. For abbreviation, take $u^0_i = u^0_{i,0}$ and $R_i = R^0_i$ for every $i \in \mathbb{N}$. We first prove that

$$R_i\big(u^0_i\big) < 0 \quad \text{for all } i \in \mathbb{N}; \tag{17}$$

$$\lim_{i\to\infty} R_i\big(u^0_i\big) = 0. \tag{18}$$

The left side of (f^0_1) implies the existence of $l_0 > 0$ and $\delta \in (0, b_1)$ such that

$$F(s) \geqslant -l_0 s^2 \quad \text{for all } s \in (0, \delta). \tag{19}$$

Let $L_0 > 0$ be large enough such that

$$\frac{1}{2}K(\rho) + l_0\|Q\|_{L^1} < L_0(\rho/2)^N \omega_N \min_{B_{\rho/2}} Q, \tag{20}$$

where $\rho > 0$ and $K(\rho)$ come from (10). Taking into account the right side of (f^0_1), there is a sequence $\{\tilde{s}_i\}_i \subset (0, \delta)$ such that $\tilde{s}_i \leqslant a_i$ and $F(\tilde{s}_i) > L_0\tilde{s}_i^2$ for all $i \in \mathbb{N}$. Let $i \in \mathbb{N}$ fixed and $w_{\tilde{s}_i} \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$ be the function from (9) corresponding to the value $\tilde{s}_i > 0$. Then $w_{\tilde{s}_i} \in W^{b_i}_{\mathrm{rad}}$, and on account of (10) and (19) one has

$$\begin{aligned}
R_i(w_{\tilde{s}_i}) &= \frac{1}{2}\|w_{\tilde{s}_i}\|^2_{H^1} - \int_{\mathbb{R}^N} Q(x)F_i\big(w_{\tilde{s}_i}(x)\big)\,dx \\
&= \frac{1}{2}\|w_{\tilde{s}_i}\|^2_{H^1} - F(\tilde{s}_i)\int_{B_{\rho/2}} Q(x)\,dx - \int_{B_\rho\setminus B_{\rho/2}} Q(x)F\big(w_{\tilde{s}_i}(x)\big)\,dx \\
&\leqslant \left[\frac{1}{2}K(\rho) - L_0(\rho/2)^N\omega_N \min_{B_{\rho/2}} Q + l_0\|Q\|_{L^1}\right]\tilde{s}_i^2.
\end{aligned}$$

Consequently, using (20), we obtain that

$$R_i\big(u^0_i\big) = \min_{W^{b_i}_{\mathrm{rad}}} R_i \leqslant R_i(w_{\tilde{s}_i}) < 0, \tag{21}$$

which proves in particular (17). Now, let us prove (18). For every $i \in \mathbb{N}$, by using the mean value theorem, (11), (13) and (14), we have

$$R_i\big(u_i^0\big) \geqslant -\int_{\mathbb{R}^N} Q(x)F_i\big(u_i^0(x)\big)\,dx \geqslant -\|Q\|_{L^1} \max_{s\in[0,1]}\big|f(s)\big|a_i.$$

Taking into account that $\lim_{i\to\infty} a_i = 0$, the above inequality and (21) leads to (18).

Due to (13) and (14), we observe that

$$R_i\big(u_i^0\big) = R_1\big(u_i^0\big) \quad \text{for all } i \in \mathbb{N}.$$

Combining this relation with (17) and (18), we see that the sequence $\{u_i^0\}_i$ contains infinitely many distinct elements.

It remains to prove relation (1). The first limit easily follows by (14), i.e. $\|u_i^0\|_{L^\infty} \leqslant a_i$ for all $i \in \mathbb{N}$, combined with $\lim_{i\to\infty} a_i = 0$. For the second limit, we use (21), (11), (13) and (14), obtaining for all $i \in \mathbb{N}$ that

$$\frac{1}{2}\big\|u_i^0\big\|_{H^1}^2 < \|Q\|_{L^1} \max_{s\in[0,1]}\big|f(s)\big|a_i,$$

