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EXISTENCE OF FIVE NONZERO SOLUTIONS WITH EXACT SIGN FOR A p-LAPLACIAN EQUATION

Michael Filippakis

National Technical University, Department of Mathematics
Zografou Campus, Athens, 15780, Greece

Alexandru Kristály

Babeş-Bolyai University, Department of Economics
400591 Cluj-Napoca, Romania

Nikolaos S. Papageorgiou

National Technical University, Department of Mathematics
Zografou Campus, Athens, 15780, Greece

Abstract. We consider nonlinear elliptic problems driven by the p-Laplacian with a nonsmooth potential depending on a parameter $\lambda > 0$. The main result guarantees the existence of two positive, two negative and a nodal (sign-changing) solution for the studied problem whenever λ belongs to a small interval $(0, \lambda^*)$ and $p \geq 2$. We do not impose any symmetry hypothesis on the nonlinear potential. The constant-sign solutions are obtained by using variational techniques based on nonsmooth critical point theory (minimization argument, Mountain Pass theorem, and a Brézis-Nirenberg type result for C^1-minimizers), while the nodal solution is constructed by an upper-lower solutions argument combined with the Zorn lemma and a nonsmooth second deformation theorem.

1. **Introduction.** Let $Z \subset \mathbb{R}^N$ be a bounded domain with C^2-boundary ∂Z and consider the nonlinear elliptic problem

$$\begin{cases} -\triangle_p x(z) = f(z, x(z), \lambda), & z \in Z; \\ x|_{\partial Z} = 0, \end{cases} \qquad (P_\lambda^o)$$

where $1 < p < \infty$, $\triangle_p(\cdot) = \operatorname{div}(\|D(\cdot)\|_{\mathbb{R}^N}^{p-2} D(\cdot))$ is the p-Laplacian, $f : Z \times \mathbb{R} \times (0, \overline{\lambda}) \to \mathbb{R}$ is a nonlinear function, $\lambda \in (0, \overline{\lambda})$ being a parameter.

The aim of this paper is to prove multiplicity results for problem (P_λ^o), establishing precisely the sign of the solutions. We emphasize that we do not impose any symmetry hypothesis on the nonlinearity f. Moreover, our study includes the case when the function $x \mapsto f(z, x, \lambda)$ has jumping discontinuities away from the origin. However, in order to materialize the type of results we obtain, let us consider here the z-independent (autonomous) case, i.e., $f : \mathbb{R} \times (0, \overline{\lambda}) \to \mathbb{R}$, and assume for the moment that f is continuous. We further assume that

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(f_1) for all $(x,\lambda) \in \mathbb{R} \times (0,\overline{\lambda})$, we have

$$|f(x,\lambda)| \le a(\lambda) + c|x|^{r-1}$$

with $a \in L^\infty(0,\overline{\lambda})_+$, $\lim_{\lambda\to 0} a(\lambda) = 0$, $c > 0$, $p < r < p^*$;

(f_2) $\lambda_2 < \lim_{x\to 0} \frac{f(x,\lambda)}{|x|^{p-2}x} < +\infty$ for every $\lambda \in (0,\overline{\lambda})$ (here, $\lambda_2 > 0$ is the second eigenvalue of $(-\triangle_p, W_0^{1,p}(Z))$);

(f_3) for every $\lambda \in (0,\overline{\lambda})$ there exist $M = M(\lambda) > 0$ and $\mu = \mu(\lambda) > p$ such that

$$0 < \mu F(x,\lambda) \le f(x,\lambda)x \quad \text{for all } |x| \ge M,$$

where $F(x,\lambda) = \int_0^x f(s,\lambda)ds$;

(f_4) for all $(x,\lambda) \in \mathbb{R} \times (0,\overline{\lambda})$, we have $f(x,\lambda)x \ge 0$ (sign condition).

As a simple consequence of our main result (Theorem 4.2), we obtain the following

Theorem 1.1. *Let $f : \mathbb{R} \times (0,\overline{\lambda}) \to \mathbb{R}$ be a continuous function which satisfies $(f_1) - (f_4)$ and $2 \le p < \infty$. Then, there exists $\lambda^* \in (0,\overline{\lambda})$ such that for every $\lambda \in (0,\lambda^*)$, the problem*

$$\begin{cases} -\triangle_p x(z) = f(x(z),\lambda), & z \in Z; \\ x|_{\partial Z} = 0, \end{cases} \tag{P'_λ}$$

has at least five nontrivial smooth solutions; namely, two positive, two negative, and a nodal (sign-changing) solution.

A simple (non-odd) function $f : \mathbb{R} \times (0,\overline{\lambda}) \to \mathbb{R}$ fulfilling the hypotheses $(f_1) - (f_4)$ is

$$f(x,\lambda) = (2 + \operatorname{sgn}(x)) \cdot \begin{cases} C_1|x|^{p-2}x, & \text{if } |x| \le \lambda \\ C_2|x|^{r-2}x + g(\lambda)\operatorname{sgn}(x), & \text{if } |x| > \lambda, \end{cases}$$

with $\lambda_2 < C_1$, $0 < C_2$, $2 \le p < r < p^*$, and $g(\lambda) = \lambda^{p-1}[C_1 - C_2\lambda^{r-p}]$. On the other hand, if $g : (0,\overline{\lambda}) \to \mathbb{R}$ is *any* continuous function, different from the above choice, f will not be continuous. In such a case, (P'_λ) need not have a solution which is not satisfactory for our purpose. In order to overcome this difficulty, we 'fill in the discontinuity gaps' of f by a well-chosen interval. In the sequel we describe roughly this procedure in the framework of the initial problem (P^0_λ).

We assume that $x \mapsto f(z,x,\lambda)$ has jumping discontinuities and f fulfills certain measurability and boundedness conditions which will be specified later (for details, see hypotheses (H_f)). We replace $f(z,x,\lambda)$ by an interval $[f_l(z,x,\lambda), f_u(z,x,\lambda)]$, where

$$f_l(z,x,\lambda) = \liminf_{x'\to x} f(z,x',\lambda) \quad \text{and} \quad f_u(z,x,\lambda) = \limsup_{x'\to x} f(z,x',\lambda).$$

In this way, instead of (P^o_λ) we are dealing with a set-valued problem. The assumptions on f allow us to define the function $j(z,x,\lambda) = \int_0^x f(z,s,\lambda)ds$, and $x \mapsto j(z,x,\lambda)$ becomes locally Lipschitz. Moreover, the generalized subdifferential of $j(z,\cdot,\lambda)$ (in the sense of Clarke) is $\partial j(z,x,\lambda) = [f_l(z,x,\lambda), f_u(z,x,\lambda)]$ for every $x \in \mathbb{R}$. (For details, see Remark 5.) This fact motivates the formulation of the differential inclusion problem (or, hemivariational inequality), whose study constitutes the main objective of our paper:

$$\begin{cases} -\triangle_p x(z) \in \partial j(z,x(z),\lambda), & z \in Z; \\ x|_{\partial Z} = 0. \end{cases} \tag{P_λ}$$

We emphasize that hemivariational inequalities are used in the study of problems with discontinuities (see Chang [12], Gasiński-Papageorgiou [26]), as well as in various engineering problems in which the corresponding energy (Euler) functional is nonsmooth and nonconvex. For various applications, we refer reader to Motreanu-Panagiotopoulos [40], Motreanu-Rădulescu [39], Naniewicz-Panagiotopoulos [41], and references therein.

Recently, multiplicity results for the p-Laplacian without any symmetry condition on the continuous right hand side nonlinearity f in (P_λ^o) were proved by Jiu-Su [31], Liu [36], and Liu-Liu [37], using Morse theory (critical groups). Their multiplicity results do not provide any information about the sign of the solutions.

The existence of multiple positive solutions in the recent decades was investigated primarily in the context of semilinear problems (i.e., $p = 2$). We mention the papers of Amann [1], Dancer [16], Dancer-Du [17], Lions [35] and references therein. For problems driven by the scalar ordinary p-Laplacian, we have the works of De Coster [22], Filippakis-Papageorgiou [24] and He-Ge [29]. For problems driven by the partial p-Laplacian, we refer the reader to the works of Ambrosetti-Garcia Azorero-Peral Alonso [3], Cammaroto-Chinnì-Di Bella [8], Garcia Azorero-Manfredi-Peral Alonso [25], Kyritsi-Papageorgiou [32] and Motreanu-Motreanu-Papageorgiou [38]. In Ambrosetti-Garcia Azorero-Peral Alonso [3] and in Garcia Azorero-Manfredi-Peral Alonso [25], the right hand side nonlinearity has the form $\lambda|x|^{q-2}x + |x|^{r-2}x$ with $1 < q < p < r < p^*$, $\lambda > 0$, and the authors prove the existence of $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ the problem has two positive solutions. In [3], the authors used the radial p-Laplacian and the main tool in their method of proof is the Leray-Schauder degree theory. In [25], $Z \subset \mathbb{R}^N$ is an arbitrary bounded domain with a smooth boundary and the approach is variational. Cammaroto-Chinnì-Di Bella [8], under a different set of rather technical hypotheses involving oscillatory nonlinearities near the origin, produce a whole sequence of small positive solutions which converges uniformly to zero. Their method of proof is completely different than those of [3], [25], and is based on an abstract variational principle of Ricceri [42]. Finally, Kyritsi-Papageorgiou [32] and Motreanu-Motreanu-Papageorgiou [38] considered problems with nonsmooth potentials (hemivariational inequality) which are non-resonance, resonance and near resonance at the principal eigenvalue $\lambda_1 > 0$. They handled the problem by using nonsmooth critical point theory.

The question of nodal (sign-changing) solutions has been investigated in detail in the semilinear case ($p = 2$). We mention the works of Dancer-Du [18, 19, 20], Dancer-Zhang [21], Li-Wang [33, 34] and Zhang-Li [46] (see also the references therein). Roughly speaking, from these works emerged two approaches to the problem: (a) combining the method of upper-lower solutions with variational techniques (see Dancer-Du [18, 19, 20]); (b) using invariance properties of the negative gradient flow associated to a well-chosen pseudo-gradient vector field (see Bartsch-Liu-Weth [5], Dancer-Zhang [21], Li-Wang [34], Zhang-Chen-Li [45], and Zhang-Li [46, 47]). Recently, Carl-Motreanu [9] and Carl-Perera [10] extended the work of Dancer-Du [19] to a nonsmooth setting which involves the p-Laplacian. The hypotheses in Carl-Motreanu [9] were given in the terms of the principal eigenvalue $\lambda_1 > 0$ as well as the Fučik spectrum of the p-Laplacian. Carl-Perera [10] assumed that the studied problem admits an ordered pair of upper and lower solution.

Note that none of the aforementioned multiplicity results for p-Laplacian produced *five* nontrivial solutions with precise sign (in smooth or nonsmooth context). Clearly, here we mean that the right hand side has no symmetry properties; otherwise, as usual, if one assumes that the nonlinear term is odd, a suitable adaptation of the Lusternik-Schnirelmann theory produces infinitely many positive/negative/nodal solutions. In the best case however, the authors obtain *three* solutions: a positive, a negative and a nodal solution (see Bartsch-Liu [4], Bartsch-Liu-Weth [5], Carl-Motreanu [9], Zhang-Chen-Li [45], and Zhang-Li [47]). Our main result complements [5], [9], [45] and [47] also from the point of view of the nonlinearity. In [9, 45, 47], the authors considered asymptotically $(p-1)$-linear problems (at infinity), while we are dealing with a superlinear (and subcritical) problem at infinity, see the Ambrosetti-Rabinowitz type hypothesis (f_3) (or $(H_f)(v)$ below). In [5], the authors assumed that $\limsup_{x\to 0}|f(z,x,\lambda)|/|x|^{p-1} < \lambda_1$ uniformly for a.a. $z \in Z$ (and all $\lambda > 0$); here, we require $\lambda_2 < \liminf_{x\to 0}|f(z,x,\lambda)|/|x|^{p-1}$ uniformly for a.a. $z \in Z$ (see $(H_f)(iv), (vi)$, or (f_2), (f_4), respectively).

Our strategy is to employ variational techniques based on the nonsmooth critical point theory (see for instance Gasiński-Papageorgiou [26]), together with the method of upper-lower solutions, which are constructed using the hypotheses on the nonsmooth potential. Our method of proof is closer to that of Dancer-Du [19] and Carl-Perera [10]. In this process, valuable tools are the nonsmooth version of the second deformation lemma (theorem) due to Corvellec [14] and an alternative variational characterization of the second eigenvalue $\lambda_2 > 0$ of $(-\triangle_p, W_0^{1,p}(Z))$, due to Cuesta-De Figueiredo-Gossez [15].

In the next section we recall various notions and results which will be used later. In §3, we first prove the existence of two constant-sign solutions for problem (P_λ), see Theorems 3.1. Next, by means of a careful modification of the energy functional associated with (P_λ) we prove the existence of two more solutions for problem (P_λ) with constant sign whenever $p \geq 2$, see Theorem 3.2. Finally, in §4, besides of the constant-sign solutions, we prove the existence of a nodal solution for problem (P_λ) by using the Zorn lemma and the second deformation theorem.

2. **Mathematical background.**

The nonsmooth critical point theory, which will be used in the variational techniques, is based mainly on the subdifferential theory for the locally Lipschitz functions. We first recall some basic notions from this theory.

Let X be a Banach space. By X^* we denote its topological dual and by $\langle\cdot,\cdot\rangle$ the duality bracket for the pair (X^*, X). If $\varphi : X \to \mathbb{R}$ is a locally Lipschitz function, the *generalized directional derivative* $\varphi^0(x;h)$ *of* φ *at* $x \in X$ *in the direction* $h \in X$, is defined by

$$\varphi^0(x;h) = \limsup_{x'\to x;\lambda\to 0^+} \frac{\varphi(x'+\lambda h) - \varphi(x')}{\lambda}.$$

It is easy to see that $h \mapsto \varphi^0(x;h)$ is sublinear continuous and so it is the support function of a nonempty, convex and w^*-compact set $\partial\varphi(x) \subseteq X^*$ defined by

$$\partial\varphi(x) = \{x^* \in X^* : \langle x^*, h\rangle \leq \varphi^0(x;h) \text{ for all } h \in X\}.$$

The multifunction $x \mapsto \partial\varphi(x)$ is called the *generalized subdifferential of* φ. Note that if $\varphi : X \to \mathbb{R}$ is continuous convex, then φ is locally Lipschitz and the generalized subdifferential of φ coincides with the subdifferential in the sense of convex analysis, given by

$$\partial_c\varphi(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq \varphi(y) - \varphi(x) \text{ for all } y \in X\}.$$

Also, if $\varphi \in C^1(X)$, then clearly φ is locally Lipschitz and $\partial\varphi(x) = \{\varphi'(x)\}$. We say that $x \in X$ is a *critical point of* φ, if $0 \in \partial\varphi(x)$; in this case, $c = \varphi(x)$ is a critical value of φ. It is easy to check that if $x \in X$ is a local extremum of φ (i.e., a local minimum or a local maximum), then x is a critical point of φ. Further details on the subdifferential theory of locally Lipschitz functions can be found in Clarke [13].

Given a locally Lipschitz function $\varphi : X \to \mathbb{R}$, we say that φ *satisfies the nonsmooth Palais-Smale condition at level* $c \in \mathbb{R}$ (the *nonsmooth* PS_c*-condition*, for short), if every sequence $\{x_n\}_{n\geq 1} \subset X$ such that $\varphi(x_n) \to c$ and $m_\varphi(x_n) = \inf\{\|x^*\| : x^* \in \partial\varphi(x_n)\} \to 0$ as $n \to \infty$, has a strongly convergent subsequence. We say that φ satisfies the *nonsmooth* PS-*condition*, if it satisfies the nonsmooth PS_c-condition at every level $c \in \mathbb{R}$.

The topological notion of linking sets is crucial in the minimax characterization of the critical values of a locally Lipschitz function.

Definition 2.1. Let Y be a Hausdorff topological space, E_0, E and D are nonempty, closed subsets of Y, with $E_0 \subset E$. We say that the pair $\{E_0, E\}$ is linking with D in Y if

(a) $E_0 \cap D = \emptyset$;

(b) for any $\gamma \in C(E, Y)$ such that $\gamma|_{E_0} = id|_{E_0}$, we have $\gamma(E) \cap D \neq \emptyset$.

Using this notion, we have the following general minimax theorem for the critical values of a locally Lipschitz function (see Gasiński-Papageorgiou [26]).

Theorem 2.2. *Let X be a Banach space, E_0, E and D nonempty, closed subsets of X such that $\{E_0, E\}$ is linking with D in X. Let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz function such that $\sup_{E_0}\varphi < \inf_D \varphi$, and $\Gamma = \{\gamma \in C(E, X) : \gamma|_{E_0} = id|_{E_0}\}$, $c = \inf_{\gamma\in\Gamma}\sup_{v\in E}\varphi(\gamma(v))$, φ satisfying the nonsmooth PS_c-condition. Then $c \geq \inf_D \varphi$ and c is a critical value of φ.*

Employing particular choices of linking sets, from the above theorem we may generate nonsmooth versions of the mountain pass theorem, of the saddle point theorem and of the generalized mountain pass theorem. For future use, let us state the nonsmooth mountain pass theorem.

Theorem 2.3. *Let X be a Banach space, $\varphi : X \to \mathbb{R}$ a locally Lipschitz function, $x_0, x_1 \in X$, $\rho > 0$ such that $\max\{\varphi(x_0), \varphi(x_1)\} < \inf\{\varphi(x) : \|x - x_0\| = \rho\} = \beta$, $\|x_1 - x_0\| > \rho$, and $\Gamma_0 = \{\gamma \in C([0,1], X) : \gamma(0) = x_0, \gamma(1) = x_1\}$, $c = \inf_{\gamma\in\Gamma_0}\sup_{t\in[0,1]}\varphi(\gamma(t))$, φ satisfying the nonsmooth PS_c-condition. Then $c \geq \beta$ and c is a critical value of φ.*

Remark 1. It is easy to see that Theorem 2.3 can be deduced from Theorem 2.2 if we choose $E_0 = \{x_0, x_1\}$, $E = [x_0, x_1] = \{x \in X : x = tx_0 + (1-t)x_1, t \in [0,1]\}$ and $D = \partial B_\rho(x_0) = \{x \in X : \|x - x_0\| = \rho\}$.

Definition 2.4. Let X be a Banach space and Y a nonempty subset of X. A deformation of Y is a continuous map $h : [0,1] \times Y \to Y$ such that $h(0, \cdot) = id|_Y$.

(a) If $Z \subset Y$, then we say that Z is a strong deformation retract of Y, if there exists a deformation h of Y such that $h(1, Y) \subseteq Z$ and $h(t, x) = x$ for all $(t, x) \in [0,1] \times Z$.

