A variational inequality on the half line

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Multiple solutions are obtained for a variational inequality defined on the half line $(0, \infty)$. Our approach is based on a key embedding result as well as on the non-smooth critical point theory for Szulkin-type functionals.

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1. Introduction

Variational inequalities either on bounded or unbounded domains describe real life phenomena from Mechanics and Mathematical Physics. A comprehensive monograph dealing with various forms of variational inequalities is due to Duvaut–Lions\textsuperscript{1}. Motivated also by some mechanical problems where certain non-differentiable term perturbs the classical function, Panagiotopoulos\textsuperscript{2} developed the so-called theory of hemivariational inequalities; see also Motreanu–Rădulescu\textsuperscript{3}.

The aim of the present paper is to study a variational inequality which is defined on the half line $(0, \infty)$ by exploiting variational arguments described below. The natural functional space we are dealing with is the well-known Sobolev space $W^{1,p}(0, \infty)$, $p > 1$. Since the domain is not bounded, the continuous embedding $W^{1,p}(0, \infty) \hookrightarrow L^\infty(0, \infty)$ is not compact. Moreover, since the domain is one-dimensional, the compactness cannot be regained from a symmetrization argument as in Esteban\textsuperscript{4}, Esteban–Lions\textsuperscript{5}, Kobayashi–Ôtani\textsuperscript{6}, Kristály\textsuperscript{7}. However, bearing in mind a specific construction from\textsuperscript{5}, it is convenient to introduce the closed, convex cone

\[ K = \{ u \in W^{1,p}(0, \infty) : u \geq 0, u \text{ is nonincreasing on } (0, \infty) \}. \]

The main result of Section 2 is to prove that the embedding $W^{1,p}(0, \infty) \hookrightarrow L^\infty(0, \infty)$ transforms the closed bounded sets from $K$ into compact sets, $p \in (1, \infty)$. This fact will be exploited (particularly, for $p = 2$) to obtain nontrivial solutions for a variational inequality defined on $(0, \infty)$, involving concave–convex nonlinearities. To be more precise, we consider the problem, denoted by $(P_\lambda)$: Find $(u, \lambda) \in K \times (0, \infty)$ such that

\[ Au(v - u) - \lambda \int_0^\infty a(x)|u(x)|^{q-2}u(x)(v(x) - u(x))\,dx - \int_0^\infty b(x)f(u(x))(v(x) - u(x))\,dx \geq 0, \quad \forall v \in K, \]

where

\[ Au(v - u) = \int_0^\infty u'(x)(v'(x) - u'(x))\,dx + \int_0^\infty u(x)(v(x) - u(x))\,dx, \]

and $q \in (1, 2)$, $a, b \in L^1(0, \infty)$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ has a suitable growth.

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By using the Ekeland variational principle and a non-smooth version of the Mountain Pass theorem for Szulkin-type functionals, we are able to guarantee the existence of $\lambda_0 > 0$ such that $(P_{\lambda_0})$ has two nontrivial solutions whenever $\lambda \in (0, \lambda_0)$.

The structure of the paper is as follows. In the next section we prove a compactness result; in Section 3 we recall some elements from the non-smooth critical point theory for Szulkin-type functionals; in Section 4 we state our main theorem and we prove some auxiliary results; and, in Section 5 we prove our main theorem.

2. A compactness result on $(0, \infty)$

We endow the space $W^{1,p}(0, \infty)$ by its natural norm
\[
\|u\| = \left( \int_0^\infty |u'|^p + \int_0^\infty |u|^p \right)^{1/p},
\]
and the space $L^\infty(0, \infty)$ by the standard sup-norm. The main result of this section is as follows.

**Proposition 2.1.** Let $p \in (1, \infty)$. The embedding $W^{1,p}(0, \infty) \hookrightarrow L^\infty(0, \infty)$ transforms the closed bounded sets from $K$ into compact sets.

**Proof.** We notice that every function $u \in W^{1,p}(0, \infty)$ ($p > 1$) admits a continuous representation, see Brézis [8]; in what follows, we will replace $u$ by this element. It is enough to consider a bounded sequence $\{u_n\}$ in $K$ and prove that there is a subsequence of it which converges strongly in $L^\infty(0, \infty)$. Taking a subsequence if necessary we may assume that $u_n \to u$ weakly in $W^{1,p}(0, \infty)$ for some $u \in W^{1,p}(0, \infty)$. Moreover, since $K$ is strongly closed and convex, then it is weakly closed; in particular $u \in K$.