which concludes the proof of Theorem 1.1. □

Proof of ($\mathbf{II_0}$); Theorem 1.2 concluded. Let $\{\theta_i\}_i$ be a sequence with negative terms such that $\lim_{i\to\infty}\theta_i = 0$. By (18) and (21), we clear have that $\lim_{i\to\infty} R_i(w_{\tilde{s}_i}) = 0$. Thus, up to a subsequence, we may assume that the sequence $\{(\theta_i, R_i(u_i^0), R_i(w_{\tilde{s}_i}), a_i)\}_i \subset \mathbb{R}^4$ which converges to $0_{\mathbb{R}^4}$, has the property that for all $i \in \mathbb{N}$,

$$\theta_i < R_i\big(u_i^0\big) \leqslant R_i(w_{\tilde{s}_i}) < \theta_{i+1}; \tag{22}$$

$$a_i < \min\left(\frac{1}{i}, \frac{1}{2i^2\|Q\|_{L^1}[\max_{[0,1]}|f| + \max_{[0,1]}|g| + 1]}\right). \tag{23}$$

Let us denote

$$\varepsilon_i' = \frac{\theta_{i+1} - R_i(w_{\tilde{s}_i})}{\|Q\|_{L^1}[\max_{[0,1]}|g| + 1]} \quad \text{and} \quad \varepsilon_i'' = \frac{R_i(u_i^0) - \theta_i}{\|Q\|_{L^1}[\max_{[0,1]}|g| + 1]}, \quad i \in \mathbb{N}.$$

Fix $k \in \mathbb{N}$. On account of (22),

$$\varepsilon_k^0 = \min\big(1, \varepsilon_1, \ldots, \varepsilon_k, \varepsilon_1', \ldots, \varepsilon_k', \varepsilon_1'', \ldots, \varepsilon_k''\big) > 0.$$

Then, for every $i \in \{1, \ldots, k\}$ and $\varepsilon \in [-\varepsilon_k^0, \varepsilon_k^0]$ we have

$$\begin{aligned} R_i^\varepsilon\big(u_{i,\varepsilon}^0\big) &\leqslant R_i^\varepsilon(w_{\tilde{s}_i}) \quad \big(\text{see (15)}\big) \\ &= R_i(w_{\tilde{s}_i}) - \varepsilon \int_{\mathbb{R}^N} Q(x)G_i(w_{\tilde{s}_i}) \\ &< \theta_{i+1} \quad \big(\text{see the choice of } \varepsilon_i' \text{ and (11)}\big), \end{aligned}$$

and taking into account that $u_{i,\varepsilon}^0$ belongs to $W_{\mathrm{rad}}^{b_i}$, and u_i^0 is the minimum point of R_i over the set $W_{\mathrm{rad}}^{b_i}$, see relation (15) for $\varepsilon = 0$, we have

$$\begin{aligned} R_i^\varepsilon\big(u_{i,\varepsilon}^0\big) &= R_i\big(u_{i,\varepsilon}^0\big) - \varepsilon \int\limits_{\mathbb{R}^N} Q(x) G_i\big(u_{i,\varepsilon}^0\big) \\ &\geqslant R_i\big(u_i^0\big) - \varepsilon \int\limits_{\mathbb{R}^N} Q(x) G_i\big(u_{i,\varepsilon}^0\big) \\ &> \theta_i \quad \big(\text{see the choice of } \varepsilon_i'' \text{ and (11)}\big). \end{aligned}$$

In conclusion, for every $i \in \{1, \dots, k\}$ and $\varepsilon \in [-\varepsilon_k^0, \varepsilon_k^0]$ we have

$$\theta_i < R_i^\varepsilon\big(u_{i,\varepsilon}^0\big) < \theta_{i+1},$$

thus

$$R_1^\varepsilon\big(u_{1,\varepsilon}^0\big) < \cdots < R_k^\varepsilon\big(u_{k,\varepsilon}^0\big).$$

But, $u_{i,\varepsilon}^0 \in W_{\mathrm{rad}}^{b_1}$ for every $i \in \{1, \dots, k\}$, so $R_i^\varepsilon(u_{i,\varepsilon}^0) = R_1^\varepsilon(u_{i,\varepsilon}^0)$, see relation (13). Therefore, from above, we obtain that for every $\varepsilon \in [-\varepsilon_k^0, \varepsilon_k^0]$,

$$R_1^\varepsilon\big(u_{1,\varepsilon}^0\big) < \cdots < R_1^\varepsilon\big(u_{k,\varepsilon}^0\big).$$

In particular, this fact shows that the elements $u_{1,\varepsilon}^0, \dots, u_{k,\varepsilon}^0$ are distinct whenever $\varepsilon \in [-\varepsilon_k^0, \varepsilon_k^0]$.