(b) If $Z \subset Y$, then we say that Z is a weak deformation retract of Y, if there exists a deformation h of Y such that $h(1, Y) \subseteq Z$ and $h(t, Z) \subseteq Z$ for all $t \in [0,1]$.

Remark 2. Clearly, every strong deformation retract of Y is a weak deformation retract of Y.

Given a locally Lipschitz function $\varphi : X \to \mathbb{R}$ and $c \in \mathbb{R}$, we define

$$\varphi^{<c} = \{x \in X : \varphi(x) < c\}, \quad \varphi^c = \{x \in X : \varphi(x) \leq c\} \text{ and}$$
$$K_c = \{x \in X : 0 \in \partial\varphi(x), \varphi(x) = c\}.$$

The next theorem is a partial extension to a nonsmooth setting of the so-called *second deformation theorem* (see Chang [11, p.23] and Gasiński-Papageorgiou [27, p. 628]) and it is due to Corvellec [14]. In fact, Corvellec's result is formulated in a more general framework; namely, for continuous functionals (or, even for lower semicontiuous functionals) on metric spaces. In particular, if X is a Banach space and $\varphi : X \to \mathbb{R}$ is continuous, one can define the set

$$K_c^{ws} = \{x \in X : |d\varphi|(x) = 0, \varphi(x) = c\},$$

where $|d\varphi|(x)$ denotes the *weak slope of* φ *at* x, see [14]. In the case when $\varphi : X \to \mathbb{R}$ is locally Lipschitz, we have

$$|d\varphi|(x) \geq \inf\{\|x^*\| : x^* \in \partial\varphi(x)\} = m_\varphi(x),$$

thus $K_c^{ws} \subseteq K_c$ (with strict inclusion, in general). For our purposes, it suffices a particular form of the result from [14], which we state next.

Theorem 2.5. *Let X be a Banach space, $\varphi : X \to \mathbb{R}$ a locally Lipschitz function which satisfies the nonsmooth* PS*-condition, $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{+\infty\}$, $a < b$, φ has no critical points in $\varphi^{-1}(a, b)$ and $K_a = K_a^{ws} \neq \emptyset$ is discrete. Then there exists a deformation $h : [0, 1] \times \varphi^{<b} \to \varphi^{<b}$ of the set $\varphi^{<b}$ such that*
(a) $h(t, \cdot)|_{K_a} = id|_{K_a}$ for all $t \in [0, 1]$;
(b) $h(1, \varphi^{<b}) \subseteq \varphi^{<a} \cup K_a$;
(c) $\varphi(h(t, x)) \leq \varphi(x)$ for all $t \in [0, 1]$ and all $x \in \varphi^{<b}$.
In particular, the set $\varphi^{<a} \cup K_a$ is a weak deformation retract of $\varphi^{<b}$.

Remark 3. In the corresponding smooth second deformation theorem, the conclusion is that φ^a is a strong deformation retract of $\varphi^b \setminus K_b$ (see Chang [11, p. 23] and Gasiński-Papageorgiou [27, p. 628]).

In the analysis of problem (P_λ) we will use some basic facts about the spectrum of the Dirichlet negative p-Laplacian. So, let $m \in L^\infty(Z)_+$, $m \neq 0$, and consider the following nonlinear weighted (with weight m) eigenvalue problem

$$-\triangle_p x(z) = \hat{\lambda} m(z)|x(z)|^{p-2}x(z) \text{ a.e. on } Z; \quad x|_{\partial Z} = 0. \tag{1}$$

The smallest number $\hat{\lambda} \in \mathbb{R}$ for which problem (1) has a nontrivial solution, is the first eigenvalue of $(-\triangle_p, W_0^{1,p}(Z), m)$ and it is denoted by $\hat{\lambda}_1(m)$. We know that $\hat{\lambda}_1(m) > 0$, it is isolated and also it is simple (i.e., the corresponding eigenspace is one-dimensional). Moreover, $\hat{\lambda}_1(m) > 0$ admits the following variational characterization

$$\hat{\lambda}_1(m) = \min\left\{\frac{\|Dx\|_p^p}{\int_Z m|x|^p dz} : x \in W_0^{1,p}(Z), x \neq 0\right\}. \tag{2}$$

In (2) the minimum is attained on the one-dimensional eigenspace corresponding to $\hat{\lambda}_1(m) > 0$. Let $u_1 \in W_0^{1,p}(Z)$ be the eigenfunction such that $\int_Z m|u_1|^p dz = 1$. Evidently, $|u_1|$ also realizes the minimum in (2) and so we may assume that

$u_1(z) \geq 0$ a.e. on Z. In fact, from the nonlinear regularity theory (see for instance Gasiński-Papageorgiou [27, p. 738]), we have

$$u_1 \in C_0^1(\overline{Z}) = \{x \in C^1(\overline{Z}) : x|_{\partial Z} = 0\}.$$

The space $C_0^1(\overline{Z})$ is an ordered Banach space with order cone

$$K_+ = \{x \in C_0^1(\overline{Z}) : x(z) \geq 0 \ \text{ for all } z \in \overline{Z}\}.$$

We know that

$$\mathrm{int}K_+ = \{x \in K_+ : x(z) > 0 \ \text{ for all } z \in Z, \text{ and } \frac{\partial x}{\partial n}(z) < 0 \ \text{ for all } z \in \partial Z\}.$$

Here, by $n(z)$ we denote the unit outward normal at $z \in \partial Z$. Using the strong maximum principle of Vázquez [43], we have $u_1 \in \mathrm{int}K_+$.

The Lusternik-Schnirelmann theory, in addition to $\hat{\lambda}_1(m) > 0$, gives a whole strictly increasing sequence $\{\hat{\lambda}_n(m)\}_{n\geq 1}$ of eigenvalues of $(-\triangle_p, W_0^{1,p}(Z), m)$ such that $\hat{\lambda}_n(m) \to +\infty$ as $n \to \infty$. These elements are the so-called *LS-eigenvalues* of $(-\triangle_p, W_0^{1,p}(Z), m)$. If $p = 2$ (linear case), then these are all the eigenvalues of $(-\triangle, H_0^1(Z), m)$. If $p \neq 2$ (nonlinear case), we do not know if this is true. However, since $\hat{\lambda}_1(m) > 0$ is isolated, we can define

$$\hat{\lambda}_2^*(m) = \{\hat{\lambda} : \hat{\lambda} \text{ is an eigenvalue of (1) and } \hat{\lambda} \neq \hat{\lambda}_1(m)\} > \hat{\lambda}_1(m).$$

Since the spectrum of $(-\triangle_p, W_0^{1,p}(Z), m)$ is a closed set, we deduce that $\hat{\lambda}_2^*(m)$ is the second eigenvalue of $(-\triangle_p, W_0^{1,p}(Z), m)$. Moreover, we have

$$\hat{\lambda}_2^*(m) = \hat{\lambda}_2(m),$$

i.e., the second eigenvalue and the second LS-eigenvalue of $(-\triangle_p, W_0^{1,p}(Z), m)$ coincide. So, for the second eigenvalue $\hat{\lambda}_2(m)$ we have a variational expression coming from the Lusternik-Schnirelmann theory. Both eigenvalues $\hat{\lambda}_1(m)$ and $\hat{\lambda}_2(m)$ exhibit certain monotonicity properties with respect to the weight function m. More precisely, we have:

(a) If $m_1(z) \leq m_2(z)$ a.e. on Z and $m_1 \neq m_2$, then $\hat{\lambda}_1(m_2) < \hat{\lambda}_1(m_1)$;

(a) If $m_1(z) < m_2(z)$ a.e. on Z, then $\hat{\lambda}_2(m_2) < \hat{\lambda}_2(m_1)$.

If $m = 1$, then we write $\hat{\lambda}_1(1) = \lambda_1$ and $\hat{\lambda}_2(1) = \lambda_2$. Recently, Cuesta-De Figueiredo-Gossez [15] presented an alternative variational characterization of λ_2. Namely, let $\partial B_1^{L^p(Z)} = \{x \in L^p(Z) : \|x\|_p = 1\}$, $U = W_0^{1,p}(Z) \cap \partial B_1^{L^p(Z)}$ and $\Gamma_U = \{\gamma_0 \in C([-1,1], U) : \gamma_0(-1) = -u_1, \gamma_0(1) = u_1\}$. Then

$$\lambda_2 = \inf_{\gamma_0 \in \Gamma_U} \sup_{x \in \gamma_0([-1,1])} \|Dx\|_p^p. \tag{3}$$

We will use (3) in the proof of the existence of a nodal solution for problem (P_λ). Finally, let us recall what we mean by upper and lower solutions for problem (P_λ).

Definition 2.6. (a) An upper solution for problem (P_λ) is a function $\overline{x} \in W^{1,p}(Z)$ such that $\overline{x}|_{\partial Z} \geq 0$ and $\int_Z \|D\overline{x}\|^{p-2}(D\overline{x}, Dy)_{\mathbb{R}^N} dz \geq \int_Z \overline{u} y dz$ for all $y \in W_0^{1,p}(Z)$, $y(z) \geq 0$ a.e. on Z for some $\overline{u} \in L^\theta(Z)$, $\overline{u}(z) \in \partial j(z, \overline{x}(z), \lambda)$ a.e. on Z and $1 < \theta < p^*$. We say that $\overline{x}$ is a strict upper solution for problem (P_λ), if it is not a solution of (P_λ).

(b) A lower solution for problem (P_λ) is a function $\underline{x} \in W^{1,p}(Z)$ such that $\underline{x}|_{\partial Z} \leq 0$ and $\int_Z \|D\underline{x}\|^{p-2}(D\underline{x}, Dy)_{\mathbb{R}^N} dz \leq \int_Z \underline{u} y dz$ for all $y \in W_0^{1,p}(Z)$, $y(z) \geq 0$

a.e. on Z for some $\underline{u} \in L^\theta(Z)$, $\underline{u}(z) \in \partial j(z, \underline{x}(z), \lambda)$ a.e. on Z and $1 < \theta < p^*$. We say that $\underline{x}$ is a strict lower solution for problem (P_λ), if it is not a solution of (P_λ).

As usual, p^* denotes the critical exponent, i.e., $p^* = Np/(N-p)$ if $p < N$, and $p^* = +\infty$ if $p \geq N$. In the sequel, we denote by "$\rightharpoonup$" and "$\rightarrow$" the weak and strong convergence, respectively. Moreover, we use the notations $r^\pm = \max\{\pm r, 0\}$ for every $r \in \mathbb{R}$, and $\|x\| = \|Dx\|_p$ for $x \in W_0^{1,p}(Z)$.

3. **Four constant-sign solutions for problem (P_λ).** In this section we establish the existence of four nontrivial solutions of constant sign for problem (P_λ) whenever λ belongs to a small interval of the form $(0, \lambda^*)$. To do this, we assume the following hypotheses on the nonsmooth potential:

$\underline{(H_j)}$: $j : Z \times \mathbb{R} \times (0, \overline{\lambda}) \to \mathbb{R}$, with $\overline{\lambda} > 0$, is a function such that

- (i) for all $(x, \lambda) \in \mathbb{R} \times (0, \overline{\lambda})$ the function $z \mapsto j(z, x, \lambda)$ is measurable;
- (ii) for almost all $z \in Z$ and all $\lambda \in (0, \overline{\lambda})$, the function $x \mapsto j(z, x, \lambda)$ is locally Lipschitz and $j(z, 0, \lambda) = 0$;
- (iii) for almost all $z \in Z$, all $(x, \lambda) \in \mathbb{R} \times (0, \overline{\lambda})$ and all $u \in \partial j(z, x, \lambda)$, we have
$$|u| \leq a(z, \lambda) + c|x|^{r-1}$$
with $a(\cdot, \lambda) \in L^\infty(Z)_+$, $\|a(\cdot, \lambda)\|_\infty \to 0$ as $\lambda \to 0^+$, $c > 0$, $p < r < p^*$;
- (iv) for every $\lambda \in (0, \overline{\lambda})$ there exists a function $\eta = \eta(\lambda) \in L^\infty(Z)_+$ such that $\eta(z) \geq \lambda_1$ a.e. on Z, $\eta \neq \lambda_1$, and
$$\eta(z) \leq \liminf_{x \to 0} \frac{u}{|x|^{p-2}x} \text{ uniformly for a.a. } z \in Z;$$
- (v) for every $\lambda \in (0, \overline{\lambda})$ there exist $M = M(\lambda) > 0$ and $\mu = \mu(\lambda) > p$ such that
$$0 < \mu j(z, x, \lambda) \leq -j^0(z, x, \lambda; -x) \text{ for a.a. } z \in Z, \text{ all } |x| \geq M;$$
- (vi) for a.a. $z \in Z$, all $x \in \mathbb{R}$, all $\lambda \in (0, \overline{\lambda})$ and all $u \in \partial j(z, x, \lambda)$, we have
$$ux \geq 0 \text{ (sign condition)}$$
and $\partial j(z, 0, \lambda) = \{0\}$.

Let $\varphi_\lambda : W_0^{1,p}(Z) \to \mathbb{R}$ be the Euler functional for problem (P_λ) which is defined by
$$\varphi_\lambda(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z j(z, x(z), \lambda)dz \text{ for all } x \in W_0^{1,p}(Z).$$
We know that φ_λ is locally Lipschitz on bounded sets of $W_0^{1,p}(Z)$, hence locally Lipschitz (see Clarke [13, p. 83]).

In the next theorem we produce the first two solutions of constant sign for problem (P_λ).

Theorem 3.1. *If hypotheses (H_j) hold, then there exists $\lambda^* \in (0, \overline{\lambda})$ such that for every $\lambda \in (0, \lambda^*)$, problem (P_λ) has two solutions $x_0 = x_0(\lambda) \in \mathrm{int}K_+$ and $v_0 = v_0(\lambda) \in -\mathrm{int}K_+$ which are local minimizers of φ_λ.*

If we assume $p \geq 2$ we have two more solutions for (P_λ). More precisely, we have

Theorem 3.2. *If hypotheses (H_j) hold and $2 \leq p < \infty$, then there exists $\lambda^* \in (0, \overline{\lambda})$ such that for every $\lambda \in (0, \lambda^*)$, problem (P_λ) has at least four solutions of constant sign $x_0 = x_0(\lambda) \in \mathrm{int}K_+$, $\hat{x} = \hat{x}(\lambda) \in \mathrm{int}K_+$, $x_0 \leq \hat{x}$, $x_0 \neq \hat{x}$, and $v_0 = v_0(\lambda) \in -\mathrm{int}K_+$, $\hat{v} = \hat{v}(\lambda) \in -\mathrm{int}K_+$, $\hat{v} \leq v_0$, $v_0 \neq \hat{v}$.*

This section deals with the proof of Theorems 3.1 and 3.2, respectively. To this end, we prove some lemmas and propositions.

Lemma 3.3. *Let X be an ordered Banach space, K is an order cone of X, $\mathrm{int}K \neq \emptyset$ and $x_0 \in \mathrm{int}K$. Then, for every $y \in X$, there exists $t = t(y) > 0$ such that $tx_0 - y \in \mathrm{int}K$.*

Proof. Since $x_0 \in \mathrm{int}K$, we can find $\delta > 0$ such that $\overline{B}_\delta(x_0) = \{x \in X : \|x - x_0\| \leq \delta\} \subset \mathrm{int}K$. Let $y \in X$, $y \neq 0$ (if $y = 0$, then clearly the lemma holds for all $t > 0$). We have $x_0 - \delta y/\|y\| \in \mathrm{int}K$. Thus, choosing $t = \|y\|/\delta$, we have $tx_0 - y \in \mathrm{int}K$. □

Let us recall the following notion from nonlinear operator theory (see Gasiński-Papageorgiou [27, p. 338] and Zeidler [44, p. 583].

Definition 3.4. Let X be a reflexiv Banach space and $A : X \to X^*$. We say that A is of type $(S)_+$, if for any sequence $\{x_n\}_{n\geq1} \subset X$ for which $x_n \rightharpoonup x$ in X and $\limsup_{n\to\infty}\langle A(x_n), x_n - x\rangle \leq 0$, one has $x_n \to x$ in X.

By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair $(W^{-1,p'}(Z), W_0^{1,p}(Z))$ $(1/p + 1/p' = 1)$. Let $A : W_0^{1,p}(Z) \to W^{-1,p'}(Z)$ be the nonlinear operator defined by

$$\langle A(x), y\rangle = \int_Z \|Dx\|_{\mathbb{R}^N}^{p-2}(Dx, Dy)_{\mathbb{R}^N}\,dz \quad \text{for all } x, y \in W_0^{1,p}(Z). \tag{4}$$

Lemma 3.5. *$A : W_0^{1,p}(Z) \to W^{-1,p'}(Z)$ defined by (4) is of type $(S)_+$.*

Proof. Let $\{x_n\}_{n\geq1} \subset W_0^{1,p}(Z)$ be a sequence such that $x_n \rightharpoonup x$ in $W_0^{1,p}(Z)$ and assume that

$$\limsup_{n\to\infty}\langle A(x_n), x_n - x\rangle \leq 0. \tag{5}$$

It is clear from (4) that A is demicontinuous monotone, hence it is maximal monotone. But a maximal monotone operator is generalized pseudomonotone (see Gasiński-Papageorgiou [27, p. 330]). So from (5) it follows that

$$\|Dx_n\|_p^p = \langle A(x_n), x_n\rangle \to \langle A(x), x\rangle = \|Dx\|_p^p.$$

Since $Dx_n \rightharpoonup Dx$ in $L^p(Z, \mathbb{R}^N)$ and the space $L^p(Z, \mathbb{R}^N)$ is uniformly convex, from the Kadec-Klee property, we have $Dx_n \to Dx$ in $L^p(Z, \mathbb{R}^N)$, hence $x_n \to x$ in $W_0^{1,p}(Z)$. □

Proof of Theorem 3.1. As we already observed, the nonlinear operator $A : W_0^{1,p}(Z) \to W^{-1,p'}(Z)$ defined by (4) is maximal monotone. Since $\langle A(x), x\rangle = \|Dx\|_p^p$, it is also coercive. Therefore, it is surjective and so we can find $e \in W_0^{1,p}(Z)$, $e \neq 0$ such that $A(e) = 1$. Acting with the test function $-e^- \in W_0^{1,p}(Z)$ we obtain $\|De^-\|_p^p \leq 0$. Hence $e^- = 0$ and so $e \geq 0$. We have

$$-\triangle_p e(z) = 1 \quad \text{a.e. on } Z, \quad e|_{\partial Z} = 0. \tag{6}$$

From nonlinear regularity theory (see for instance Gasiński-Papageorgiou [27, p. 738]) we have $e \in C_0^1(\overline{Z})$ and then by the strong maximum principle of Vázquez [43], we have $e \in \mathrm{int}K_+$.