Let us fix $y > 0$. Then
\[
|u_n(y) - u(y)|y \leq 2p|u_n(y) + u(y)|y \\
\leq 2p \int_0^y [u_n(x) + u(x)]dx \\
< 2p \left( \|u_n\|_{W^{1,p}} + \|u\|_{W^{1,p}} \right)^{1/p} y^{-1/p} < \varepsilon/2.
\]
Since $\{u_n\}$ is bounded in $W^{1,p}(0, \infty)$, dividing by $y > 0$ the above inequality, then for every $\varepsilon > 0$ there exits $R_\varepsilon > 0$ such that
\[
|u_n(y) - u(y)| < \|u_n\|_{W^{1,p}} + \|u\|_{W^{1,p}} y^{-1/p} < \varepsilon/2
\]
for every $y > R_\varepsilon$ and for every $n \in \mathbb{N}$. Thus
\[
\|u_n - u\|_{L^\infty(R_\varepsilon, \infty)} < \varepsilon, \quad \forall n \in \mathbb{N}. \tag{2.1}
\]
On the other hand, by Rellich theorem, $W^{1,p}(0, R_\varepsilon) \hookrightarrow C^0[0, R_\varepsilon]$ ($p > 1$) is compact. Since $u_n \to u$ in $W^{1,p}(0, \infty)$, in particular, $u_n \to u$ (strongly) in $C^0[0, R_\varepsilon]$, up to a subsequence, i.e., there exists $n_\varepsilon \in \mathbb{N}$ such that
\[
\|u_n - u\|_{C^0[0, R_\varepsilon]} < \varepsilon, \quad \forall n \geq n_\varepsilon.
\]
Combining this fact with (2.1), we obtain
\[
\|u_n - u\|_{L^\infty(0, \infty)} < \varepsilon, \quad \forall n \geq n_\varepsilon,
\]
and thus the claim is proven. □

3. Szulkin-type functionals

Let $X$ be a real Banach space and $X^*$ its dual. Let $E : X \to \mathbb{R}$ be a functional of class $C^1$ and let $\psi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper (i.e., $\not\equiv +\infty$), convex, lower semicontinuous function. Then, $I = E + \psi$ is a Szulkin-type functional, see [9]. An element $u \in X$ is called a critical point of $I = E + \psi$ if
\[
E'(u)(v - u) + \psi'(u)(v - u) \geq 0 \quad \text{for all } v \in X,
\]
or equivalently,
\[
0 \in E'(u) + \partial \psi(u) \quad \text{in } X^*.
\]
where $\partial \psi(u)$ stands for the subdifferential of the convex functional $\psi$ at $u \in X$.

**Proposition 3.1** ([9, p. 80]). Every local minimum point of $I = E + \psi$ is a critical point of $I$ in the sense of (3.1).

**Definition 3.1.** The functional $I = E + \psi$ satisfies the Palais–Smale condition at level $c \in \mathbb{R}$, (shortly, (PSZ)$_c$-condition) if every sequence $\{u_n\} \subset X$ such that $\lim_{n} I(u_n) = c$ and
\[
\langle E'(u_n), v - u_n \rangle_{X'} + \psi'(u_n) \geq -\varepsilon_n \| v - u_n \| \quad \text{for all } v \in X,
\]
where $\varepsilon_n \to 0$, possesses a convergent subsequence.
Let $X$ be a Banach space, $I = E + \psi : X \to \mathbb{R} \cup \{+\infty\}$ a Szulkin-type functional and we assume that

(i) $I(x) \geq \alpha$ for all $\|x\| = \rho$ with $\alpha, \rho > 0$, and $I(0) = 0$;

(ii) there is $e \in X$ with $\|e\| > \rho$ and $I(e) \leq 0$.

If $I$ satisfies the $(PS)_c$-condition for

$$ c = \inf \{ \rho \in [0,1] \mid \gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e \} $$

then $c$ is a critical value of $I$ and $c \geq \alpha$.

4. Main theorem and related results

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. We denote by $F(s) = \int_0^s f(t)dt$. We assume that

(f1): There exists $p > 2$ such that $f(s) = O(|s|^{p-1})$ as $s \to 0$.

(f2): There exists $v > p$ such that

$$ vF(s) - f(s)s \leq 0, \quad \forall s \in \mathbb{R}. $$

(f3): There exists $R > 0$ such that

$$ \max_{s \in [0,1]} F(s) > 0. $$

We shall prove the following theorem which represents the main result of this paper.

**Theorem 4.1.** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies (f1)-(f3), $q \in (1, 2)$, and $a, b \in L^1(0, \infty)$ with $a, b > 0$. Then there exists $\lambda_0 > 0$ such that $(P_\lambda)$ has at least two nontrivial, distinct solutions $u^\lambda_1, u^\lambda_2 \in K$ whenever $\lambda \in (0, \lambda_0)$.

For every $\lambda > 0$, we define the functional $I_\lambda : W^{1,2}(0, \infty) \to \mathbb{R}$ by

$$ I_\lambda(u) = \frac{1}{2} \|u\|^2 - \frac{\lambda}{q} \int_0^\infty a(x)|u|^q dx - F(u), $$

where

$$ F(u) = \int_0^\infty b(x)F(u(x))dx. $$

Due to the continuous embedding $W^{1,2}(0, \infty) \hookrightarrow L^\infty(0, \infty)$, and $a, b \in L^1(0, \infty)$, the functional $I_\lambda$ is well defined and of class $C^1$ on $W^{1,2}(0, \infty)$.