It remains to prove relation (2). The first relation directly follows by (14) and (23). To check the second limit, we observe that for every $i \in \{1, \dots, k\}$ and $\varepsilon \in [-\varepsilon_k^0, \varepsilon_k^0]$,

$$R_1^\varepsilon\big(u_{i,\varepsilon}^0\big) = R_i^\varepsilon\big(u_{i,\varepsilon}^0\big) < \theta_{i+1} < 0.$$

Consequently, for every $i \in \{1, \dots, k\}$ and $\varepsilon \in [-\varepsilon_k^0, \varepsilon_k^0]$, by a mean value theorem we obtain

$$\begin{aligned} \frac{1}{2}\big\|u_{i,\varepsilon}^0\big\|_{H^1}^2 &< \int\limits_{\mathbb{R}^N} Q(x)\big[F_i\big(u_{i,\varepsilon}^0\big) + \varepsilon G_i\big(u_{i,\varepsilon}^0\big)\big] \\ &\leqslant \|Q\|_{L^1}\Big[\max_{[0,1]}|f| + \max_{[0,1]}|g|\Big]a_i \quad \big(\text{see (11), (14) and } \varepsilon_k^0 \leqslant 1\big) \\ &< \frac{1}{2i^2} \quad \big(\text{see (23)}\big), \end{aligned}$$

which concludes the proof of Theorem 1.2. □

4. Proof of Theorems 1.3 and 1.4

The left side of (f_1^∞) implies the existence of $l_\infty > 0$ and $\delta > 0$ such that

$$F(s) \geqslant -l_\infty s^2 \quad \text{for all } s > \delta. \tag{24}$$

Fix a number $L_\infty > 0$ large enough such that

$$\frac{1}{2}K(\rho) + l_\infty \|Q\|_{L^1} < L_\infty (\rho/2)^N \omega_N \min_{B_{\rho/2}} Q, \tag{25}$$

where $\rho > 0$ and $K(\rho)$ are from (10). The right side of (f_1^∞) implies the existence of a sequence $\{\tilde{s}_i\}_i \subset (0,\infty)$ such that $\lim_{i\to\infty} \tilde{s}_i = \infty$, and

$$F(\tilde{s}_i) > L_\infty \tilde{s}_i^2 \quad \text{for all } i \in \mathbb{N}. \tag{26}$$

Since $\lim_{i\to\infty} s_i = \infty$, see (f_2^∞), we may fix a subsequence $\{s_{m_i}\}_i$ of $\{s_i\}_i$ such that $\tilde{s}_i \leqslant s_{m_i}$ for all $i \in \mathbb{N}$. Due to the continuity of f and g, we may fix the positive sequences $\{a_i\}_i$, $\{b_i\}_i$, and $\{\varepsilon_i\}_i$ such that $\lim_{i\to\infty} a_i = \lim_{i\to\infty} b_i = +\infty$, and for all $i \in \mathbb{N}$,

$$a_i < s_{m_i} < b_i < a_{i+1}; \tag{27}$$

$$f(s) + \varepsilon g(s) \leqslant 0 \quad \text{for all } s \in [a_i, b_i] \text{ and } \varepsilon \in [-\varepsilon_i, \varepsilon_i]. \tag{28}$$

In the same way as we did in (13), let us define the truncation functions $f_i, g_i : [0,\infty) \to \mathbb{R}$ by

$$f_i(s) = f\big(\min(s, b_i)\big) \quad \text{and} \quad g_i(s) = g\big(\min(s, b_i)\big). \tag{29}$$

Since $f_i(0) = g_i(0) = 0$, we may extend continuously the functions f_i and g_i to the whole real line, taking 0 for negative arguments. For every $s \in \mathbb{R}$ and $i \in \mathbb{N}$, let $F_i(s) = \int_0^s f_i(t)\,dt$ and $G_i(s) = \int_0^s g_i(t)\,dt$.