We claim that we can find $\lambda^* \in (0, \overline{\lambda})$ such that for every $\lambda \in (0, \lambda^*)$ we may choose $\xi_1 = \xi_1(\lambda) > 0$ satisfying

$$\|a(\cdot,\lambda)\|_\infty + c(\xi_1\|e\|_\infty)^{r-1} < \xi_1^{p-1}. \tag{7}$$

We argue by contradiction. So, suppose that we cannot find $\xi_1 > 0$ for which (7) holds. This means that there exists a sequence $\{\lambda_n\}_{n\geq 1} \subset (0, \overline{\lambda})$ such that $\lambda_n \to 0^+$ and

$$\xi^{p-1} \leq \|a(\cdot, \lambda_n)\|_\infty + c(\xi\|e\|_\infty)^{r-1} \text{ for all } n \geq 1, \text{ and all } \xi > 0.$$

We let $n \to \infty$, and using hypothesis $(H_j)(iii)$, we get

$$1 \leq c\xi^{r-p}\|e\|_\infty^{r-1} \text{ for all } \xi > 0.$$

Since $r > p$, letting $\xi \to 0^+$, we have a contradiction. This shows that (7) is true.

We fix $\lambda \in (0, \lambda^*)$ and let $\xi_1 = \xi_1(\lambda) > 0$ be as in (7). We set $\overline{x} = \xi_1 e \in \mathrm{int}K_+$. Then

$$\begin{aligned} -\triangle_p \overline{x}(z) &= -\xi_1^{p-1}\triangle_p e(z) \\ &= \xi_1^{p-1} && \text{(see (6))} \\ &> \|a(\cdot, \lambda)\|_\infty + c(\xi_1\|e\|_\infty)^{r-1} && \text{(see (7))} \\ &\geq \overline{u}(z) \text{ a.e. on } Z, \end{aligned} \tag{8}$$

for all $\overline{u} \in L^{r'}(Z)$ $(1/r + 1/r' = 1)$, $\overline{u} \in \partial j(z, \overline{x}(z), \lambda)$ a.e. on Z (see hypothesis $(H_j)(iii)$). Due to (8), $\overline{x} \in \mathrm{int}K_+$ is a strict upper solution for problem (P_λ).

Evidently, $\underline{x} = 0$ is a lower solution for problem (P_λ) (since $\partial j(z, 0, \lambda) = \{0\}$, see $(H_j)(vi)$). We introduce the following truncation of j:

$$\tilde{j}_+(z, x, \lambda) = \begin{cases} 0, & \text{if } x < 0 \\ j(z, x, \lambda), & \text{if } 0 \leq x \leq \overline{x}(z) \\ j(z, \overline{x}(z), \lambda), & \text{if } \overline{x}(z) < x. \end{cases} \tag{9}$$

From the nonsmooth chain rule (see Clarke [13, p. 42]), we have

$$\partial\tilde{j}_+(z, x, \lambda) \subseteq \begin{cases} \{0\}, & \text{if } x < 0 \\ \{\tau\partial j(z, 0, \lambda) : \tau \in [0, 1]\} = \{0\}, & \text{if } x = 0 \\ \partial j(z, x, \lambda), & \text{if } 0 < x < \overline{x}(z) \\ \{\tau\partial j(z, \overline{x}(z), \lambda) : \tau \in [0, 1]\}, & \text{if } x = \overline{x}(z) \\ \{0\}, & \text{if } \overline{x}(z) < x. \end{cases} \tag{10}$$

We consider the functional $\tilde{\varphi}_\lambda : W_0^{1,p}(Z) \to \mathbb{R}$ defined by

$$\tilde{\varphi}_\lambda(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z \tilde{j}_+(z, x(z), \lambda) \text{ for all } x \in W_0^{1,p}(Z).$$

We know that $\tilde{\varphi}_\lambda$ is locally Lipschitz. Also exploiting the compact embedding of $W_0^{1,p}(Z)$ into $L^p(Z)$, we can easily check that $\tilde{\varphi}_\lambda$ is sequentially weakly lower semicontinuous. Moreover, from (9) and hypothesis $(H_j)(iii)$, we see that for every $x \in W_0^{1,p}(Z)$ one has

$$\tilde{\varphi}_\lambda(x) \geq \frac{1}{p}\|Dx\|_p^p - c_1\|Dx\|_p$$

for some $c_1 > 0$. Consequently, $\tilde{\varphi}_\lambda$ is coercive and bounded from below.

Let $I_+ = [0, \overline{x}] = \{x \in W_0^{1,p}(Z) : 0 \leq x(z) \leq \overline{x}(z) \text{ a.e. on } Z\}$. In a standard way, we can find $x_0 \in I_+$ such that

$$\tilde{\varphi}_\lambda(x_0) = \inf_{I_+} \tilde{\varphi}_\lambda = \tilde{m}_\lambda \leq \tilde{\varphi}_\lambda(0) = 0.$$

We claim that $\tilde{m}_\lambda < 0$. To this end, let $\beta_0 = \int_Z (\lambda_1 - \eta(z))u_1^p(z)dz$. Note that $\beta_0 < 0$ (see hypothesis $(H_j)(iv)$). Now, fix $\varepsilon \in (0, -\beta_0)$. By virtue of the hypothesis $(H_j)(iv)$, we can find $\delta = \delta(\varepsilon, \lambda) > 0$ such that

$$(\eta(z) - \varepsilon)x^{p-1} \leq u \text{ for a.a. } z \in Z, \text{all } x \in [0, \delta] \text{ and all } u \in \partial j(z, x, \lambda). \tag{11}$$

Because of hypothesis $(H_j)(ii)$ and Rademacher's theorem, for almost all $z \in Z$, the function $x \mapsto j(z,x,\lambda)$ is differentiable at almost every $x \in \mathbb{R}$. Moreover, at any such point of differentiability, we have $\frac{d}{dx}j(z,x,\lambda) \in \partial j(z,x,\lambda)$. So, from (11), we have

$$(\eta(z)-\varepsilon)x^{p-1} \leq \frac{d}{dx}j(z,x,\lambda) \text{ for a.a. } z \in Z \text{ and a.a. } x \in [0,\delta].$$

Integrating and taking into account that $j(z,0,\lambda)=0$, we have

$$\frac{1}{p}(\eta(z)-\varepsilon)x^{p} \leq j(z,x,\lambda) \text{ for a.a. } z \in Z \text{ and all } x \in [0,\delta]. \tag{12}$$

Since $\overline{x}$, $u_1 \in \mathrm{int}K_+$, using Lemma 3.3 we can find $t>0$ small such that

$$tu_1(z) \leq \overline{x}(z) \text{ and } tu_1(z) \in [0,\delta] \text{ for all } z \in \overline{Z}. \tag{13}$$

By using (12), (13) and (9), it follows that

$$\frac{t^p}{p}(\eta(z)-\varepsilon)u_1(z)^p \leq j(z,tu_1(z),\lambda) = \tilde{j}_+(z,tu_1(z),\lambda) \text{ for a.a. } z \in Z. \tag{14}$$

Hence we have

$$\begin{array}{rcll}
\tilde{\varphi}_\lambda(tu_1) & = & \frac{t^p}{p}\|Du_1\|_p^p - \int_Z \tilde{j}_+(z,tu_1(z),\lambda)dz & \\
& \leq & \frac{t^p}{p}\lambda_1\|u_1\|_p^p - \frac{t^p}{p}\int_Z \eta u_1^p dz + \frac{t^p}{p}\varepsilon\|u_1\|_p^p & \text{(see (14))} \\
& = & \frac{t^p}{p}\left[\int_Z (\lambda_1-\eta(z))u_1^p(z)dz + \varepsilon\right] & \text{(since } \|u_1\|_p = 1) \\
& = & \frac{t^p}{p}[\beta_0+\varepsilon] & \\
& < & 0. &
\end{array}$$

In conclusion, $\tilde{\varphi}_\lambda(x_0) = \tilde{m}_\lambda < 0 = \tilde{\varphi}_\lambda(0)$, thus, $x_0 \neq 0$.

Given $y \in I_+$, we consider the function $\beta_1 : [0,1] \to \mathbb{R}$ defined by

$$\beta_1(t) = \tilde{\varphi}_\lambda(ty+(1-t)x_0).$$

Evidently, β_1 is a locally Lipschitz function and 0 is a local minimum of it. In particular, we have

$$\tilde{\varphi}_\lambda^0(x_0; y-x_0) \geq 0.$$

Moreover, due to Chang [12, p. 106], we find $u_0 \in L^{r'}(Z)$, $u_0(z) \in \partial\tilde{j}_+(z,x_0(z),\lambda)$ a.e. on Z, such that

$$0 \leq \langle A(x_0), y-x_0\rangle - \int_Z u_0(z)(y-u_0)(z)dz. \tag{15}$$

For every $h \in W_0^{1,p}(Z)$ and $\varepsilon>0$ we define

$$y_{\varepsilon,h}(z) = \left\{ \begin{array}{ccl} 0, & \text{if} & z \in \{x_0+\varepsilon h \leq 0\} \\ x_0(z)+\varepsilon h(z), & \text{if} & z \in \{0 < x_0+\varepsilon h < \overline{x}\} \\ \overline{x}(z), & \text{if} & z \in \{\overline{x} \leq x_0+\varepsilon h\}. \end{array}\right.$$

Clearly, $y_{\varepsilon,h} \in I_+$. Using $y = y_{\varepsilon,h}$ in (15), we have

$$\begin{aligned}
0 \quad \leq \quad & \varepsilon \int_{\{0<x_0+\varepsilon h<\overline{x}\}} \|Dx_0\|_{\mathbb{R}^N}^{p-2}(Dx_0, Dh)_{\mathbb{R}^N}\,dz - \varepsilon \int_{\{0<x_0+\varepsilon h<\overline{x}\}} u_0 h dz \\
& - \int_{\{x_0+\varepsilon h\leq 0\}} \|Dx_0\|_{\mathbb{R}^N}^{p}\,dz + \int_{\{x_0+\varepsilon h\leq 0\}} u_0 x_0 dz \\
& + \int_{\{\overline{x}\leq x_0+\varepsilon h\}} \|Dx_0\|_{\mathbb{R}^N}^{p-2}(Dx_0, D(\overline{x}-x_0))_{\mathbb{R}^N}\,dz - \int_{\{\overline{x}\leq x_0+\varepsilon h\}} u_0(\overline{x}-x_0)dz \\
= \quad & \varepsilon \int_Z \|Dx_0\|_{\mathbb{R}^N}^{p-2}(Dx_0, Dh)_{\mathbb{R}^N}\,dz - \varepsilon \int_Z u_0 h dz \\
& - \int_{\{\overline{x}\leq x_0+\varepsilon h\}} \|D\overline{x}\|_{\mathbb{R}^N}^{p-2}(D\overline{x}, D(x_0+\varepsilon h-\overline{x}))_{\mathbb{R}^N}\,dz \\
& + \int_{\{\overline{x}\leq x_0+\varepsilon h\}} \overline{u}(x_0+\varepsilon h-\overline{x})dz \\
& \left(\begin{array}{c} \text{with } \overline{u}\in L^{r'}(Z),\ \overline{u}(z)\in\partial j(z,\overline{x}(z),\lambda) \text{ a.e. on } Z, \\ u_0(z)=\tau(z)\overline{u}(z) \text{ a.e. on } \{x_0=\overline{x}\},\ \tau: Z\to[0,1] \text{ measurable, see (10)} \end{array} \right) \\
& + \int_{\{x_0+\varepsilon h\leq 0\}} u_0(x_0+\varepsilon h)dz + \int_{\{\overline{x}\leq x_0+\varepsilon h\}} (\overline{u}-u_0)(\overline{x}-x_0-\varepsilon h)dz \\
& - \int_{\{x_0+\varepsilon h\leq 0\}} \|Dx_0\|_{\mathbb{R}^N}^{p}\,dz - \varepsilon \int_{\{x_0+\varepsilon h\leq 0\}} \|Dx_0\|_{\mathbb{R}^N}^{p-2}(Dx_0, Dh)_{\mathbb{R}^N}\,dz \\
& + \int_{\{\overline{x}\leq x_0+\varepsilon h\}} (\|D\overline{x}\|_{\mathbb{R}^N}^{p-2}D\overline{x} - \|Dx_0\|_{\mathbb{R}^N}^{p-2}Dx_0, D(x_0-\overline{x}))_{\mathbb{R}^N}\,dz \\
& +\varepsilon \int_{\{\overline{x}\leq x_0+\varepsilon h\}} (\|D\overline{x}\|_{\mathbb{R}^N}^{p-2}D\overline{x} - \|Dx_0\|_{\mathbb{R}^N}^{p-2}Dx_0, Dh)_{\mathbb{R}^N}\,dz.
\end{aligned} \tag{16}$$

Since $\overline{x} \in \mathrm{int}K_+$ is an upper solution for the problem (P_λ), we have

$$-\int_{\{\overline{x}\leq x_0+\varepsilon h\}} \|D\overline{x}\|_{\mathbb{R}^N}^{p-2}(D\overline{x}, D(x_0+\varepsilon h-\overline{x}))_{\mathbb{R}^N}\,dz + \int_{\{\overline{x}\leq x_0+\varepsilon h\}} \overline{u}(x_0+\varepsilon h-\overline{x})dz \leq 0. \tag{17}$$

Due to hypothesis $(H_j)(vi)$ (sign condition) and (10), we have

$$\int_{\{x_0+\varepsilon h\leq 0\}} u_0(x_0+\varepsilon h)dz \leq 0. \tag{18}$$

Recall that $\overline{x} \in \mathrm{int}K_+$ and $x_0 \leq \overline{x}$. Hence

$$\begin{aligned}
\int_{\{\overline{x}\leq x_0+\varepsilon h\}} (\overline{u}-u_0)(\overline{x}-x_0-\varepsilon h)dz &= \int_{\{x_0\leq\overline{x}\leq x_0+\varepsilon h\}} (\overline{u}-u_0)(\overline{x}-x_0-\varepsilon h)dz = \\
&= \int_{\{x_0=\overline{x};\ \overline{x}\leq x_0+\varepsilon h\}} (\overline{u}-u_0)(\overline{x}-x_0-\varepsilon h)dz + \\
&\quad + \int_{\{x_0<\overline{x}\leq x_0+\varepsilon h\}} (\overline{u}-u_0)(\overline{x}-x_0-\varepsilon h)dz \\
&\leq \int_{\{x_0<\overline{x}\leq x_0+\varepsilon h\}} (\overline{u}-u_0)(\overline{x}-x_0-\varepsilon h)dz
\end{aligned}$$

(since $u_0(z) = \tau(z)\overline{u}(z)$ a.e. on $\{x_0 = \overline{x}\}$ with $\tau : Z \to [0,1]$ measurable)

$$\begin{aligned}
&\leq c_1 \int_{\{x_0<\overline{x}\leq x_0+\varepsilon h\}} (x_0+\varepsilon h-\overline{x})dz \quad \text{for some } c_1>0 \text{ (see } (H_j)(iii)) \\
&\qquad \leq \varepsilon c_1 \int_{\{x_0<\overline{x}\leq x_0+\varepsilon h\}} h dz \quad (\text{since } x_0\leq\overline{x}).
\end{aligned} \tag{19}$$

Recalling that the operator A is monotone (in fact, strictly monotone), we have

$$\int_{\{\overline{x}\leq x_0+\varepsilon h\}}(\|D\overline{x}\|_{\mathbb{R}^N}^{p-2}D\overline{x}-\|Dx_0\|_{\mathbb{R}^N}^{p-2}Dx_0,D(x_0-\overline{x}))_{\mathbb{R}^N}dz\leq 0. \tag{20}$$

We return to the estimation (16) and use (17)-(20), obtaining

$$\begin{aligned} 0 \quad \leq \quad & \varepsilon\int_Z\|Dx_0\|_{\mathbb{R}^N}^{p-2}(Dx_0,Dh)_{\mathbb{R}^N}dz-\varepsilon\int_Z u_0h \\ & +\varepsilon c_1\int_{\{x_0<\overline{x}\leq x_0+\varepsilon h\}}hdz-\varepsilon\int_{\{x_0+\varepsilon h\leq 0\}}\|Dx_0\|_{\mathbb{R}^N}^{p-2}(Dx_0,Dh)_{\mathbb{R}^N}dz \\ & +\varepsilon\int_{\{\overline{x}\leq x_0+\varepsilon h\}}(\|D\overline{x}\|_{\mathbb{R}^N}^{p-2}D\overline{x}-\|Dx_0\|_{\mathbb{R}^N}^{p-2}Dx_0,Dh)_{\mathbb{R}^N}dz. \end{aligned} \tag{21}$$

By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. We have

$$|\{x_0+\varepsilon h\geq\overline{x}>x_0\}|_N\to 0 \quad \text{as } \varepsilon\to 0^+. \tag{22}$$

By applying Stampacchia's theorem (see Gasiński-Papageorgiou [27, p. 195-196]), we know that

$$Dx_0(z)=0 \text{ a.e. on } \{x_0=0\} \text{ and } Dx_0(z)=D\overline{x}(z) \text{ a.e. on } \{x_0=\overline{x}\}. \tag{23}$$

So, if we divide (21) with $\varepsilon>0$ and let $\varepsilon\to 0^+$, using (22) and (23), we obtain

$$0\leq\langle A(x_0),h\rangle-\int_Z u_0hdz. \tag{24}$$

Since $h\in W_0^{1,p}(Z)$ was arbitrary, from (24) we infer that $A(x_0)=u_0$, i.e.,

$$-\triangle_p x_0(z)=u_0(z) \text{ a.e. on } Z,\quad x_0|_{\partial Z}=0,\quad x_0\neq 0.$$

As before, nonlinear regularity theory and the strong maximum principle imply that $x_0\in\operatorname{int}K_+$ (note that by hypothesis $(H_j)(vi)$ and (10), $u_0(z)\geq 0$ a.e. on Z). Since $\lambda\in(0,\lambda^*)$ and $u_0(z)\in\partial\tilde{j}_+(z,x_0(z),\lambda)$ a.e. on Z, from (10) and hypothesis $(H_j)(iii)$ we have

$$\begin{aligned} u_0(z) \quad & \leq \quad a(z,\lambda)+c|x_0(z)|^{r-1} \\ & \leq \quad \|a(\cdot,\lambda)\|_\infty+c\|x_0\|_\infty^{r-1} \\ & \leq \quad \|a(\cdot,\lambda)\|_\infty+c\|\overline{x}\|_\infty^{r-1} \\ & < \quad \xi_1^{p-1} \quad \text{(see (7))}. \end{aligned} \tag{25}$$

We know that

$$-\triangle_p x_0(z)=u_0(z) \quad\text{and}\quad -\triangle_p\overline{x}(z)=\xi_1^{p-1} \text{ a.e. on } Z. \tag{26}$$

By (25), (26) and Guedda-Véron [28, Proposition 2.2], we have

$$x_0(z)<\overline{x}(z) \text{ for all } z\in Z \text{ and } \frac{\partial\overline{x}}{\partial n}(z)<\frac{\partial x_0}{\partial n}(z) \text{ for all } z\in\partial Z,$$

i.e., $\overline{x}-x_0\in\operatorname{int}K_+$. Also recall that $x_0\in\operatorname{int}K_+$. Therefore, we can find $\delta>0$ such that

$$\overline{x}-(x_0+B_\delta^{C_0^1(\overline{Z})})\subset\operatorname{int}K_+ \tag{27}$$

and

$$x_0+B_\delta^{C_0^1(\overline{Z})}\subset\operatorname{int}K_+, \tag{28}$$

where $B_\delta^{C_0^1(\overline{Z})}=\{x\in C_0^1(\overline{Z}):\|x\|_{C_0^1(\overline{Z})}<\delta\}$.

Combining (27), (28) and (9), we deduce that $x_0 \in \mathrm{int}K_+$ is a local $C_0^1(\overline{Z})$-minimizer of φ_λ. Therefore, by Kyritsi-Papageorgiou [32, Proposition 3] (which is actually a nonsmooth extension of the well-known result of Brézis-Nirenberg [7]) we infer that $x_0 \in \mathrm{int}K_+$ is also a local $W_0^{1,p}(Z)$-minimizer of φ_λ, so a solution of (P_λ).

Similarly, working on the negative semiaxis, we produce $\underline{v} \in -\mathrm{int}K_+$ a lower solution for problem (P_λ). Now, $\overline{v} = 0$ is an upper solution (in fact, it is a solution due to $(H_j)(vi)$). Thus, we fix the pair $\{\underline{v}, 0\}$. Introducing a function $\tilde{j_-}$ in a similar way as we did in (9) for $\tilde{j}_+$, and defining its corresponding functional $\tilde{\varphi}_\lambda$, we obtain a second solution $v_0 = v_0(\lambda) \in -\mathrm{int}K_+$ of problem (P_λ) which is a local minimizer of φ_λ. This concludes the proof of Theorem 3.1. $\square$

Using the two solutions x_0 and v_0 from Theorem 3.1 and considering a suitable modification of the Euler functional φ_λ, we will produce two more solutions of constant sign, one positive and the other negative, as we stated in Theorem 3.2. Our goal will be achieved by proving three Propositions.

For this purpose, let $\sigma_\pm : \mathbb{R} \to \mathbb{R}$ be the truncation functions defined by

$$\sigma_+(x) = \begin{cases} 0, & \text{if } x \le 0 \\ x, & \text{if } x > 0 \end{cases} \quad \text{and} \quad \sigma_-(x) = \begin{cases} x, & \text{if } x < 0 \\ 0, & \text{if } x \ge 0. \end{cases}$$

Let

$$\hat{j}_+(z, x, \lambda) = j(z, \sigma_+(x) + x_0(z), \lambda) \quad \text{and} \quad \hat{j}_-(z, x, \lambda) = j(z, \sigma_-(x) + v_0(z), \lambda).$$

Clearly,

- for all $(x, \lambda) \in \mathbb{R} \times (0, \overline{\lambda})$, $z \mapsto \hat{j}_\pm(z, x, \lambda)$ are both measurable;
- for a.a. $z \in Z$ and all $\lambda \in (0, \overline{\lambda})$, $x \mapsto \hat{j}_\pm(z, x, \lambda)$ are both locally Lipschitz.

Moreover, the nonsmooth chain rule implies that

$$\partial \hat{j}_+(z, x, \lambda) \subseteq \begin{cases} \{0\}, & \text{if} \quad x < 0 \\ \{\tau j(z, x_0(z), \lambda) : \tau \in [0, 1]\}, & \text{if} \quad x = 0 \\ \partial j(z, x + x_0(z), \lambda), & \text{if} \quad x > 0 \end{cases} \tag{29}$$

and

$$\partial \hat{j}_-(z, x, \lambda) \subseteq \begin{cases} \partial j(z, x + v_0(z), \lambda), & \text{if} \quad x < 0 \\ \{\tau j(z, v_0(z), \lambda) : \tau \in [0, 1]\}, & \text{if} \quad x = 0 \\ \{0\}, & \text{if} \quad x > 0. \end{cases} \tag{30}$$

For every $\lambda \in (0, \overline{\lambda})$ we consider the locally Lipschitz functions $\psi_\lambda^\pm : W_0^{1,p}(Z) \to \mathbb{R}$ defined by

$$\psi_\lambda^+(x) = \frac{1}{p}\left[\|D(x + x_0)\|_p^p - \|Dx_0\|_p^p\right] - \int_Z \hat{j}_+(z, x(z), \lambda)dz \; + \int_Z u_0 x^- dz + \xi_\lambda^0,$$

and

$$\psi_\lambda^-(x) = \frac{1}{p}\left[\|D(x + v_0)\|_p^p - \|Dv_0\|_p^p\right] - \int_Z \hat{j}_-(z, x(z), \lambda)dz \; - \int_Z w_0 x^+ dz + \xi_\lambda^1,$$

where

$$\xi_\lambda^0 = \int_Z j(z, x_0(z), \lambda)dz, \qquad \xi_\lambda^1 = \int_Z j(z, v_0(z), \lambda)dz$$

and

$$A(x_0) = u_0, \quad A(v_0) = w_0,$$

with $u_0 \in L^{r'}(Z)$, $u_0(z) \in \partial j(z, x_0(z), \lambda)$ a.e. on Z (as in (26)), and $w_0 \in L^{r'}(Z)$, $w_0(z) \in \partial j(z, v_0(z), \lambda)$ a.e. on Z.

Proposition 3.1. *If hypotheses* (H_j) *hold and* $\lambda \in (0, \lambda^*)$, *then* $\psi_\lambda^\pm$ *both satisfy the nonsmooth* PS-*condition.*

Proof. We do the proof for ψ_λ^+, the proof for ψ_λ^- being similar. Let $\{x_n\}_{n\geq 1} \subset W_0^{1,p}(Z)$ be a sequence such that $|\psi_\lambda^+(x_n)| \leq M_1$ for some $M_1 > 0$, all $n \geq 1$ and $m_{\psi_\lambda^+}(x_n) \to 0$. Clearly, we can find $x_n^* \in \partial\psi_\lambda^+(x_n)$ such that $m_{\psi_\lambda^+}(x_n) = \|x_n^*\|$. Then

$$x_n^* = A(x_n + x_0) - u_n - \hat{u}_n,$$

with $u_n \in L^{r'}(Z)$, $u_n(z) \in \partial\hat{j}_+(z, x_n(z), \lambda)$ a.e. on Z and

$$\hat{u}_n(z) = \begin{cases} u_0(z), & \text{if} \quad x_n(z) < 0 \\ \{\tau u_0(z) : \tau \in [0,1]\}, & \text{if} \quad x_n(z) = 0 \\ 0, & \text{if} \quad x_n(z) > 0. \end{cases} \tag{31}$$

Hence we have $|\langle x_n^*, v\rangle| \leq \varepsilon_n\|v\|$ for all $v \in W_0^{1,p}(Z)$ with $\varepsilon_n \to 0^+$, i.e.,

$$\left|\langle A(x_n + x_0), v\rangle - \int_Z u_n v dz - \int_Z \hat{u}_n v dz\right| \leq \varepsilon_n\|v\|. \tag{32}$$

Put the test function $v = -x_n^- \in W_0^{1,p}(Z)$ in (32). Then

$$\left|\langle A(x_n + x_0), -x_n^-\rangle + \int_Z u_n x_n^- dz + \int_Z \hat{u}_n x_n^- dz\right| \leq \varepsilon_n\|x_n^-\|. \tag{33}$$

Note that

$$A(x_n + x_0) = A(x_n^+ + x_0) + A(x_0 - x_n^-) - A(x_0). \tag{34}$$

Recall that

$$\begin{aligned} Dx_n^+(z) &= \begin{cases} Dx_n(z), & \text{for a.a. } z \in \{x_n > 0\} \\ 0, & \text{for a.a. } z \in \{x_n \leq 0\} \end{cases} \\ Dx_n^-(z) &= \begin{cases} 0, & \text{for a.a. } z \in \{x_n \geq 0\} \\ -Dx_n(z), & \text{for a.a. } z \in \{x_n < 0\}. \end{cases} \end{aligned} \tag{35}$$

Due to (34), we have

$$\begin{aligned} \langle A(x_n + x_0), -x_n^-\rangle &= \langle A(x_n^+ + x_0), -x_n^-\rangle + \langle A(x_0 - x_n^-), -x_n^-\rangle \\ &\quad -\langle A(x_0), -x_n^-\rangle \\ &= \langle A(x_0 - x_n^-), -x_n^-\rangle \quad \text{(see (35))} \\ &\geq \|D(x_0 - x_n^-)\|_p^p - \|D(x_0 - x_n^-)\|_p^{p-1}\|Dx_0\|_p. \end{aligned} \tag{36}$$

From (29), we have

$$\int_Z u_n(z) x_n^-(z) dz = 0. \tag{37}$$

Returning to (33) and using (36), (31) and (37), we obtain

$$\|D(x_0 - x_n^-)\|_p^p \leq \|D(x_0 - x_n^-)\|_p^{p-1}\|Dx_0\|_p + c_2\|Dx_n^-\|_p \tag{38}$$

for some $c_2 > 0$ and all $n \geq 1$. From (38) it follows that $\{x_n^-\}_{n\geq 1} \subset W_0^{1,p}(Z)$ is bounded. From the choice of the sequence $\{x_n\}_{n\geq 1} \subset W_0^{1,p}(Z)$ we have

$$\frac{\mu}{p}\left[\|D(x_n + x_0)\|_p^p - \|Dx_0\|_p^p\right] - \int_Z \mu j(z, x_n^+ + x_0, \lambda) dz + \int_Z \mu u_0 x_n^- dz + \mu\xi_\lambda^0 \leq \mu M_1 \tag{39}$$

for all $n \geq 1$. Note that

$$\|D(x_n + x_0)\|_p^p - \|Dx_0\|_p^p =$$

$$= [\|D(x_n^+ + x_0)\|_p^p - \|Dx_0\|_p^p] + [\|D(x_0 - x_n^-)\|_p^p - \|Dx_0\|_p^p] \tag{40}$$

We use (40) in (39). Because $\{x_n^-\}_{n\geq 1} \subset W_0^{1,p}(Z)$ is bounded, we have

$$\frac{\mu}{p}\|D(x_n^+ + x_0)\|_p^p - \int_Z \mu j(z, x_n^+ + x_0, \lambda)dz \leq M_2 \tag{41}$$

for some $M_2 > 0$ and all $n \geq 1$. Now, we put the test function $x_n^+ + x_0 \in W_0^{1,p}(Z)$ in (32). Then

$$-\|D(x_n^+ + x_0)\|_p^p + \int_Z u_n(x_n^+ + x_0)dz \leq c_3\|x_n^+ + x_0\| + c_4 \tag{42}$$

for some $c_3, c_4 > 0$ and all $n \geq 1$. Since $u_n(z) \in \partial\hat{j}_+(z, x_n(z), \lambda)$ a.e. on Z, because of (29), we have

$$u_n(z)(-(x_n^+ + x_0)) \leq j^0(z, (x_n^+ + x_0)(z), \lambda; -(x_n^+ + x_0)) \ \text{ a.e. on } \{x_n > 0\}.$$

Therefore,

$$u_n(z)(x_n^+ + x_0) \geq -j^0(z, (x_n^+ + x_0)(z), \lambda; -(x_n^+ + x_0)) \ \text{ a.e. on } \{x_n > 0\}. \tag{43}$$

From hypothesis $(H_j)(vi)$ (sign condition) and (29), we have

$$u_n(z)x_0(z) \geq 0 \ \text{ a.e. on } \{x_n = 0\}, \tag{44}$$

and

$$u_n(z)(-(x_n^+ + x_0)(z)) = 0 \ \text{ a.e. on } \{x_n < 0\}. \tag{45}$$

Adding (41) and (42) and using (43)-(45), we obtain

$$\left(\frac{\mu}{p} - 1\right)\|D(x_n^+ + x_0)\|_p^p - \int_{\{x_n>0\}} [\mu j(z, x_n + x_0, \lambda) + j^0(z, (x_n + x_0), \lambda; -(x_n + x_0))]dz$$
$$\leq M_3 + c_5\|x_n^+ + x_0\|$$

for some $M_3, c_5 > 0$ and all $n \geq 1$. Using hypotheses $(H_j)(iii), (v)$, it follows that

$$\left(\frac{\mu}{p} - 1\right)\|D(x_n^+ + x_0)\|_p^p \leq M_4 + c_5\|x_n^+ + x_0\|$$

for some $M_4 > 0$ and all $n \geq 1$. Thus, the sequence $\{x_n^+\}_{n\geq 1} \subset W_0^{1,p}(Z)$ is bounded (recall $\mu > p$). Consequently, the sequence $\{x_n\}_{n\geq 1} \subset W_0^{1,p}(Z)$ is bounded.

Now, passing to a suitable subsequence if necessarily, we may assume that $x_n \rightharpoonup x$ in $W_0^{1,p}(Z)$, $x_n \to x$ in $L^r(Z)$, $x_n(z) \to x(z)$ a.e. on Z and $|x_n(z)| \leq k(z)$ for a.a. $z \in Z$ and all $n \geq 1$, with $k \in L^r(Z)_+$. From the choice of the sequence $\{x_n\}_{n\geq 1} \subset W_0^{1,p}(Z)$ (see (32)), we have

$$|\langle A(x_n + x_0), x_n - x\rangle - \int_Z u_n(x_n - x)dz - \int_Z \hat{u}_n(x_n - x)dz| \leq \varepsilon_n\|x_n - x\|, \tag{46}$$

with $\varepsilon_n \to 0^+$. Clearly

$$\int_Z u_n(x_n - x)dz \to 0 \ \text{ and } \int_Z \hat{u}_n(x_n - x)dz \to 0 \ \text{ as } n \to \infty.$$

Therefore, (46) implies

$$\lim_{n\to\infty}\langle A(x_n + x_0), x_n - x\rangle = 0.$$

Since $x_n \rightharpoonup x$ in $W_0^{1,p}(Z)$, from Lemma 3.5, we infer that $x_n \to x$ in $W_0^{1,p}(Z)$, i.e. ψ_λ^+ satisfies the nonsmooth PS-condition. $\square$

Proposition 3.2. *If hypotheses* (H_j) *hold,* $\lambda \in (0, \lambda^*)$ *and* $2 \leq p$, *then we can find* $\rho > 0$ *such that*

$$\inf\{\psi_\lambda^+(x) : \|x\| = \rho\} = \gamma_\lambda^+(\rho) > 0 \quad \textit{and} \quad \inf\{\psi_\lambda^-(x) : \|x\| = \rho\} = \gamma_\lambda^-(\rho) > 0.$$

Proof. Again we do the proof for ψ_λ^+, the proof for ψ_λ^- being similar. From Theorem 3.1, we know that $x_0 \in \mathrm{int}K_+$ is a local minimizer of φ_λ. We can always assume that x_0 is an isolated critical point of φ_λ (otherwise, we have a whole sequence of distinct positive solutions of (P_λ)). Hence we can find a number $\rho_0 > 0$ such that

$$\varphi_\lambda(x_0) < \varphi_\lambda(y) \ \text{ and } \ 0 \notin \partial\varphi_\lambda(y) \ \text{ for all } \ y \in \overline{B}_{\rho_0}(x_0) \setminus \{x_0\}. \tag{47}$$

We claim that for all $\rho \in (0, \rho_0)$, we have

$$\inf\{\varphi_\lambda(x) : \overline{B}_{\rho_0}(x_0) \setminus B_\rho(x_0)\} > \varphi_\lambda(x_0). \tag{48}$$

We proceed indirectly. So suppose that the claim is not true. Then we can find a sequence $\{x_n\}_{n\geq 1} \subset \overline{B}_{\rho_0}(x_0) \setminus B_\rho(x_0)$ such that $\lim_{n\to\infty} \varphi_\lambda(x_n) = \varphi_\lambda(x_0)$. Clearly, $\{x_n\}_{n\geq 1} \subset W_0^{1,p}(Z)$ is bounded and so we may assume that $x_n \rightharpoonup \tilde{x}$ in $W_0^{1,p}(Z)$ and $x_n \to \tilde{x}$ in $L^r(Z)$ (recall that $r < p^*$). We have $\tilde{x} \in \overline{B}_{\rho_0}(x_0)$. Recalling that φ_λ is sequentially weakly lower semicontinuous, we have

$$\varphi_\lambda(\tilde{x}) \leq \lim_{n\to\infty} \varphi_\lambda(x_n) = \varphi_\lambda(x_0).$$

Due to (47), $\tilde{x} = x_0$.

By virtue of the Lebourg mean value theorem (see Clarke [13, Theorem 2.3.7]), we have

$$\varphi_\lambda(x_n) - \varphi_\lambda\left(\frac{x_n + x_0}{2}\right) = \left\langle x_n^*, \frac{x_n - x_0}{2}\right\rangle$$

with $x_n^* \in \partial\varphi_\lambda\left(t_n x_n + (1-t_n)\frac{x_n + x_0}{2}\right)$, and $t_n \in (0,1)$, $n \geq 1$. We know that

$$x_n^* = A\left(t_n x_n + (1-t_n)\frac{x_n + x_0}{2}\right) - \tilde{u}_n$$

with $\tilde{u}_n(z) \in \partial j\left(z, t_n x_n(z) + (1-t_n)(\frac{x_n + x_0}{2})(z), \lambda\right)$ a.a on Z, and $\tilde{u}_n \in L^{r'}(Z)$ $(1/r + 1/r' = 1)$. Hence

$$\begin{aligned}\varphi_\lambda(x_n) - \varphi_\lambda\left(\frac{x_n + x_0}{2}\right) \quad &= \quad \left\langle A\left(t_n x_n + (1-t_n)\frac{x_n + x_0}{2}\right), \frac{x_n - x_0}{2}\right\rangle \\ &\quad - \int_Z \tilde{u}_n \frac{x_n - x_0}{2} dz.\end{aligned} \tag{49}$$

Since $x_n \to x_0$ in $L^r(Z)$ and by hypothesis $(H_j)(iii)$, $\{\tilde{u}_n\}_{n\geq 1} \subset L^{r'}(Z)$ is bounded, we have

$$\lim_{n\to\infty} \int_Z \tilde{u}_n \frac{x_n - x_0}{2} dz = 0. \tag{50}$$

Also because $\frac{x_n + x_0}{2} \rightharpoonup x_0$ in $W_0^{1,p}(Z)$ and φ_λ is sequentially weakly lower semicontinuous, we have

$$\varphi_\lambda(x_0) \leq \liminf_{n\to\infty} \varphi_\lambda\left(\frac{x_n + x_0}{2}\right). \tag{51}$$

Returning to (49), passing to the limit as $n \to \infty$ and using (50), (51) and the fact that $\lim_{n\to\infty} \varphi_\lambda(x_n) = \varphi_\lambda(x_0)$, we obtain

$$\limsup_{n\to\infty} \left\langle A\left(t_n x_n + (1-t_n)\frac{x_n + x_0}{2}\right), \frac{x_n - x_0}{2}\right\rangle \leq 0.$$

Multiplying the above inequality by the term $(1+t_n)$ we obtain

$$\limsup_{n\to\infty}\left\langle A\left(t_n x_n+(1-t_n)\frac{x_n+x_0}{2}\right), t_n x_n+(1-t_n)\frac{x_n+x_0}{2}-x_0\right\rangle\leq 0. \tag{52}$$

We may assume that $t_n\to t^*\in[0,1]$. Therefore, $t_n x_n+(1-t_n)\frac{x_n+x_0}{2}\rightharpoonup x_0$ in $W_0^{1,p}(Z)$. From (52) and Lemma 3.5 if follows that

$$t_n x_n+(1-t_n)\frac{x_n+x_0}{2}\to x_0 \ \text{ in } \ W_0^{1,p}(Z). \tag{53}$$

But note that

$$\left\|t_n x_n+(1-t_n)\frac{x_n+x_0}{2}-x_0\right\|=(1+t_n)\left\|\frac{x_n-x_0}{2}\right\|\geq\frac{\rho}{2}, \tag{54}$$

since $\{x_n\}_{n\geq1}\subset\overline{B}_{\rho_0}(x_0)\setminus B_\rho(x_0)$. Comparing (53) with (54), we have a contradiction. This proves that (48) is true.