For every $i \in \mathbb{N}$ fixed and $\varepsilon \in [-\varepsilon_i, \varepsilon_i]$ the function $h_{i,\varepsilon}^\infty : [0,\infty) \to \mathbb{R}$ defined by $h_{i,\varepsilon}^\infty = f_i + \varepsilon g_i$ is continuous, bounded, and $h_{i,\varepsilon}^\infty(0) = 0$. On account of relations (28) and (29), one has $h_{i,\varepsilon}^\infty(s) \leqslant 0$ for all $s \in [a_i, b_i]$. Consequently, we may apply Theorem 2.1 to the function $h_{i,\varepsilon}^\infty$ obtaining that for every $i \in \mathbb{N}$ and $\varepsilon \in [-\varepsilon_i, \varepsilon_i]$, the problem

$$\begin{cases} -\Delta u + u = Q(x) h_{i,\varepsilon}^\infty(u), & x \in \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{$\mathrm{P}_{i,\varepsilon}^\infty$}$$

has a radially symmetric, weak solution $u_{i,\varepsilon}^\infty \in H^1(\mathbb{R}^N)$ such that

$$u_{i,\varepsilon}^\infty \in [0, a_i] \quad \text{for a.e. } x \in \mathbb{R}^N; \tag{30}$$

$$u_{i,\varepsilon}^\infty \text{ is the infimum of the functional } R_i^\varepsilon \text{ on } W_{\mathrm{rad}}^{b_i}, \tag{31}$$

where R_i^ε is defined exactly as in (16). Due to (29) and (30), $u_{i,\varepsilon}^\infty$ is a weak solution not only for $(\mathrm{P}_{i,\varepsilon}^\infty)$ but also for the initial problem (P_ε). Consequently, we have to prove that

(I_∞) there are infinitely many distinct elements in the sequence $\{u_{i,0}^\infty\}_i$ verifying (3), see Theorem 1.3;

(II_∞) for every $k \in \mathbb{N}$, there are at least k distinct elements $u_{i,\varepsilon}^\infty$ verifying (5) when ε belongs to a certain interval around the origin, see Theorem 1.4.

Proof of (I_∞); Theorem 1.3 concluded. Let $u_i^\infty = u_{i,0}^\infty$ and $R_i = R_i^0$ for every $i \in \mathbb{N}$. We prove that

$$\lim_{i\to\infty} R_i\big(u_i^\infty\big) = -\infty. \tag{32}$$

Let $i \in \mathbb{N}$ be fixed and $w_{\tilde{s}_i} \in H^1_{\mathrm{rad}}(\mathbb{R}^N)$ be the function from (9) corresponding to the value $\tilde{s}_i > 0$. Then $w_{\tilde{s}_i} \in W_{\mathrm{rad}}^{b_i}$, and on account of (10), (24) and (26), one has

$$\begin{aligned}
R_i(w_{\tilde{s}_i}) &= \frac{1}{2}\|w_{\tilde{s}_i}\|_{H^1}^2 - \int_{\mathbb{R}^N} Q(x)F_i\big(w_{\tilde{s}_i}(x)\big)\,dx \\
&= \frac{1}{2}\|w_{\tilde{s}_i}\|_{H^1}^2 - F(\tilde{s}_i)\int_{B_{\rho/2}} Q(x)\,dx - \int_{(B_\rho\setminus B_{\rho/2})\cap\{w_{\tilde{s}_i}>\delta\}} Q(x)F\big(w_{\tilde{s}_i}(x)\big)\,dx \\
&\quad - \int_{(B_\rho\setminus B_{\rho/2})\cap\{w_{\tilde{s}_i}\leqslant\delta\}} Q(x)F\big(w_{\tilde{s}_i}(x)\big)\,dx \\
&\leqslant \left[\frac{1}{2}K(\rho) - L_\infty(\rho/2)^N\omega_N \min_{B_{\rho/2}} Q + l_\infty\|Q\|_{L^1}\right]\tilde{s}_i^2 + \|Q\|_{L^1}\max_{s\in[0,\delta]}\big|F(s)\big|.
\end{aligned}$$