By definition, for every $x\in W_0^{1,p}(Z)$ we have

$$\begin{aligned}\psi_\lambda^+(x) &= \frac{1}{p}\left[\|D(x+x_0)\|_p^p-\|Dx_0\|_p^p\right]-\int_Z\hat{j}_+(z,x(z),\lambda)dz\ +\int_Z u_0x^-dz+\xi_\lambda^0\\ &= \frac{1}{p}\left[\|D(x^++x_0)\|_p^p-\|Dx_0\|_p^p\right]+\frac{1}{p}\left[\|D(x_0-x^-)\|_p^p-\|Dx_0\|_p^p\right]\\ &\quad -\int_Z\hat{j}_+(z,x(z),\lambda)dz\ +\int_Z u_0x^-dz+\xi_\lambda^0. \end{aligned}\tag{55}$$

Recall that $A(x_0)=u_0$. Hence

$$\int_Z u_0x^-dz=\int_Z\|Dx_0\|_{\mathbb{R}^N}^{p-2}(Dx_0,Dx^-)_{\mathbb{R}^N}dz. \tag{56}$$

We use (56) and (55), obtaining

$$\begin{aligned}\psi_\lambda^+(x) &= \frac{1}{p}\left[\|D(x_0-x^-)\|_p^p-\|Dx_0\|_p^p\right]+\int_Z\|Dx_0\|_{\mathbb{R}^N}^{p-2}(Dx_0,Dx^-)_{\mathbb{R}^N}dz\\ &\quad+\frac{1}{p}\left[\|D(x^++x_0)\|_p^p-\|Dx_0\|_p^p\right]-\int_Z j(z,x^++x_0,\lambda)dz\ +\xi_\lambda^0\\ &= \frac{1}{p}\left[\|D(x_0-x^-)\|_p^p-\|Dx_0\|_p^p\right]+\int_Z\|Dx_0\|_{\mathbb{R}^N}^{p-2}(Dx_0,Dx^-)_{\mathbb{R}^N}dz\\ &\quad+\varphi_\lambda(x^++x_0)-\varphi_\lambda(x_0).\end{aligned}\tag{57}$$

Let $\rho\in(0,\rho_0)$ and suppose that $\|x\|=\rho$. Then we must have either $\|x^+\|\geq\rho/2$ or $\|x^-\|\geq\rho/2$.

First, we assume that $\|x^+\|\geq\rho/2$. Then from (48) we can find $\beta_\lambda^1=\beta_\lambda^1(\rho)>0$ such that

$$\varphi_\lambda(x^++x_0)-\varphi_\lambda(x_0)\geq\beta_\lambda^1>0. \tag{58}$$

Also from the monotonicity of the gradient of a convex function, we have

$$\frac{1}{p}\left[\|D(x_0-x^-)\|_p^p-\|Dx_0\|_p^p\right]+\int_Z\|Dx_0\|_{\mathbb{R}^N}^{p-2}(Dx_0,Dx^-)_{\mathbb{R}^N}dz\geq0. \tag{59}$$

Using (58) and (59) in (57), we obtain

$$\psi_\lambda^+(x)\geq\beta_\lambda^1>0 \ \text{ for all } \|x\|=\rho \text{ with } \|x^+\|\geq\frac{\rho}{2}\ \ (\rho\in(0,\rho_0)). \tag{60}$$

Now, we assume that $\|x^-\|\geq\rho/2$. Then since $\|x\|=\rho<\rho_0$, we have

$$\varphi_\lambda(x^++x_0)\geq\varphi_\lambda(x_0)\qquad\text{(see (47))}. \tag{61}$$

Also, recall that for $2 \leq p$ and $\xi_1, \xi_2 \in \mathbb{R}^N$, we have a Clarkson type inequality

$$\|\xi_2\|^p_{\mathbb{R}^N} - \|\xi_1\|^p_{\mathbb{R}^N} \geq p\|\xi_1\|^{p-2}_{\mathbb{R}^N}(\xi_1, \xi_2 - \xi_1)_{\mathbb{R}^N} + \frac{1}{2^{p-1}-1}\|\xi_1 - \xi_2\|^p_{\mathbb{R}^N}. \tag{62}$$

Applying (62) with

$$\xi_1 = Dx_0(z) \quad \text{and} \quad \xi_2 = D(x_0 - x^-)(z)$$

and integrating on Z, we may fix $\beta^2_\lambda = \beta^2_\lambda(\rho) > 0$ such that

$$\frac{1}{p}\left[\|D(x_0 - x^-)\|^p_p - \|Dx_0\|^p_p\right] + \int_Z \|Dx_0\|^{p-2}_{\mathbb{R}^N}(Dx_0, Dx^-)_{\mathbb{R}^N}\,dz \geq \beta^2_\lambda > 0. \tag{63}$$

Thus, using (61) and (63) in (57), then

$$\psi^+_\lambda(x) \geq \beta^2_\lambda > 0 \ \text{ for all } \|x\| = \rho \text{ with } \|x^-\| \geq \frac{\rho}{2} \ \ (\rho \in (0, \rho_0)). \tag{64}$$

From (60) and (64) we conclude that

$$\psi^+_\lambda(x) \geq \gamma^+_\lambda = \min\{\beta^1_\lambda, \beta^2_\lambda\} \ \text{ for all } x \in W^{1,p}_0(Z) \text{ with } \|x\| = \rho.$$

Similarly, we may show that there exists $\gamma^-_\lambda = \gamma^-_\lambda(\rho) > 0$ such that

$$\psi^-_\lambda(x) \geq \gamma^-_\lambda \quad \text{for all } x \in W^{1,p}_0(Z) \text{ with } \|x\| = \rho.$$

This concludes our proof. $\square$

Proposition 3.3. *If hypotheses (H_j) hold and $\lambda \in (0, \lambda^*)$, then there exist $y_+ = y_+(\lambda, \rho)$, $y_- = y_-(\lambda, \rho) \in W^{1,p}_0(Z)$ such that $\|y_+\| > \rho$, $\|y_-\| > \rho$, and*

$$\psi^+_\lambda(y_+) < \gamma^+_\lambda \quad \textit{and} \quad \psi^-_\lambda(y_-) < \gamma^-_\lambda,$$

where $\rho > 0$ and $\gamma^\pm_\lambda$ are from Proposition 3.2.

Proof. Let N_0 be the Lebesgue-null set, outside of which hypotheses (H_j) (ii), $(iii), (v)$ hold. Let $z \in Z \setminus N_0$ and $|x| \geq M$. We set

$$k_\lambda(z, t) = j(z, tx, \lambda).$$

It is clear that $t \mapsto k_\lambda(z, t)$ is a locally Lipschitz function and from the nonsmooth chain rule (see Clarke [13, p. 45]), we have $\partial k_\lambda(z, t) = \partial_x j(z, tx, \lambda)x$, thus

$$t\partial k_\lambda(z, t) = \partial_x j(z, tx, \lambda)tx.$$

Then from [12, p. 106] and hypothesis $(H_j)(v)$, we have $\mu k_\lambda(z, t) \leq tk'_\lambda(z, t)$ for all $z \in Z \setminus N_0$ and almost all $t \geq 1$. Consequently,

$$\frac{\mu}{t} \leq \frac{k'_\lambda(z, t)}{k_\lambda(z, t)} \ \text{ for all } z \in Z \setminus N_0 \text{ and almost all } t \geq 1.$$

Integrating this last inequality from 1 to $t_0 > 1$, we obtain $t^\mu_0 k_\lambda(z, 1) \leq k_\lambda(z, t_0)$. Consequently, we have

$$t^\mu j(z, x, \lambda) \leq j(z, tx, \lambda) \quad \text{for all } z \in Z \setminus N_0 \text{ all } |x| \geq M \text{ and all } t \geq 1. \tag{65}$$

For $x \geq M$, due to (65), we have

$$j(z, x, \lambda) = j(z, \frac{x}{M}M, \lambda) \geq \left(\frac{x}{M}\right)^\mu j(z, M, \lambda) \tag{66}$$

and for $x \leq -M$,

$$j(z, x, \lambda) = j(z, \frac{x}{-M}(-M), \lambda) \geq \left(\frac{|x|}{M}\right)^\mu j(z, -M, \lambda). \tag{67}$$

By virtue of hypothesis $(H_j)(iii)$ we can find $c_6 = c_6(\lambda) > 0$ such that

$$|j(z,x,\lambda)| \leq c_6 \quad \text{for all } z \in Z \setminus N_0 \text{ and all } |x| \leq M. \tag{68}$$

Combining (66), (67) and (68), we conclude that

$$j(z,x,\lambda) \geq c_7|x|^\mu - c_6 \quad \text{for a.a. } z \in Z, \text{ all } x \in \mathbb{R} \text{ and some } c_7 > 0. \tag{69}$$

Recall that $u_1 \in \mathrm{int}K_+$. Thus, using Lemma 3.3, we can find $t > 0$ large enough such that

$$tu_1 - x_0 \in K_+ \ \text{ and } \ \|tu_1 - x_0\| > \rho.$$

Then

$$\begin{aligned}
\psi_\lambda^+(tu_1 - x_0) &= \frac{1}{p}\left[t^p\|Du_1\|_p^p - \|Dx_0\|_p^p\right] - \int_Z j(z, tu_1, \lambda)dz \ + \xi_\lambda^0 \\
&\leq \frac{t^p}{p}\|Du_1\|_p^p - c_7 t^\mu \|u_1\|_\mu^\mu + c_8 \quad \text{for some } c_8 > 0 \text{ (see (69))} \\
&\leq \frac{t^p}{p}\lambda_1\|u_1\|_p^p - c_9 t^\mu\|u_1\|_p^\mu + c_8 \quad \text{for some } c_9 > 0 \text{ (since } \mu > p).
\end{aligned}$$

Since $\mu > p$, for $t > 0$ large, we have $\|y_+\| > \rho$ and

$$\psi_\lambda^+(tu_1 - x_0) < 0.$$

Thus, for $t > 0$ large and $y_+ = tu_1 - x_0$, we have

$$\psi_\lambda^+(y_+) < 0 < \gamma_\lambda^+.$$

Similarly, we obtain $y_- \in W_0^{1,p}(Z)$ such that $\psi_\lambda^-(y_-) < 0 < \gamma_\lambda^-$. □

Proof of Theorem 3.2. From Theorem 3.1, we already have two solutions $x_0 \in \mathrm{int}K_+$ and $v_0 \in -\mathrm{int}K_+$ when $\lambda \in (0, \lambda^*)$. Note that Propositions 3.1, 3.2 and 3.3 permit the use of Theorem 2.3 for the functionals $\psi_\lambda^\pm$. As before, we consider only the case of ψ_λ^+, the other one is analogous. Therefore, we can find $\tilde{x} \in W_0^{1,p}(Z)$ such that

$$0 \in \partial\psi_\lambda^+(\tilde{x}) \ \text{ and } \ \psi_\lambda^+(0) = 0 < \gamma_\lambda^+ \leq \psi_\lambda^+(\tilde{x}). \tag{70}$$

In particular, $\tilde{x} \neq 0$. From the inclusion in (70), we have

$$A(\tilde{x} + x_0) = \tilde{u} + \hat{u}_0 \tag{71}$$

with

$$\tilde{u} \in L^{r'}(Z), \ \tilde{u}(z) \in \partial\hat{j}_+(z, \tilde{x}(z), \lambda) \ \text{ a.e. on } Z \tag{72}$$

and

$$\hat{u}_0(z) = \begin{cases} u_0(z), & \text{if } \ \tilde{x}(z) < 0 \\ \{\tau u_0(z) : \tau \in [0,1]\}, & \text{if } \ \tilde{x}(z) = 0 \\ 0, & \text{if } \ \tilde{x}(z) > 0. \end{cases} \tag{73}$$

In (71) we act with the test function $-\tilde{x}^- \in W_0^{1,p}(Z)$. Hence

$$\langle A(\tilde{x} + x_0), -\tilde{x}^-\rangle = \int_Z \tilde{u}(-\tilde{x}^-)dz + \int_Z \hat{u}_0(-\tilde{x}^-)dz. \tag{74}$$

By using (72) and (29), we infer that

$$\int_Z \tilde{u}(-\tilde{x}^-)dz = 0. \tag{75}$$

On th other hand, recall that $A(x_0) = u_0$. From this fact and (73), it follows that

$$\int_Z \hat{u}_0(-\tilde{x}^-)dz = \int_Z u_0(-\tilde{x}^-)dz = \langle A(x_0), -\tilde{x}^-\rangle. \tag{76}$$

Returning to (74) and use (75) and (76), we obtain

$$\langle A(\tilde{x}+x_0)-A(x_0),-\tilde{x}^-\rangle = 0. \tag{77}$$

Now, recall that (see also (34))

$$A(\tilde{x}+x_0)=A(\tilde{x}^+ + x_0)+A(x_0-\tilde{x}^-)-A(x_0). \tag{78}$$

Using (78) and (77) one has

$$\langle A(\tilde{x}^+ + x_0)-A(x_0),-\tilde{x}^-\rangle + \langle A(x_0-\tilde{x}^-)-A(x_0),-\tilde{x}^-\rangle = 0. \tag{79}$$

Due to (35), we have

$$\langle A(\tilde{x}^+ + x_0)-A(x_0),-\tilde{x}^-\rangle = 0.$$

So, by (79), it follows that

$$\langle A(x_0-\tilde{x}^-)-A(x_0),-\tilde{x}^-\rangle = 0. \tag{80}$$

On the other hand, as we already mentioned, the nonlinear operator A is strictly monotone. Hence from (80) we deduce that

$$x_0-\tilde{x}^- = x_0, \text{ i.e., } \tilde{x}^- = 0 \text{ and so } \tilde{x}\geq 0, \ \tilde{x}\neq 0.$$

Let $\hat{x}=\tilde{x}+x_0$. It is clear that $\hat{x}\geq x_0$ and $\hat{x}\neq x_0$. We have two possibilities:

1) $z\in\{\tilde{x}=0\}$. Then

$$\begin{aligned} -\triangle_p\hat{x}(z) &= -\triangle_p x_0(z)=A(x_0)(z)=u_0(z)\\ &\in \partial j(z,x_0(z),\lambda)\\ &= \partial j(z,\hat{x}(z),\lambda). \end{aligned}$$

2) $z\in\{\tilde{x}>0\}$. In this case we have

$$\begin{aligned} -\triangle_p\hat{x}(z) &= \tilde{u}(z)+\hat{u}_0(z) && \text{(see (71))}\\ &= \tilde{u}(z) && \text{(see (73))}\\ &\in \partial\hat{j}_+(z,\tilde{x}(z),\lambda) && \text{(see (72))}\\ &\subseteq \partial j(z,\tilde{x}(z)+x_0(z),\lambda) && \text{(see (29))}\\ &= \partial j(z,\hat{x}(z),\lambda). \end{aligned}$$

Therefore, in both cases we have

$$-\triangle_p\hat{x}(z)=\hat{u}(z)\in\partial j(z,\hat{x}(z),\lambda) \text{ a.e. on } Z, \quad \hat{x}|_{\partial Z}=0,$$

so, $\hat{x}\in \mathrm{int}K_+$ (from the strong maximum principle) and it is a solution for (P_λ).

Similarly working with ψ_λ^- and using this time (30), we obtain $\hat{v}\in -\mathrm{int}K_+$, $\hat{v}\leq v_0$, $\hat{v}\neq v_0$ another constant sign solution for (P_λ). □

Remark 4. Let $j(z,x,\lambda)=\frac{\lambda}{q}|x|^q+\frac{1}{r}|x|^r$ with $1<q<p<r<p^*$. Clearly, j satisfies hypotheses (H_j). Now, $\partial j(z,x,\lambda)=\{\lambda|x|^{q-2}x+|x|^{r-2}x\}$ is the classical convex-concave nonlinearity. This problem has been extensively studied by Ambrosetti-Brézis-Cerami [2], Bartsch-Willem [6] ($p=2$), Ambrosetti-Garcia Azorero-Peral Alonso [3] (for problem with radial p - Laplacian).

Now, let $j(z,x,\lambda)=(2+\mathrm{sgn}(x))\left(\frac{\lambda}{q}|x|^q+\frac{1}{r}|x|^r\right)$. It also verifies the hypotheses (H_j); in addition, it has no symmetry property.

4. **A nodal solution for problem (P_λ).** In this section we establish the existence of a nodal (sign-changing) solution for problem (P_λ) besides the two (respectively, four) constant-sign solutions from Theorem 3.1 and 3.2, respectively. Here, we consider the special case when the potential $j(z,x,\lambda)$ has the form

$$j(z,x,\lambda)=\int_0^x f(z,s,\lambda)ds, \tag{81}$$

where the function $x\mapsto f(z,x,\lambda)$ has possible jumping discontinuities. To be precise, we assume

$\underline{(H_f)}$: $f:Z\times\mathbb{R}\times(0,\overline{\lambda})\to\mathbb{R}$, with $\overline{\lambda}>0$, is a function such that

- (i) for all $\lambda\in(0,\overline{\lambda})$ the function $(z,x)\mapsto f(z,x,\lambda)$ is measurable;
- (ii) for almost all $z\in Z$ and all $\lambda\in(0,\overline{\lambda})$, the function $x\mapsto f(z,x,\lambda)$ has jumping discontinuities and it is continuous at 0;
- (iii) for almost all $z\in Z$, all $(x,\lambda)\in\mathbb{R}\times(0,\overline{\lambda})$, we have
$$|f(z,x,\lambda)|\le a(z,\lambda)+c|x|^{r-1}$$
with $a(\cdot,\lambda)\in L^\infty(Z)_+$, $\|a(\cdot,\lambda)\|_\infty\to0$ as $\lambda\to0^+$, $c>0$, $p<r<p^*$;
- (iv) for every $\lambda\in(0,\overline{\lambda})$ there exists a function $\hat\eta=\hat\eta(\lambda)\in L^\infty(Z)_+$ such that
$$\lambda_2<\liminf_{x\to0}\frac{f_l(z,x,\lambda)}{|x|^{p-2}x}\le\limsup_{x\to0}\frac{f_u(z,x,\lambda)}{|x|^{p-2}x}\le\hat\eta(z)$$
uniformly for a.a. $z\in Z$, where $f_l(z,x,\lambda)=\liminf_{x'\to x}f(z,x',\lambda)$ and $f_u(z,x,\lambda)=\limsup_{x'\to x}f(z,x',\lambda)$.
- (v) for every $\lambda\in(0,\overline{\lambda})$ there exist $M=M(\lambda)>0$ and $\mu=\mu(\lambda)>p$ such that
$$0<\mu j(z,x,\lambda)\le\max\{f_l(z,x,\lambda)x,f_u(z,x,\lambda)x\}$$
for a.a. $z\in Z$, all $|x|\ge M$ (j being from (81));
- (vi) for a.a. $z\in Z$, all $x\in\mathbb{R}$ and all $\lambda\in(0,\overline{\lambda})$, we have
$$f(z,x,\lambda)x\ge0\ \ \text{(sign condition)}.$$

Remark 5. Under hypotheses (H_f), the function j defined in (81) is well-defined, $(z,x)\mapsto j(z,x,\lambda)$ is measurable, and for almost all $z\in Z$ and all $\lambda\in(0,\overline{\lambda})$, $x\mapsto j(z,x,\lambda)$ is locally Lipschitz. Moreover, for a.a. $z\in Z$, all $x\in\mathbb{R}$ and all $\lambda\in(0,\overline{\lambda})$ we have

$$\partial j(z,x,\lambda)=[f_l(z,x,\lambda),f_u(z,x,\lambda)], \tag{82}$$

(see Chang [12]). In particular, since $f(z,\cdot,\lambda)$ is continuous in the origin, we have $\partial j(z,0,\lambda)=\{0\}$. Consequently, if f verifies (H_f), then the function j defined by (81) fulfills (H_j) as well.

Now, we are ready to state the theorems for nodal (sign-changing) solutions. Let $\lambda^*>0$ from Theorem 3.1.

Theorem 4.1. *If hypotheses (H_f) hold and $\lambda\in(0,\lambda^*)$, problem (P_λ) has at least three nontrivial solutions $x_0=x_0(\lambda)\in\mathrm{int}K_+$, $v_0=v_0(\lambda)\in-\mathrm{int}K_+$, and $y_0\in C_0^1(\overline{Z})$ a nodal solution.*

Theorem 4.2. *If hypotheses (H_f) hold, $2\le p<\infty$, and $\lambda\in(0,\lambda^*)$, problem (P_λ) has at least five nontrivial solutions $x_0=x_0(\lambda)\in\mathrm{int}K_+$, $\hat x=\hat x(\lambda)\in\mathrm{int}K_+$, $x_0\le\hat x$, $x_0\ne\hat x$, $v_0=v_0(\lambda)\in-\mathrm{int}K_+$, $\hat v=\hat v(\lambda)\in-\mathrm{int}K_+$, $\hat v\le v_0$, $v_0\ne\hat v$, and $y_0\in C_0^1(\overline{Z})$ a nodal solution.*

In this section we deal with the proof of these results. To do this, we follow the approach first suggested by Dancer-Du [19] for semilinear problems (i.e., $p = 2$), with a smooth potential (i.e., $j(z, \cdot, \lambda) \in C^1(\mathbb{R})$).

The strategy of our proof is the following. Using the method of upper-lower solutions, we produce a smallest positive solution y_+ and a biggest negative solution y_- (see Proposition 4.3). Then we form the order interval $[y_-, y_+]$ and using variational techniques on suitable truncated functionals, we generate a solution y_0 in $[y_-, y_+]$, different than the two endpoints y_-, y_+. If $y_0 \neq 0$, then necessarily y_0 will be sign-changing. In order to show the nontriviality of y_0, we will use the alternative characterization of the second eigenvalue $\lambda_2 > 0$ given by Cuesta-De Figueiredo-Gossez [15] and also Theorem 2.5 (the nonsmooth second deformation theorem).

Now, we start to implement the strategy outlined above. We say that a nonempty set $S \subseteq W^{1,p}(Z)$ is *downward (upward) directed* if for every elements $y_1, y_2 \in S$ there exists $y \in S$ $(z \in S)$ such that $y \leq y_1$ and $y \leq y_2$ $(y_1 \leq z$ and $y_2 \leq z)$.

Let us fix $\lambda \in (0, \lambda^*)$.

Lemma 4.3. *The set of upper solutions for problem (P_λ) is downward directed. Moreover, if y_1 and y_2 are upper solutions for problem (P_λ) then $\min\{y_1, y_2\}$ is also an upper solution for problem (P_λ).*

Proof. Let y_1 and y_2 two upper solutions for problem (P_λ). Given $\varepsilon > 0$, we consider the truncation function $\xi_\varepsilon : \mathbb{R} \to \mathbb{R}$ defined by

$$\xi_\varepsilon(s) = \begin{cases} -\varepsilon, & \text{if} \quad s \leq -\varepsilon \\ s, & \text{if} \quad s \in [-\varepsilon, \varepsilon] \\ \varepsilon, & \text{if} \quad s \geq \varepsilon. \end{cases}$$

The function ξ_ε is clearly Lipschitz continuous, thus the well-known theorem of Marcus-Mizel implies that $\xi_\varepsilon((y_1 - y_2)^-) \in W^{1,p}(Z)$. By the chain rule for Sobolev functions one has

$$D\xi_\varepsilon((y_1 - y_2)^-) = \xi'_\varepsilon((y_1 - y_2)^-) D(y_1 - y_2)^-. \tag{83}$$

Consider the test function $\psi \in C_c^1(Z)$ with $\psi \geq 0$. Then $\xi_\varepsilon((y_1 - y_2)^-)\psi \in W^{1,p}(Z) \cap L^\infty(Z)$ and

$$D\xi_\varepsilon((y_1 - y_2)^-\psi) = \psi D\xi_\varepsilon((y_1 - y_2)^-) + \xi_\varepsilon((y_1 - y_2)^-) D\psi. \tag{84}$$

Since y_1 and y_2 are upper solutions for problem (P_λ), from Definition 2.6 (a), we have

$$\langle A(y_1), \xi_\varepsilon((y_1 - y_2)^-)\psi\rangle \geq \int_Z u_1 \xi_\varepsilon((y_1 - y_2)^-)\psi dz$$

and

$$\langle A(y_2), (\varepsilon - \xi_\varepsilon((y_1 - y_2)^-))\psi\rangle \geq \int_Z u_2(\varepsilon - \xi_\varepsilon((y_1 - y_2)^-))\psi dz$$

for some $u_k \in L^{r'}(Z)$ with $u_k \in \partial j(z, y_k(z), \lambda)$ a.e. on Z, $k = 1, 2$. Adding the above inequalities, we obtain

$$\begin{aligned} &\langle A(y_1), \xi_\varepsilon((y_1 - y_2)^-)\psi\rangle + \langle A(y_2), (\varepsilon - \xi_\varepsilon((y_1 - y_2)^-))\psi\rangle \\ &\qquad \geq \int_Z u_1 \xi_\varepsilon((y_1 - y_2)^-)\psi dz + \int_Z u_2(\varepsilon - \xi_\varepsilon((y_1 - y_2)^-))\psi dz. \end{aligned} \tag{85}$$

Using (83) and (84), we have

$$\begin{aligned}
&\langle A(y_1), \xi_\varepsilon((y_1-y_2)^-)\psi\rangle \\
&\qquad = \int_Z \|Dy_1\|_{\mathbb{R}^N}^{p-2}(Dy_1, D((y_1-y_2)^-))_{\mathbb{R}^N}\xi'_\varepsilon((y_1-y_2)^-)\psi dz \\
&\qquad\quad + \int_Z \|Dy_1\|_{\mathbb{R}^N}^{p-2}(Dy_1, D\psi)_{\mathbb{R}^N}\xi_\varepsilon((y_1-y_2)^-)dz \\
&\qquad = -\int_{\{-\varepsilon\le y_1-y_2\le 0\}} \|Dy_1\|_{\mathbb{R}^N}^{p-2}(Dy_1, D(y_1-y_2))_{\mathbb{R}^N}\psi dz \\
&\qquad\quad + \int_Z \|Dy_1\|_{\mathbb{R}^N}^{p-2}(Dy_1, D\psi)_{\mathbb{R}^N}\xi_\varepsilon((y_1-y_2)^-)dz \qquad (86)
\end{aligned}$$

and in a similar way

$$\begin{aligned}
&\langle A(y_2), (\varepsilon-\xi_\varepsilon((y_1-y_2)^-))\psi\rangle \\
&\qquad = \int_{\{-\varepsilon\le y_1-y_2\le 0\}} \|Dy_2\|_{\mathbb{R}^N}^{p-2}(Dy_2, D(y_1-y_2))_{\mathbb{R}^N}\psi dz \\
&\qquad\quad + \int_Z \|Dy_2\|_{\mathbb{R}^N}^{p-2}(Dy_2, D\psi)_{\mathbb{R}^N}(\varepsilon-\xi_\varepsilon((y_1-y_2)^-))dz. \qquad (87)
\end{aligned}$$

According to (86) and (87), and recalling that $\psi \ge 0$, we obtain

$$\begin{aligned}
&\langle A(y_1), \xi_\varepsilon((y_1-y_2)^-)\psi\rangle + \langle A(y_2), (\varepsilon-\xi_\varepsilon((y_1-y_2)^-))\psi\rangle \\
&\quad = \int_{\{-\varepsilon\le y_1-y_2\le 0\}} (\|Dy_2\|_{\mathbb{R}^N}^{p-2}Dy_2 - \|Dy_1\|_{\mathbb{R}^N}^{p-2}Dy_1, D(y_1-y_2))_{\mathbb{R}^N}\psi dz \\
&\qquad + \int_Z \|Dy_1\|_{\mathbb{R}^N}^{p-2}(Dy_1, D\psi)_{\mathbb{R}^N}\xi_\varepsilon((y_1-y_2)^-)dz \\
&\qquad + \int_Z \|Dy_2\|_{\mathbb{R}^N}^{p-2}(Dy_2, D\psi)_{\mathbb{R}^N}(\varepsilon-\xi_\varepsilon((y_1-y_2)^-))dz \\
&\quad \le \int_Z \|Dy_1\|_{\mathbb{R}^N}^{p-2}(Dy_1, D\psi)_{\mathbb{R}^N}\xi_\varepsilon((y_1-y_2)^-)dz \\
&\qquad + \int_Z \|Dy_2\|_{\mathbb{R}^N}^{p-2}(Dy_2, D\psi)_{\mathbb{R}^N}(\varepsilon-\xi_\varepsilon((y_1-y_2)^-))dz. \qquad (88)
\end{aligned}$$

We return to (85) and (88) and dividing by $\varepsilon > 0$, we obtain

$$\begin{aligned}
&\int_Z \|Dy_1\|_{\mathbb{R}^N}^{p-2}(Dy_1, D\psi)_{\mathbb{R}^N}\frac{1}{\varepsilon}\xi_\varepsilon((y_1-y_2)^-)dz \\
&\quad + \int_Z \|Dy_2\|_{\mathbb{R}^N}^{p-2}(Dy_2, D\psi)_{\mathbb{R}^N}(1-\frac{1}{\varepsilon}\xi_\varepsilon((y_1-y_2)^-))dz \\
&\ge \int_Z u_1\frac{1}{\varepsilon}\xi_\varepsilon((y_1-y_2)^-)\psi dz + \int_Z u_2\left(1-\frac{1}{\varepsilon}\xi_\varepsilon((y_1-y_2)^-)\right)\psi dz. \qquad (89)
\end{aligned}$$

One can observe that

$$\frac{1}{\varepsilon}\xi_\varepsilon((y_1-y_2)^-(z)) \to \chi_{\{y_1<y_2\}}(z) \text{ a.e. on } Z \text{ as } \varepsilon \to 0$$

and

$$\chi_{\{y_1\ge y_2\}} = 1 - \chi_{\{y_1<y_2\}}.$$

Passing to the limit as $\varepsilon \to 0^+$ in (89), we obtain

$$\int_{\{y_1<y_2\}} \|Dy_1\|_{\mathbb{R}^N}^{p-2}(Dy_1, D\psi)_{\mathbb{R}^N}dz + \int_{\{y_1\geq y_2\}} \|Dy_2\|_{\mathbb{R}^N}^{p-2}(Dy_2, D\psi)_{\mathbb{R}^N}dz$$
$$\geq \int_{\{y_1<y_2\}} u_1\psi dz + \int_{\{y_1\geq y_2\}} u_2\psi dz. \tag{90}$$

Since $y = \min\{y_1, y_2\} \in W^{1,p}(Z)$, we have

$$Dy(z) = \begin{cases} Dy_1(z), & \text{for a.a. } z \in \{y_1 < y_2\} \\ Dy_2(z), & \text{for a.a. } z \in \{y_1 \geq y_2\}. \end{cases}$$

Also let $u = \chi_{\{y_1<y_2\}}u_1 + \chi_{\{y_1\geq y_2\}}u_2$. Then $u \in L^{r'}(Z)$ and $u(z) \in \partial j(z, y(z), \lambda)$ a.e. on Z. Consequently, with these notations, (90) is equivalent to

$$\int_Z \|Dy\|_{\mathbb{R}^N}^{p-2}(Dy, D\psi)_{\mathbb{R}^N}dz \geq \int_Z u\psi dz. \tag{91}$$

But $\psi \in C_c^1(Z)_+$ was arbitrary and $C_c^1(Z)_+$ is dense in $W_0^{1,p}(Z)_+$. So, (91) holds for all $\psi \in W_0^{1,p}(Z)$, $\psi \geq 0$ and this implies that $y = \min\{y_1, y_2\}$ is an upper solution for problem (P_λ), which concludes the proof. $\square$

In a similar manner we can prove

Lemma 4.4. *The set of lower solutions for problem (P_λ) is upward directed. Moreover, if z_1 and z_2 are lower solutions for problem (P_λ) then $\max\{z_1, z_2\}$ is also a lower solution for problem (P_λ).*

Proposition 4.1. *If hypotheses (H_f) hold and $\lambda \in (0, \lambda^*)$, then there exists $\varepsilon_0 = \varepsilon_0(\lambda) > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, the element $\underline{x}_\varepsilon = \varepsilon u_1 \in \mathrm{int}K_+$ is a strict lower solution for problem (P_λ). Similarly, $\overline{v}_\varepsilon = -\varepsilon u_1 \in -\mathrm{int}K_+$ is a strict upper solution for problem (P_λ).*

Proof. We fix $\lambda \in (0, \lambda^*)$. Hypothesis $(H_f)(iv)$ implies that we can find $\xi_0 > \lambda_2$ and $\delta_0 > 0$ such that

$$\xi_0 x^{p-1} \leq f_l(z, x, \lambda) \text{ for a.a. } z \in Z \text{ and all } x \in [0, \delta_0]. \tag{92}$$

Recall that $u_1 \in \mathrm{int}K_+$ and let $\gamma_0 \in (0, 1)$ be small such that $0 < \gamma_0 u_1(z) \leq \delta_0$ for all $z \in Z$. Let $\overline{x} \in \mathrm{int}K_+$ be the strict upper solution produced in the proof of Theorem 3.1. Using Lemma 3.3 we can find $t_0 > \xi_0$ such that $t_0\overline{x} - \gamma_0\xi_0 u_1 \in \mathrm{int}K_+$. Now, we choose $\varepsilon_0 = \gamma_0\xi_0 t_0^{-1} > 0$ and fix $\varepsilon \in (0, \varepsilon_0)$. Let $\underline{x}_\varepsilon = \varepsilon u_1 \in \mathrm{int}K_+$. It is clear that $0 < \underline{x}_\varepsilon(z) \leq \delta_0$ for all $z \in Z$. Hence

$$\begin{aligned} -\triangle_p \underline{x}_\varepsilon(z) &= \lambda_1|\underline{x}_\varepsilon(z)|^{p-2}\underline{x}_\varepsilon(z) = \lambda_1\underline{x}_\varepsilon(z)^{p-1} \\ &< \xi_0\underline{x}_\varepsilon(z)^{p-1} \qquad (\text{since } \lambda_1 < \lambda_2 < \xi_0 \text{ and } \underline{x}_\varepsilon \geq 0, \underline{x}_\varepsilon \neq 0) \\ &\leq u(z) \quad \text{a.e. on } Z \qquad (\text{see (92) and (82)}) \end{aligned} \tag{93}$$

for all $u \in L^{r'}(Z)$ with $u(z) \in \partial j(z, \underline{x}_\varepsilon(z), \lambda)$ a.e. on Z. From (93) it is clear that $\underline{x}_\varepsilon \in \mathrm{int}K_+$ is a strict lower solution for problem (P_λ). Evidently, $\overline{x} - \underline{x}_\varepsilon \in \mathrm{int}K_+$.

A similar reasoning shows that $\overline{v}_\varepsilon = -\varepsilon u_1 \in -\mathrm{int}K_+$ is a strict upper solution for problem (P_λ). $\square$

In the sequel, we will use the following two intervals in $W_0^{1,p}(Z)$:

$$[\underline{x}, \overline{x}] = \{x \in W_0^{1,p}(Z) : \underline{x}(z) \leq x(z) \leq \overline{x}(z) \text{ a.e. on } Z\};$$
$$[\underline{v}, \overline{v}] = \{v \in W_0^{1,p}(Z) : \underline{v}(z) \leq v(z) \leq \overline{v}(z) \text{ a.e. on } Z\},$$

where $\underline{x}$ is a fixed strict lower solution for problem (P_λ) (similarly, $\overline{v}$ is a strict upper solution for problem (P_λ)), obtained in Proposition 4.1.

Proposition 4.2. *If hypotheses (H_f) hold and $\lambda \in (0, \lambda^*)$, then problem (P_λ) admits a smallest solution in $[\underline{x}, \overline{x}]$ and a biggest solution in $[\underline{v}, \overline{v}]$.*

Proof. We deal with the existence of a smallest solution in $[\underline{x}, \overline{x}]$; the proof concerning a biggest solution in $[\underline{v}, \overline{v}]$ is similar.