Using the fact that $\lim_{i\to\infty}\tilde{s}_i = \infty$ and (25), we have that $\lim_{i\to\infty} R_i(w_{\tilde{s}_i}) = -\infty$. But, $R_i(u_i^\infty) \leqslant R_i(w_{\tilde{s}_i})$ for all $i \in \mathbb{N}$, which implies (32).

Now, let us assume that in the sequence $\{u_i^\infty\}_i$ there are only finitely many distinct elements, say $\{u_1^\infty, \ldots, u_{i_0}^\infty\}$ for some $i_0 \in \mathbb{N}$. Consequently, the sequence $\{R_i(u_i^\infty)\}_i$ reduces mostly to the finite set $\{R_{i_0}(u_1^\infty), \ldots, R_{i_0}(u_{i_0}^\infty)\}$, which contradicts relation (32).

It remains to prove (3). Arguing by contradiction assume there exists a subsequence $\{u_{k_i}^\infty\}_i$ of $\{u_i^\infty\}_i$ such that for all $i \in \mathbb{N}$, we have $\|u_{k_i}^\infty\|_{L^\infty} \leqslant M$ for some $M > 0$. In particular, $\{u_{k_i}^\infty\}_i \subset W_{\mathrm{rad}}^{b_l}$ for some $l \in \mathbb{N}$. Thus, for every $k_i \geqslant l$, we have

$$\begin{aligned}
R_l\big(u_l^\infty\big) &= \min_{W_{\mathrm{rad}}^{b_l}} R_l = \min_{W_{\mathrm{rad}}^{b_l}} R_{k_i} \\
&\geqslant \min_{W_{\mathrm{rad}}^{b_{k_i}}} R_{k_i} = R_{k_i}\big(u_{k_i}^\infty\big) \\
&\geqslant \min_{W_{\mathrm{rad}}^{b_l}} R_{k_i} \quad \big(\text{cf. hypothesis, } u_{k_i}^\infty \in W_{\mathrm{rad}}^{b_l}\big) \\
&= R_l\big(u_l^\infty\big).
\end{aligned}$$

As a consequence,

$$R_{k_i}\big(u_{k_i}^{\infty}\big)=R_l\big(u_l^{\infty}\big)\quad \text{for all } i\in\mathbb{N}. \tag{33}$$

But, the sequence $\{R_i(u_i^{\infty})\}_i$ is non-increasing; indeed, due to (31) and (29), for all $i\in\mathbb{N}$, one has

$$R_{i+1}\big(u_{i+1}^{\infty}\big)=\min_{W_{\mathrm{rad}}^{b_{i+1}}} R_{i+1}\leqslant \min_{W_{\mathrm{rad}}^{b_i}} R_{i+1}=\min_{W_{\mathrm{rad}}^{b_i}} R_i=R_i\big(u_i^{\infty}\big).$$

Combining this latter fact with (33), one can find a number $i_0\in\mathbb{N}$ such that $R_i(u_i^{\infty})=R_l(u_l^{\infty})$ for all $i\geqslant i_0$. This fact contradicts (32) which concludes the proof of Theorem 1.3. □

Proof of (3′) from Remark 1.4. Assume that (4) holds for f with $q\in(2,2^*/2)$. By contradiction, we assume that there exists a subsequence $\{u_{k_i}^{\infty}\}_i$ of $\{u_i^{\infty}\}_i$ such that for all $i\in\mathbb{N}$, we have $\|u_{k_i}^{\infty}\|_{H^1}\leqslant\tilde{M}$ for some $\tilde{M}>0$. Now, let us fix $\alpha\in[2q,2^*)$. On account of (4) and the mean value theorem, we have

$$\begin{aligned}\bigg|\int_{\mathbb{R}^N} Q(x)F_{k_i}\big(u_{k_i}^{\infty}(x)\big)\,dx\bigg| &\leqslant c\big(\|Q\|_{L^2}\big\|u_{k_i}^{\infty}\big\|_{L^2}+\|Q\|_{L^{\alpha/(\alpha-q)}}\big\|u_{k_i}^{\infty}\big\|_{L^{\alpha}}^{q}\big)\\ &\leqslant c\big(\|Q\|_{L^2}\tilde{M}+\|Q\|_{L^{\alpha/(\alpha-q)}}C_{\alpha}^{q}\tilde{M}^{q}\big)<\infty,\end{aligned}$$

where $C_\alpha>0$ is the Sobolev embedding constant in $H^1(\mathbb{R}^N)\hookrightarrow L^{\alpha}(\mathbb{R}^N)$. Consequently, the sequence $\{R_{k_i}(u_{k_i}^{\infty})\}_i$ is bounded. Since the sequence $\{R_i(u_i^{\infty})\}_i$ is non-increasing, it will be bounded as well, which contradicts (32). □

Proof of (II_∞); Theorem 1.4 concluded. Let $\{\theta_i\}_i$ be a sequence with negative terms such that $\lim_{i\to\infty}\theta_i=-\infty$. On account of the proof of Theorem 1.3, up to a subsequence, we may assume that the sequence $\{(\theta_i, R_i(u_i^{\infty}), R_i(w_{\tilde{s}_i}), a_i)\}_i\subset\mathbb{R}^4$ which converges to $(-\infty,-\infty,-\infty,\infty)$, has the property that for all $i\in\mathbb{N}$,

$$\theta_{i+1}<R_i\big(u_i^{\infty}\big)\leqslant R_i(w_{\tilde{s}_i})<\theta_i; \tag{34}$$

$$a_i\geqslant i. \tag{35}$$

Let us denote

$$\varepsilon_i'=\frac{\theta_i-R_i(w_{\tilde{s}_i})}{\|Q\|_{L^1}[\max_{[0,b_i]}|g|+1]b_i}\quad\text{and}\quad \varepsilon_i''=\frac{R_i(u_i^{\infty})-\theta_{i+1}}{\|Q\|_{L^1}[\max_{[0,b_i]}|g|+1]b_i},\quad i\in\mathbb{N}.$$

Fix $k\in\mathbb{N}$. Due to (34), we have

$$\varepsilon_k^{\infty}=\min\big(1,\varepsilon_1,\ldots,\varepsilon_k,\varepsilon_1',\ldots,\varepsilon_k',\varepsilon_1'',\ldots,\varepsilon_k''\big)>0.$$

Then, for every $i\in\{1,\ldots,k\}$ and $\varepsilon\in[-\varepsilon_k^{\infty},\varepsilon_k^{\infty}]$ we have

$$\begin{aligned} R_i^\varepsilon\bigl(u_{i,\varepsilon}^\infty\bigr) &\leqslant R_i^\varepsilon(w_{\tilde{s}_i}) \quad \bigl(\text{see (31)}\bigr) \\ &= R_i(w_{\tilde{s}_i}) - \varepsilon \int_{\mathbb{R}^N} Q(x) G_i(w_{\tilde{s}_i}) \\ &< \theta_i \quad \bigl(\text{see the choice of } \varepsilon_i', \tilde{s}_i \leqslant s_{m_i} \text{ and (27)}\bigr), \end{aligned}$$

and since $u_{i,\varepsilon}^\infty$ belongs to $W_{\mathrm{rad}}^{b_i}$, and u_i^∞ is the minimum point of R_i on the set $W_{\mathrm{rad}}^{b_i}$, see relation (31) for $\varepsilon = 0$, we have

$$\begin{aligned} R_i^\varepsilon\bigl(u_{i,\varepsilon}^\infty\bigr) &= R_i\bigl(u_{i,\varepsilon}^\infty\bigr) - \varepsilon \int_{\mathbb{R}^N} Q(x) G_i\bigl(u_{i,\varepsilon}^\infty\bigr) \\ &\geqslant R_i\bigl(u_i^\infty\bigr) - \varepsilon \int_{\mathbb{R}^N} Q(x) G_i\bigl(u_{i,\varepsilon}^\infty\bigr) \\ &> \theta_{i+1} \quad \bigl(\text{see the choice of } \varepsilon_i'', \tilde{s}_i \leqslant s_{m_i}, \text{ and (27)}\bigr). \end{aligned}$$