Let S_+ be the set of the solutions for problem (P_λ) belonging to the order interval $T_+ = [\underline{x}, \overline{x}]$. Without loosing the generality, we may assume that $S_+ \neq \emptyset$. Indeed, taking into account Proposition 4.1, we may fix $\underline{x}$ so small such that $x_0 - \underline{x} \in \mathrm{int}K_+$, where $x_0 \in \mathrm{int}K_+$ is a solution for problem (P_λ) (see Theorem 3.1). In this way, $x_0 \in S_+$. We divide the proof into three steps.

Step 1. The set S_+ is downward directed.
To prove this, we fix $x_1, x_2 \in S_+$. Then x_1, x_2 are both upper solutions for problem (P_λ) and so by virtue of Lemma 4.3, $\tilde{x} = \min\{x_1, x_2\} \in W_0^{1,p}(Z)$ is also an upper solution for problem (P_λ). Let

$$\tilde{T}_+ = [\underline{x}, \tilde{x}] = \{x \in W_0^{1,p}(Z) : \underline{x}(z) \leq x(z) \leq \tilde{x}(z) \ \text{ a.e. on } Z\}.$$

Using standard truncation and penalization techniques, we may find $\tilde{x}_0 \in \tilde{T}_+$ a solution for problem (P_λ) (see Gasiński-Papageorgiou [27] and references therein). Moreover, $\tilde{x}_0 \in \mathrm{int}K_+$ and

$$\underline{x} \leq \tilde{x}_0 \leq \min\{x_1, x_2\} \leq \overline{x},$$

i.e., $\tilde{x}_0 \in S_+$ and so S_+ is downward directed.

Step 2. The set S_+ contains a minimal element.
Let D be a chain in S_+ (i.e., a totally ordered subset of S_+). By Dunford-Schwartz [23, Corollary 7, p. 336], we can find $\{x_n\}_{n\geq 1} \subseteq D$ such that

$$\inf_{n\geq 1} x_n = \inf D.$$

We may assume that $\{x_n\}_{n\geq 1}$ is decreasing. Since $\{x_n\}_{n\geq 1} \subseteq S_+$, we see that

$$\|Dx_n\|_p \leq M_2 \ \text{ for some } M_2 > 0 \text{ and all } n \geq 1.$$

Hence $\{x_n\}_{n\geq 1} \subset W_0^{1,p}(Z)$ is bounded and we may assume that $x_n \rightharpoonup \hat{y}$ in $W_0^{1,p}(Z)$ and $x_n \to \hat{y}$ in $L^r(Z)$ as $n \to \infty$. Since $\{x_n\}_{n\geq 1} \subseteq S_+$, we have

$$A(x_n) = u_n, \quad n \geq 1, \tag{94}$$

with $u_n \in L^{r'}(Z)$, $u_n(z) \in \partial j(z, x_n(z), \lambda)$ a.e. on Z. Hypothesis $(H_f)(iii)$ implies that $\{u_n\} \subset L^{r'}(Z)$ is bounded. So, we may assume that

$$u_n \rightharpoonup \hat{u} \ \text{ in } L^{r'}(Z) \ \text{ as } n \to \infty.$$

From Hu-Papageorgiou [30, p. 695] and since the multifunction $x \mapsto \partial j(z, x, \lambda)$ has closed graph, we have

$$\hat{u}(z) \in \partial j(z, \hat{y}(z), \lambda) \ \text{ a.e. on } Z.$$

In (94) we act with the test function $x_n - \hat{y} \in W_0^{1,p}(Z)$ obtaining

$$\langle A(x_n), x_n - \hat{y}\rangle = \int_Z u_n(x_n - \hat{y})dz \to 0. \tag{95}$$

Invoking Lemma 3.5, from (95), we infer that $x_n \to \hat{y}$ in $W_0^{1,p}(Z)$. So, if in (94) we pass to the limit as $n \to \infty$, we obtain

$$A(\hat{y}) = \hat{u}$$

with $\hat{u} \in L^{r'}(Z)$, $\hat{u}(z) \in \partial j(z, \hat{y}(z), \lambda)$ a.e. on Z. Hence $\hat{y} \in S_+$ and $\hat{y} = \inf D$, i.e., $\hat{y}$ is a lower bound of D. Since $D \subseteq S_+$ was fixed arbitrarily, invoking Zorn's lemma, we obtain a minimal element $x_* \in S_+$ of the set S_+.

Step 3. The minimal element $x_* \in S_+$ is the smallest element of S_+.
Suppose the contrary, i.e., there exists an element $\underline{x}_* \in S_+$ such that $x_* \not\leq \underline{x}_*$. Since S_+ is downward directed (Step 1), there exists $x \in S_+$ such that $x \leq x_*$ and $x \leq \underline{x}_*$. Since x_* is a minimal element of S_+ (Step 2), we have necessarily $x = x_*$, so, $x_* \leq \underline{x}_*$. But this contradicts $x_* \not\leq \underline{x}_*$. This concludes our proof. ☐

Proposition 4.3. *If hypotheses (H_f) hold and $\lambda \in (0, \lambda^*)$, then problem (P_λ) has a smallest positive solution $y_+ \in \mathrm{int}K_+$ and a biggest negative solution $y_- \in -\mathrm{int}K_+$.*

Proof. As before, we deal only with existence of the smallest positive solution $y_+ \in \mathrm{int}K_+$; the other part goes in a similar way.

Let $\underline{x}_n = \varepsilon_n u_1$ with $\varepsilon_n \to 0^+$ and set $T_+^n = [\underline{x}_n, \overline{x}]$. Using Proposition 4.2 we may fix $x_n^* \in T_+^n$ as the smallest solution for problem (P_λ) in the order interval T_+^n. Recall that $\{x_n^*\}_{n \geq 1} \subset W_0^{1,p}(Z)$ is bounded and so we may assume that

$$x_n^* \rightharpoonup y_+ \ \text{ in } W_0^{1,p}(Z) \text{ and } x_n^* \to y_+ \ \text{ in } L^p(Z) \text{ as } n \to \infty.$$

We have

$$A(x_n^*) = u_n^*, \quad n \geq 1, \tag{96}$$

with $u_n^* \in L^{r'}(Z)$, $u_n^*(z) \in \partial j(z, x_n^*(z), \lambda)$ a.e. on Z. In (96) we act with the test function $x_n^* - y_+$ and as before, passing to the limit as $n \to \infty$ and using Lemma 3.5, we have

$$x_n^* \to y_+ \ \text{ in } W_0^{1,p}(Z) \ \text{ as } n \to \infty.$$

We divide the proof into three steps.

Step 1. $y_+ \neq 0$.
Suppose that $y_+ = 0$. Then $\|x_n^*\| \to 0$ as $n \to \infty$. Since $x_n^* \neq 0$, we may set $w_n = \frac{x_n^*}{\|x_n^*\|}$, $n \geq 1$. By passing to a suitable subsequence if necessary, we may assume that

$$w_n \rightharpoonup w \ \text{ in } W_0^{1,p}(Z) \text{ and } w_n \to w \ \text{ in } L^p(Z) \text{ as } n \to \infty.$$

From (96), we have

$$A(w_n) = \frac{u_n^*}{\|x_n^*\|^{p-1}}, \ n \geq 1. \tag{97}$$

First, we show that the sequence $\left\{\frac{u_n^*}{\|x_n^*\|^{p-1}}\right\}_{n \geq 1} \subset L^{p'}(Z)$ is bounded. Hypothesis $(H_f)(iv)$ implies that we can find $\delta > 0$ such that

$$\eta \leq \frac{u}{|x|^{p-2}x} \leq \hat{\eta}(z) + 1 \tag{98}$$

for a.a. $z \in Z$, $0 < |x| \leq \delta$, all $u \in \partial j(z, x, \lambda)$ (see relation (82)) with $\eta > \lambda_2$. Moreover, due to hypothesis $(H_f)(iii)$, we have

$$|u| \leq \left(\frac{a(z, \lambda)}{\delta^{r-1}} + c\right) |x|^{r-1} \tag{99}$$

for a.a. $z \in Z$, all $|x| \geq \delta$ and all $u \in \partial j(z,x,\lambda)$. Combining (98) and (99), we have

$$|u| \leq c_{10}(|x|^{p-1} + |x|^{r-1})$$

for a.a. $z \in Z$, all $x \in \mathbb{R}$, all $u \in \partial j(z,x,\lambda)$, with $c_{10} > 0$. Hence

$$\begin{aligned} \frac{|u_n^*(z)|}{\|x_n^*\|^{p-1}} &\leq c_{10}\left(1 + |x_n^*(z)|^{r-p}\right)|w_n(z)|^{p-1} \\ &\leq c_{11}|w_n(z)|^{p-1} = c_{11}w_n(z)^{p-1} \end{aligned}$$

a.e. on Z, with $c_{11} > 0$ (recall that $0 \leq x_n^* \leq \overline{x}$). Consequently, the sequence is $\left\{\frac{u_n^*}{\|x_n^*\|^{p-1}}\right\}_{n\geq 1}$ is bounded in $L^{p'}(Z)$.

Now, acting with the test function $w_n - w$ in (97), passing to the limit as $n \to \infty$ and using Lemma 3.5, we obtain $w_n \to w$ in $W_0^{1,p}(Z)$. In particular, we have that $w \geq 0$ and $\|w\| = 1$.

Since the sequence $\left\{\frac{u_n^*}{\|x_n^*\|^{p-1}}\right\}_{n\geq 1} \subset L^{p'}(Z)$ is bounded, we may assume that

$$\theta_n = \frac{u_n^*}{\|x_n^*\|^{p-1}} \rightharpoonup \theta \ \text{ in } L^{p'}(Z) \text{ as } n \to \infty. \tag{100}$$

From the above facts we deduce that

$$\theta(z) = 0 \ \text{ a.e. on } \{w = 0\}. \tag{101}$$

Now, we are interested on the behaviour of θ on the set $\{w > 0\}$. To do this, for given $\varepsilon > 0$ and $n \geq 1$, we introduce the set

$$E_+^n = \left\{z \in Z : \eta - \varepsilon \leq \frac{u_n^*}{x_n^*(z)^{p-1}} \leq \hat{\eta}(z) + \varepsilon\right\}.$$

Recall that $\|x_n^*\| \to 0$ and, so, at least for a subsequence, we have $x_n^*(z) \to 0^+$ a.e. on $\{w > 0\}$. By virtue of hypothesis $(H_f)(iv)$ we have

$$\chi_{E_+^n}(z) \to 1 \ \text{ a.e. on } \ \{w > 0\}$$

By the dominated convergence theorem we have

$$\left\|(1 - \chi_{E_+^n})\frac{u_n^*}{\|x_n^*\|^{p-1}}\right\|_{L^{p'}(\{w>0\})} \to 0$$

as $n \to \infty$. Therefore, from (100) it follows that

$$\chi_{E_+^n}\frac{u_n^*}{\|x_n^*\|^{p-1}} \rightharpoonup \theta \ \text{ in } L^{p'}(\{w > 0\}).$$

From the definition of the set E_+^n, we have

$$\begin{aligned} \chi_{E_+^n}(z)(\eta - \varepsilon)w_n(z)^{p-1} &\leq \chi_{E_+^n}(z)\frac{u_n^*}{x_n^*(z)^{p-1}}w_n(z)^{p-1} = \chi_{E_+^n}(z)\theta_n(z) \\ &\leq \chi_{E_+^n}(z)(\hat{\eta}(z) + \varepsilon)w_n(z)^{p-1} \ \text{ a.e. on } \{w > 0\}. \end{aligned}$$

Taking weak limits in $L^{p'}(\{w > 0\})$, using Mazur's lemma and letting $\varepsilon \to 0^+$, we obtain

$$\eta w(z)^{p-1} \leq \theta(z) \leq \hat{\eta}(z)w(z)^{p-1} \ \text{ a.e. on } \{w > 0\}. \tag{102}$$

Consequently, from (102) and (101), we infer that

$$\theta(z) = \hat{\xi}(z)w(z)^{p-1} \ \text{ a.e. on } Z, \tag{103}$$

with $\hat{\xi} \in L^\infty(Z)_+$, $\eta \leq \hat{\xi}(z) \leq \hat{\eta}(z)$ a.e. on Z. We pass to the limit as $n \to \infty$ in (97) and due to (103), we obtain

$$A(w) = \hat{\xi} w^{p-1}.$$

Hence, we have that

$$\begin{cases} -\triangle_p w(z) = \hat{\xi}(z) w(z)^{p-1} & \text{a.e. on } Z, \\ w|_{\partial Z} = 0, \ w \geq 0. \end{cases} \tag{104}$$

Note that $\hat{\lambda}_1(\hat{\xi}) < \hat{\lambda}_1(\lambda_2) < \hat{\lambda}_1(\lambda_1) = 1$ and so from (104), we see that $w \in C_0^1(\overline{Z})$ (nonlinear regularity theory) must be 0. But we know that $\|w\| = 1$, so we have a contradiction. This proves that we cannot have $y_+ = 0$, i.e., $y_+ \geq 0$, $y_+ \neq 0$.

Step 2. $y_+ \in \text{int}K_+$.

A similar argument as in Proposition 4.2 (Step 2) shows that

$$A(y_+) = u_+ \tag{105}$$

with $u_+ \in L^{r'}(Z)$, $u_+(z) \in \partial j(z, y_+(z), \lambda)$ a.e. on Z. Consequently, from (105) we have

$$\begin{cases} -\triangle_p y_+(z) = u_+(z) & \text{a.e. on } Z, \\ y_+|_{\partial Z} = 0. \end{cases}$$

Then $y_+ \in K_+ \setminus \{0\}$ (nonlinear regularity theory) and by virtue of hypothesis $(H_f)(vi)$ we have $u_+(z) \geq 0$ a.e. on Z. So $\triangle_p y_+(z) \leq 0$ a.e. on Z. Invoking the strong maximum principle of Vázquez [43], we conclude that $y_+ \in \text{int}K_+$.

Step 3. y_+ is the smallest positive solution for problem (P_λ).

Let us assume that $\hat{y}$ is a nontrivial positive solution with $\hat{y} \leq \overline{x}$. As before, via the sign changing condition (hypothesis $(H_f)(vi)$) and the strong maximum principle, we check that $\hat{y} \in \text{int}K_+$. Then Lemma 3.3 implies that we can find $\varepsilon_0 > 0$ small such that $\varepsilon_0 u_1 \leq \hat{y}$. Consequently, for large $n \geq 1$, we have $\varepsilon_n u_1 \leq \varepsilon_0 u_1 \leq \hat{y} \leq \overline{x}$. This fact implies that $\hat{y}$ is a solution for problem (P_λ) in the ordered interval T_+^n. On the other hand, x_n^* is the smallest solution for problem (P_λ) in T_+^n. Thus, $x_n^* \leq \hat{y}$ for enough large $n \geq 1$. Passing to the limit, we have $y_+ \leq \hat{y}$, i.e., y_+ is indeed the smallest positive solution for problem (P_λ).

In a similar way, working on $T_-^n = [\underline{v}, \overline{v}_n]$ with $\overline{v}_n = \varepsilon_n(-u_1)$, we obtain $y_- \in -\text{int}K_+$, the biggest negative solution for problem (P_λ). In this case, if we denote by v_n^* the biggest solution for the problem (P_λ) in T_-^n, instead of $w_n = \frac{x_n^*}{\|x_n^*\|}$ and E_+^n, we have to work with $\tilde{w}_n = \frac{v_n^*}{\|v_n^*\|}$ and

$$E_-^n = \left\{ z \in Z : \ \eta - \varepsilon \leq \frac{u_n^*}{|v_n^*(z)|^{p-2} v_n^*(z)} \leq \hat{\eta}(z) + \varepsilon \right\},$$

respectively, where $u_n^* = A(v_n^*)$ with $u_n^* \in L^{r'}(Z)$, $u_n^*(z) \in \partial j(z, v_n^*(z), \lambda)$ a.e. on Z. $\square$

Remark 6. In order to prove Propositions 4.1 and 4.3, in hypothesis $(H_f)(iv)$ it is enough to have the first eigenvalue λ_1 instead of λ_2. However, the presence of λ_2 in $(H_f)(iv)$ is crucial in the proof of the main results (Theorems 4.1 and 4.2), see below.

Proof of Theorem 4.1. Fix $\lambda \in (0, \lambda^*)$. From Theorem 3.1 we already have two solutions $x_0 = x_0(\lambda) \in \text{int}K_+$ and $v_0 = v_0(\lambda) \in -\text{int}K_+$.

Now, let $y_+ \in \mathrm{int}K_+$ be the smallest positive solution and $y_- \in \mathrm{int}K_+$the biggest negative solution for problem (P_λ) obtained in Proposition 4.3. We have

$$A(y_\pm) = u_\pm \tag{106}$$

with $u_\pm \in L^{r'}(Z)$, $u_\pm(z) \in \partial j(z, y_\pm(z), \lambda)$ a.e. on Z. Then, we introduce the modifications of the discontinuous nonlinearity f as follows:

$$f_+(z,x,\lambda) = \begin{cases} 0, & \text{if} \quad x < 0 \\ f(z,x,\lambda), & \text{if} \quad 0 \le x \le y_+(z) \\ u_+(z), & \text{if} \quad y_+(z) < x, \end{cases}$$

$$f_-(z,x,\lambda) = \begin{cases} u_-(z), & \text{if} \quad x < y_-(z) \\ f(z,x,\lambda), & \text{if} \quad y_-(z) \le x \le 0 \\ 0, & \text{if} \quad 0 < x, \end{cases}$$

and

$$\hat{f}(z,x,\lambda) = \begin{cases} u_-(z), & \text{if} \quad x < y_-(z) \\ f(z,x,\lambda), & \text{if} \quad y_-(z) \le x \le y_+(z) \\ u_+(z), & \text{if} \quad \overline{y}_+(z) < x. \end{cases}$$

Using these functions, we introduce the corresponding potentials, defined by

$$j_\pm(z,x,\lambda) = \int_0^x f_\pm(z,s,\lambda)ds \quad \text{and} \quad \hat{j}(z,x,\lambda) = \int_0^x \hat{f}(z,s,\lambda)ds.$$

Finally, using these potentials we introduce the corresponding locally Lipschitz Euler functionals defined on $W_0^{1,p}(Z)$:

$$\varphi_\pm(x,\lambda) = \frac{1}{p}\|Dx\|_p^p - \int_Z j_\pm(z,x(z),\lambda)dz$$

and

$$\hat{\varphi}(x,\lambda) = \frac{1}{p}\|Dx\|_p^p - \int_Z \hat{j}(z,x(z),\lambda)dz$$

for all $x \in W_0^{1,p}(Z)$ and all $\lambda \in (0, \lambda^*)$.