Thus, for every $i \in \{1, \dots, k\}$ and $\varepsilon \in [-\varepsilon_k^\infty, \varepsilon_k^\infty]$ we have

$$\theta_{i+1} < R_i^\varepsilon\bigl(u_{i,\varepsilon}^\infty\bigr) < \theta_i. \tag{36}$$

In particular,

$$R_k^\varepsilon\bigl(u_{k,\varepsilon}^\infty\bigr) < \cdots < R_1^\varepsilon\bigl(u_{1,\varepsilon}^\infty\bigr) < 0. \tag{37}$$

By construction, $u_{i,\varepsilon}^\infty \in W_{\mathrm{rad}}^{b_k}$ for every $i \in \{1, \dots, k\}$, see (27); thus, $R_i^\varepsilon(u_{i,\varepsilon}^\infty) = R_k^\varepsilon(u_{i,\varepsilon}^\infty)$, see relation (29). Therefore, (37) implies that for every $\varepsilon \in [-\varepsilon_k^\infty, \varepsilon_k^\infty]$,

$$R_k^\varepsilon\bigl(u_{k,\varepsilon}^\infty\bigr) < \cdots < R_k^\varepsilon\bigl(u_{1,\varepsilon}^\infty\bigr) < 0.$$

In particular, the elements $u_{1,\varepsilon}^\infty, \dots, u_{k,\varepsilon}^\infty$ are distinct whenever $\varepsilon \in [-\varepsilon_k^\infty, \varepsilon_k^\infty]$.

Now, we prove relation (5). Fix $\varepsilon \in [-\varepsilon_k^\infty, \varepsilon_k^\infty]$. First, since $R_1^\varepsilon(u_{1,\varepsilon}^\infty) < 0 = R_1^\varepsilon(0)$, then $\|u_{1,\varepsilon}^\infty\|_{L^\infty} > 0$ which proves (5) for $i = 1$. We further prove that

$$\bigl\|u_{i,\varepsilon}^\infty\bigr\|_{L^\infty} > a_{i-1} \quad \text{for all } i \in \{2, \dots, k\}. \tag{38}$$

Let us assume that there exists an element $i_0 \in \{2, \dots, k\}$ such that $\|u_{i_0,\varepsilon}^\infty\|_{L^\infty} \leqslant a_{i_0-1}$. Since $a_{i_0-1} < b_{i_0-1}$, then $u_{i_0,\varepsilon}^\infty \in W_{\mathrm{rad}}^{b_{i_0-1}}$. Thus, on account of (31) and (29), we have

$$R_{i_0-1}^\varepsilon\bigl(u_{i_0-1,\varepsilon}^\infty\bigr) = \min_{W_{\mathrm{rad}}^{b_{i_0-1}}} R_{i_0-1}^\varepsilon \leqslant R_{i_0-1}^\varepsilon\bigl(u_{i_0,\varepsilon}^\infty\bigr) = R_{i_0}^\varepsilon\bigl(u_{i_0,\varepsilon}^\infty\bigr),$$

which contradicts (37). Thus, (38) holds true which can be combined with (35), obtaining relation (5). This ends the proof of Theorem 1.4. □

Proof of (5′) from Remark 1.5. Assume that both f and g verify (4) with $q \in (2, 2^*/2)$. Fix $\alpha \in [2q, 2^*)$. We may assume that the sequence $\{\theta_i\}_i$ from (34) fulfills

$$\theta_i < -2c\|Q\|_{L^2}(i-1) - 2c\|Q\|_{L^{\alpha/(\alpha-q)}} C_\alpha^q (i-1)^q \quad \text{for all } i \in \mathbb{N}, \tag{39}$$

where $c > 0$ comes from (4). Let us fix $\varepsilon \in [-\varepsilon_k^\infty, \varepsilon_k^\infty]$. We assume that $\|u_{i_0,\varepsilon}^\infty\|_{H^1} \leqslant i_0 - 1$ for some $i_0 \in \{1, \dots, k\}$. Then, we have

$$\begin{aligned}
\frac{1}{2}\|u_{i_0,\varepsilon}^\infty\|_{H^1}^2 &= R_{i_0}^\varepsilon(u_{i_0,\varepsilon}^\infty) + \int_{\mathbb{R}^N} Q(x)\big[F_{i_0}(u_{i_0,\varepsilon}^\infty) + \varepsilon G_{i_0}(u_{i_0,\varepsilon}^\infty)\big] \\
&< \theta_{i_0} + c(1+|\varepsilon|)\big[\|Q\|_{L^2}\|u_{i_0,\varepsilon}^\infty\|_{H^1} + \|Q\|_{L^{\alpha/(\alpha-q)}} C_\alpha^q \|u_{i_0,\varepsilon}^\infty\|_{H^1}^q\big] \quad \big(\text{see (36)}\big) \\
&\leqslant \theta_{i_0} + 2c\big[\|Q\|_{L^2}(i_0-1) + \|Q\|_{L^{\alpha/(\alpha-q)}} C_\alpha^q (i_0-1)^q\big] \quad \big(\varepsilon_k^\infty \leqslant 1\big) \\
&< 0 \quad \big(\text{see (39)}\big),
\end{aligned}$$

contradiction. Therefore, relation (5′) is proved. □

Acknowledgment

The author would like to thank the referee for his/her valuable observations.

Appendix A. Principle of symmetric criticality for Szulkin-type functionals

The proof of our main results rely heavily in the principle of symmetric criticality for Szulkin-type functionals, which we state here for the sake of completeness. For further details, see the paper of Kobayashi and Ôtani [5].

Let X be a real Banach space and X^* its dual. Let $E : X \to \mathbb{R}$ be a functional of class C^1 and let $\zeta : X \to \mathbb{R} \cup \{+\infty\}$ be a proper (i.e. $\not\equiv +\infty$), convex, lower semicontinuous function. Then, $I = E + \zeta$ is a *Szulkin-type functional*, see [15]. An element $u \in X$ is called a *critical point of* $I = E + \zeta$ if

$$E'(u)(v-u) + \zeta(v) - \zeta(u) \geqslant 0 \quad \text{for all } v \in X, \tag{40}$$

or equivalently,

$$0 \in E'(u) + \partial\zeta(u) \quad \text{in } X^*,$$

where $\partial\zeta(u)$ stands for the subdifferential of the convex functional ζ at $u \in X$.

Proposition A.1. *(See [15, p. 80].) Every local minimum point of $I = E + \zeta$ is a critical point of I in the sense of* (40).

Let G be a topological group acting linearly on X. We say that G acts *continuously on* X if the map $(g, u) \mapsto gu$ from $G \times X$ into X is continuous. A set M is called *G-invariant* if

$gM = \{gu\colon u \in M\} \subseteq M$ for every $g \in G$. A function h on X is called *G-invariant* if $h(gu) = h(u)$ for every $u \in X$ and $g \in G$. The linear subspace of G-symmetric points of X is defined by

$$\Sigma = \mathrm{Fix}_G(X) = \{u \in X\colon\ gu = u \text{ for all } g \in G\}.$$

A special form of [5, Theorem 3.16] is the following result, known as the *principle of symmetric criticality for Szulkin functionals*.

Theorem A.1. *Let X be a reflexive Banach space and let $I = E + \zeta : X \to \mathbb{R} \cup \{+\infty\}$ be a Szulkin-type functional on X. If a compact group G acts linearly and continuously on X, and the functionals E and ζ are G-invariant, then the principle of symmetric criticality holds, i.e., fixing $u \in \Sigma$, we have*

$$0 \in (E|_\Sigma)'(u) + \partial(\zeta|_\Sigma)(u) \quad \text{in } \Sigma^* \quad \Longrightarrow \quad 0 \in E'(u) + \partial\zeta(u) \quad \text{in } X^*.$$

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