We also introduce the following order intervals in $W_0^{1,p}(Z)$

$$J_\pm = [0, y_\pm], \quad \text{and} \quad \hat{J} = [y_-, y_+].$$

We divide the proof into four steps.

Step 1. The critical points of $\varphi_\pm(\cdot,\lambda)$ are in $J_\pm$. Similarly, the critical points of $\hat{\varphi}(\cdot,\lambda)$ are in $\hat{J}$.

We show this for $\varphi_+(\cdot,\lambda)$, the proof for $\varphi_-(\cdot,\lambda)$ and $\hat{\varphi}(\cdot,\lambda)$ being similar. So, let $x \in W_0^{1,p}(Z)$ be a critical point of $\varphi_+(\cdot,\lambda)$, i.e., $0 \in \partial\varphi_+(x,\lambda)$. Then

$$A(x) = u \tag{107}$$

with $u \in L^{r'}(Z)$, $u(z) \in \partial j_+(z,x(z),\lambda)$ a.e. on Z. Acting with the test function $(x-y_+)^+ \in W_0^{1,p}(Z)$ in (107) we obtain

$$\begin{aligned} \langle A(x), (x-y_+)^+\rangle &= \int_Z u(x-y_+)^+ dz \\ &= \int_{\{x>y_+\}} u(x-y_+)dz \\ &= \int_Z u_+(x-y_+)^+ dz \quad \text{(recall the definition of } f_+) \\ &= \langle A(y_+), (x-y_+)^+\rangle \quad \text{(see (106))}. \end{aligned}$$

Consequently, $\langle A(x)-A(y_+),(x-y_+)^+\rangle = 0$. Thus,

$$\int_{\{x>y_+\}} (\|Dx\|_{\mathbb{R}^N}^{p-2}Dx - \|Dy_+\|_{\mathbb{R}^N}^{p-2}Dy_+, Dx-Dy_+)_{\mathbb{R}^N}dz = 0.$$

Due to strict monotonicity, we should have $|\{x>y_+\}|_N = 0$, i.e., $x \le y_+$.

Now, acting with the test function x^- in (107) we have $\langle A(x), x^-\rangle = 0$, due to the definition of f_+. Therefore, $x^- = 0$, i.e., $x \ge 0$.

Step 2. The set of critical points of $\varphi_\pm(\cdot,\lambda)$ is $\{0, y_\pm\}$.
Again, we deal only with $\varphi_+(\cdot,\lambda)$. Assume the contrary, i.e., there exists a critical point y of $\varphi_+(\cdot,\lambda)$ different of 0 and y_+. Using Step 1, the element y belongs to $J_+ = [0,y_+]$. In particular, by $(H_f)(vi)$ and the strong maximum principle, y is a positive solution for (P_λ). But this contradicts the fact that y_+ is the smallest positive solution for (P_λ).

Step 3. The elements y_+ and y_- are local minimizers of $\hat{\varphi}(\cdot,\lambda)$.
Because of hypothesis $(H_f)(iv)$ and relation (82) we can find $\delta>0$ small such that

$$\lambda_2 x^{p-1} \le u \quad \text{for a.a. } z\in Z, \text{ all } 0\le x\le \delta \text{ and all } u\in\partial j(z,x,\lambda). \tag{108}$$

We choose $\varepsilon>0$ small such that

$$\varepsilon u_1(z) \le \min\{y_+(z),\delta\} \ \text{ for all } z\in\overline{Z}. \tag{109}$$

Using (108), (109), (82) and recalling the definition of j_+, we have

$$j_+(z,\varepsilon u_1(z),\lambda) = \int_0^{\varepsilon u_1(z)} f(z,s,\lambda)ds \ge \frac{\lambda_2}{p}\varepsilon^p u_1(z)^p \tag{110}$$

a.e. on Z and all $\lambda\in(0,\lambda^*)$. Hence

$$\begin{aligned}\varphi_+(\varepsilon u_1,\lambda) &= \frac{\varepsilon^p}{p}\|Du_1\|_p^p - \int_Z j_+(z,\varepsilon u_1(z),\lambda)dz\\ &\le \frac{\varepsilon^p}{p}\int_Z(\lambda_1-\lambda_2)u_1(z)^p dz < 0 \quad (\text{see } (110)).\end{aligned}$$

Consequently,

$$\inf\varphi_+(\cdot,\lambda) < 0 = \varphi_+(0,\lambda). \tag{111}$$

From hypothesis $(H_f)(iii)$ and the definition of $j_+(z,x,\lambda)$, for some $a\in L^\infty(Z)_+$, we have

$$|j_+(z,x,\lambda)| \le a(z)|x| \quad \text{for a.a. } z\in Z, \text{ all } x\in\mathbb{R} \text{ and all } \lambda\in(0,\lambda^*). \tag{112}$$

From (112) and Poincaré's inequality we deduce that the functional $\varphi_+(\cdot,\lambda)$ is coercive. It is easy to see that $\varphi_+(\cdot,\lambda)$ is sequentially weakly lower semicontinuous on $W_0^{1,p}(Z)$. Thus, we can find $y_+^0\in W_0^{1,p}(Z)$ a minimizer of $\varphi_+(\cdot,\lambda)$. We have

$$\varphi_+(y_+^0,\lambda) < 0 = \varphi_+(0,\lambda) \quad (\text{see } (111)),$$

i.e., $y_+^0 \ne 0$.

Since y_+^0 is a nonzero critical point of $\varphi_+(\cdot,\lambda)$, on account of Step 2, we must have $y_+^0 = y_+$. On the other hand, since $y_+\in \mathrm{int}K_+$, we see that $y_+ = y_+^0$ is a local minimizer of $\varphi_+(\cdot,\lambda)$ in the $C_0^1(\overline{Z})$-topology, and on a small $C_0^1(\overline{Z})$-neighborhood of $y_+ = y_+^0$ the function $\varphi_+(\cdot,\lambda)$ coincides with $\hat{\varphi}(\cdot,\lambda)$, see the definitions of f_+ and $\hat{f}$. Therefore, y_+ is a local minimizer of $\hat{\varphi}(\cdot,\lambda)$ in the $C_0^1(\overline{Z})$-topology. Hence, invoking again the result of Kyritsi-Papageorgiou [32, Proposition 3], y_+ is also a local $W_0^{1,p}(Z)$-minimizer of $\hat{\varphi}(\cdot,\lambda)$. The same property holds for y_-, which concludes the proof of Step 3.

Step 4. There exists a critical point y_0 of $\hat{\varphi}(\cdot,\lambda)$, different from 0 and $y_{\pm}$. In particular, y_0 is a nodal solution for problem (P_λ).
We may assume that y_+ and y_- are isolated critical points (actually, minimizers) of $\hat{\varphi}(\cdot,\lambda)$. Indeed, let us assume the contrary for y_+ (a similar argument works clearly for y_-). So, let $\{x_n\}_{n\geq 1} \subset W_0^{1,p}(Z)$ be such that $x_n \notin \{0, y_+, y_-\}$ for all $n \geq 1$, $x_n \to y_+$ in $W_0^{1,p}(Z)$ and $0 \in \partial\hat{\varphi}(x_n,\lambda)$ for all $n \geq 1$. Due to Step 1, $x_n \in \hat{J}$ for all $n \geq 1$. Moreover, x_n should be nodal for all $n \geq 1$. Otherwise, let $x_{n_0} \geq 0$. By the strong maximum principle, we have $x_{n_0} \in \mathrm{int}K_+$ and by the definition of $\hat{f}$, x_{n_0} is a positive solution for (P_λ). Since $x_{n_0} \leq y_+$, and $x_{n_0} \neq y_+$, we have a contradiction to the fact that y_+ is the smallest positive solution for problem (P_λ). Therefore, in this way we produced a whole sequence of distinct nodal solutions for problem (P_λ) and Theorem 4.1 is proved.

Assume without loosing the generality that $\hat{\varphi}(y_-,\lambda) \leq \hat{\varphi}(y_+,\lambda)$. Arguing exactly as we did in the proof of Proposition 3.2 (see (47)-(54), and note that the hypothesis $2 \leq p$ was not used there), we may choose $\rho > 0$ small enough that

$$\hat{\varphi}(y_+,\lambda) < \inf\{\hat{\varphi}(x,\lambda) : x \in \partial B_\rho(y_+)\} \leq 0,$$

where $\partial B_\rho(y_+) = \{x \in W_0^{1,p}(Z) : \|x - y_+\| = \rho\}$. Now, we consider the sets

$$D = \partial B_\rho(y_+), \quad J = [y_-, y_+], \quad \text{and} \quad J_0 = \{y_-, y_+\}.$$

Clearly, $\{J_0, J\}$ is linking with D in $W_0^{1,p}(Z)$ (see Definition 2.1). Moreover, $\hat{\varphi}(\cdot,\lambda)$ is coercive and from this it follows easily that $\hat{\varphi}(\cdot,\lambda)$ satisfies the nonsmooth PS-condition. So, we can apply Theorem 2.2 and obtain a critical point y_0 of $\hat{\varphi}(\cdot,\lambda)$ such that

$$\hat{\varphi}(y_-,\lambda) \leq \hat{\varphi}(y_+,\lambda) < \hat{\varphi}(y_0,\lambda) = \inf_{\overline{\gamma}\in\overline{\Gamma}} \max_{t\in[-1,1]} \hat{\varphi}(\overline{\gamma}(t),\lambda), \tag{113}$$

where

$$\overline{\Gamma} = \{\overline{\gamma} \in C([-1,1], W_0^{1,p}(Z)) : \overline{\gamma}(-1) = y_-, \overline{\gamma}(1) = y_+\}.$$

It is clear from (113) that $y_0 \neq y_\pm$, while from Step 1, we have $y_0 \in J$. In particular, due to the definition of $\hat{f}$, the element y_0 is a solution for problem (P_λ).

It remains to prove that $y_0 \neq 0$; in this way, y_0 is a nodal solution for problem (P_λ), since it is located between y_- and y_+, and $y_0 \neq y_\pm$. To this end, it is enough to show that $\hat{\varphi}(y_0,\lambda) < 0 = \hat{\varphi}(0,\lambda)$. This will be achieved once we produce a path $\overline{\gamma}_0 \in \overline{\Gamma}$ such that

$$\hat{\varphi}(\overline{\gamma}_0(t),\lambda) < 0 \quad \text{for all} \quad t \in [-1,1].$$

In what follows, we construct such a path $\overline{\gamma}_0 \in \overline{\Gamma}$. First, because of hypothesis $(H_f)(iv)$ we can find $\delta_0 > 0$ such that

$$\lambda_2 + \delta_0 < \frac{f_l(z,x,\lambda)}{|x|^{p-2}x} \quad \text{for a.a } z \in Z, \text{ all } 0 < |x| \leq \delta_0. \tag{114}$$

By Rademacher's theorem, $\frac{d}{dx}j(z,x,\lambda)$ exists for a.a. $z \in Z$ and all $x \in \mathbb{R}$. Moreover, $\frac{d}{dx}j(z,x,\lambda) \in \partial j(z,x,\lambda) = [f_l(z,x,\lambda), f_u(z,x,\lambda)]$. So, integrating in (114), we obtain

$$\frac{1}{p}(\lambda_2 + \delta_0)|x|^p < j(z,x,\lambda) \quad \text{for a.a } z \in Z \text{ and all } 0 < |x| \leq \delta_0. \tag{115}$$

Let $\partial B_1^{L^p(Z)} = \{x \in L^p(Z) : \|x\|_p = 1\}$ and $U = W_0^{1,p}(Z) \cap \partial B_1^{L^p(Z)}$ endowed with relative $W_0^{1,p}(Z)$-topology. Also let $U_c = U \cap C_0^1(\overline{Z})$ equipped with the relative

$C_0^1(\overline{Z})$-topology. Recall that $\Gamma_U = \{\gamma_0 \in C([-1,1],U) : \gamma_0(-1) = -u_1, \gamma_0(1) = u_1\}$, see (3). Also, let

$$\Gamma_{U_c} = \{\gamma_0 \in C([-1,1],U_c) : \gamma_0(-1) = -u_1, \gamma_0(1) = u_1\}.$$

The density of U_c in U (for the $W_0^{1,p}(Z)$-topology) implies the density of Γ_{U_c} in Γ_U (for the $C([-1,1],U)$-topology). Due to (3), we can find $\hat{\gamma}_0 \in \Gamma_{U_c}$ such that

$$\max\{\|Dx\|_p^p : x \in \hat{\gamma}_0([-1,1])\} \leq \lambda_2 + \delta_0, \tag{116}$$

$\delta_0 > 0$ being from (114). Since $\hat{\gamma}_0 \in C([-1,1],U_c)$ and $-y_-, y_+ \in \operatorname{int}K_+$, we can find $\varepsilon > 0$ small such that

$$\varepsilon|x(z)| \leq \delta_0 \ \text{ for all } z \in \overline{Z}, \text{ all } x \in \hat{\gamma}_0([-1,1])$$

and

$$\varepsilon x \in [y_-, y_+] \ \text{ for all } \ x \in \hat{\gamma}_0([-1,1]).$$

If $x \in \hat{\gamma}_0([-1,1])$, then

$$\begin{aligned}
\hat{\varphi}(\varepsilon x, \lambda) &= \varphi_\lambda(\varepsilon x) \\
&= \frac{\varepsilon^p}{p}\|Dx\|_p^p - \int_Z j(z, \varepsilon x(z), \lambda)dz \\
&< \frac{\varepsilon^p}{p}\|Dx\|_p^p - \frac{\varepsilon^p}{p}(\lambda_2 + \delta_0)\|x\|_p^p \quad \text{(see (115))} \\
&\leq 0 \qquad \text{(see (116) and recall that } \|x\|_p = 1).
\end{aligned}$$

So, if we consider the continuous path $\gamma_0 = \varepsilon\hat{\gamma}_0$ joining $-\varepsilon u_1$ and εu_1, we have

$$\hat{\varphi}(\cdot, \lambda)|_{\gamma_0} < 0. \tag{117}$$

Now, we will produce a continuous path joining εu_1 and y_+ along which the functional $\hat{\varphi}(\cdot, \lambda)$ is also strictly negative. Due to Step 2, we know that the set of critical points of $\varphi_+(\cdot, \lambda)$ is $\{0, y_+\}$. We will apply Theorem 2.5 for the functional $\varphi_+(\cdot, \lambda)$, by choosing

$$a_+ = \varphi_+(y_+, \lambda) = \inf \varphi_+(\cdot, \lambda) < 0 \ \text{ and } \ b_+ = \varphi_+(0, \lambda) = 0.$$

Since $\varphi_+(\cdot, \lambda)$ is coercive, it satisfies the nonsmooth PS-condition. It is clear that $K_{a_+} = \{y_+\}$. Moreover, since y_+ is a local minimizer of (the continuous function) $\varphi_+(\cdot, \lambda)$, then the weak slope of $\varphi_+(\cdot, \lambda)$ at the point y_+ should be 0, i.e., $|d\varphi_+(\cdot, \lambda)|(y_+) = 0$. Thus, $y_+ \in K_{a_+}^{ws}$. Consequently, $K_{a_+} = K_{a_+}^{ws} = \{y_+\}$, and one can find a deformation $h : [0,1] \times \varphi_+(\cdot, \lambda)^{<b_+} \to \varphi_+(\cdot, \lambda)^{<b_+}$ of the set $\varphi_+(\cdot, \lambda)^{<b_+} = \{x \in W_0^{1,p}(Z) : \varphi_+(x, \lambda) < b_+ = 0\}$ such that

(a) $h(t, \cdot)|_{K_{a_+}} = id|_{K_{a_+}}$ for all $t \in [0,1]$;

(b) $h(1, \varphi_+(\cdot, \lambda)^{<b_+}) \subseteq \varphi_+(\cdot, \lambda)^{<a_+} \cup K_{a_+}$;

(c) $\varphi_+(h(t,x), \lambda) \leq \varphi_+(x, \lambda)$ for all $t \in [0,1]$ and all $x \in \varphi_+(\cdot, \lambda)^{<b_+}$.

Now, we define the path

$$\gamma_+(t) = h(t, \varepsilon u_1) \quad \text{for all } t \in [0,1].$$

First of all, γ_+ is well-defined. Indeed, $\varepsilon u_1 \in \varphi_+(\cdot, \lambda)^{<b_+}$, since $\varphi_+(\varepsilon u_1, \lambda) = \hat{\varphi}(\varepsilon u_1, \lambda) < 0$, see (117). Clearly, γ_+ is a continuous path and

- $\gamma_+(0) = h(0, \varepsilon u_1) = \varepsilon u_1$ (since h is a deformation, see Definition 2.4);
- $\gamma_+(1) = h(1, \varepsilon u_1) = y_+$ (note that $\varphi_+(\cdot, \lambda)^{<a_+} = \emptyset$ and $K_{a_+} = \{y_+\}$);
- $\varphi_+(\gamma_+(t), \lambda) = \varphi_+(h(t, \varepsilon u_1), \lambda) \leq \varphi_+(\varepsilon u_1, \lambda) < 0$ for all $t \in [0,1]$.

In this way, we produced a continuous path γ_+ joining εu_1 and y_+ such that

$$\varphi_+(\cdot, \lambda)|_{\gamma_+} < 0.$$

On the other hand, taking into account the definition of the functions f_+, j_+ and using hypothesis $(H_f)(vi)$ (sign condition), a simple computation shows that

$$\hat{\varphi}(x, \lambda) \leq \varphi_+(x, \lambda) \quad \text{for all } x \in W_0^{1,p}(Z).$$

Therefore,

$$\hat{\varphi}(\cdot, \lambda)|_{\gamma_+} < 0. \tag{118}$$

In a similar fashion, we can produce a continuous path γ_- joining $-\varepsilon u_1$ and y_- such that

$$\hat{\varphi}(\cdot, \lambda)|_{\gamma_-} < 0. \tag{119}$$

Concatinating the paths γ_-, γ_0, γ_+, we obtain a continuous path $\overline{\gamma}_0 \in \overline{\Gamma}$ and due to (117), (118) and (119), we have

$$\hat{\varphi}(\cdot, \lambda)|_{\overline{\gamma}_0} < 0.$$

From (113) it follows that $\hat{\varphi}(y_0, \lambda) < 0 = \hat{\varphi}(0, \lambda)$, thus, $y_0 \neq 0$.

Therefore, we have a nontrivial nodal solution for problem (P_λ) and by nonlinear regular theory, we have $y_0 \in C_0^1(\overline{Z})$. □

Proof of Theorem 4.2. We combine the proof of Theorems 3.2 and 4.1, respectively. □

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E-mail address: mfilip@math.ntua.gr
E-mail address: alexandrukristaly@yahoo.com
E-mail address: npapg@math.ntua.gr