We define the indicator function of the set $K$, i.e.

$$ \psi_K(u) = \begin{cases} 0, & \text{if } u \in K, \\ +\infty, & \text{if } u \not\in K. \end{cases} $$

The function $\psi_K$ is convex, proper, and lower semicontinuous. In conclusion, $I_\lambda = E_\lambda + \psi_K$ is a Szulkin-type functional. Moreover, one easily obtains the following

**Proposition 4.1.** Fix $\lambda > 0$ arbitrarily. Every critical point $u \in W^{1,2}(0, \infty)$ of $I_\lambda = E_\lambda + \psi_K$ is a solution of $(P_\lambda)$.

**Proof.** Since $u \in W^{1,2}(0, \infty)$ is a critical point of $I_\lambda = E_\lambda + \psi_K$, one has

$$ E'_{\lambda}(u)(v - u) + \psi_K(v) - \psi_K(u) \geq 0, \quad \forall v \in W^{1,2}(0, \infty). $$

In particular, $u$ necessarily belongs to $K$. In case $u$ does not belong to $K$ we get $\psi_K(u) = +\infty$. Taking then, for instance $v = 0 \in K$ in the above inequality, we reach a contradiction. Now, we fix $v \in K$ arbitrarily. Since

$$ E'_{\lambda}(u)(v - u) = Au(v - u) - \lambda \int_0^\infty a(x)|u|^q u^q dx - \int_0^\infty b(x)f(u(x))(v(x) - u(x))dx, $$

the desired inequality follows. \qed

We shall show next that $I_\lambda = E_\lambda + \psi_K$ fulfills the $(PS)_c$-condition for every $c \in \mathbb{R}$.

**Proposition 4.2.** If the continuous function $f : \mathbb{R} \to \mathbb{R}$ verifies (f2) then $I_\lambda = E_\lambda + \psi_K$ satisfies $(PS)_c$, for every $\lambda > 0$ and $c \in \mathbb{R}$.

**Proof.** Let $\lambda > 0$ and $c \in \mathbb{R}$ be some fixed numbers. Let $\{u_n\}$ be a sequence in $W^{1,2}(0, \infty)$ such that

$$ I_\lambda(u_n) = E_\lambda(u_n) + \psi_K(u_n) \to c; \quad E'_{\lambda}(u_n)(v - u_n) + \psi_K(v) - \psi_K(u_n) \geq -\varepsilon_n \|v - u_n\|, \quad \forall v \in W^{1,2}(0, \infty). $$

(4.1)

(4.2)
There exist a sequence \( \epsilon_n \) being a sequence in \([0, \infty)\) with \( \epsilon_n \to 0 \). By (4.1) one concludes that the sequence \( \{u_n\} \) belongs entirely to \( K \). Setting \( v = 2u_n \) in (4.2), we obtain
\[
E_1(u_n)(u_n) \geq -\epsilon_n \| u_n \|.
\]
Therefore, we derive
\[
\| u_n \|^2 - \lambda \int_0^\infty a(x)|u_n|^q\,dx - \int_0^\infty b(x)f(u_n(x))u_n(x)\,dx \geq -\epsilon_n \| u_n \|.
\]
(4.3)

By (4.1) for large \( n \in \mathbb{N} \) we get
\[
c + 1 \geq \frac{1}{2} \| u_n \|^2 - \lambda \int_0^\infty a(x)|u_n|^q\,dx - \int_0^\infty b(x)f(u_n(x))\,dx.
\]
(4.4)

Multiplying (4.3) by \( v^{-1} \), adding this one to (4.4) and applying the Hölder inequality, for large \( n \in \mathbb{N} \) we obtain
\[
c + 1 + \frac{1}{v} \| u_n \| \geq \left( \frac{1}{2} - \frac{1}{v} \right) \| u_n \|^2 - \lambda \left( \frac{1}{q} - \frac{1}{v} \right) \int_0^\infty a(x)|u_n|^q\,dx
\]
\[
- \frac{1}{v} \int_0^\infty b(x) [f(u_n(x))u_n(x) + v F(u_n(x))]\,dx
\]
\[
\geq \left( \frac{1}{2} - \frac{1}{v} \right) \| u_n \|^2 - \lambda \left( \frac{1}{q} - \frac{1}{v} \right) \| a \|_{L^q} \| u_n \|^q
\]
\[
\geq \left( \frac{1}{2} - \frac{1}{v} \right) \| u_n \|^2 - \lambda \left( \frac{1}{q} - \frac{1}{v} \right) \| a \|_{L^q} k_\infty^q \| u_n \|^q,
\]
where \( k_\infty > 0 \) is the best Sobolev constant of the embedding \( W^{1,2}(0, \infty) \hookrightarrow L^\infty(0, \infty) \). Since \( q < 2 < v \), from the above estimation we derive that the sequence \( \{u_n\} \) is bounded in \( K \). Therefore, due to Proposition 2.1, up to a subsequence, we can suppose that
\[
\begin{align*}
    u_n &\to u \quad \text{weakly in } W^{1,2}(0, \infty); \\
    u_n &\to u \quad \text{strongly in } L^\infty(0, \infty).
\end{align*}
\]
(4.5)

As \( K \) is (weakly) closed, \( u \in K \). Setting \( v = u \) in (4.2), we obtain
\[
Au_n(u_n - u) + \int_0^\infty b(x)f(u_n(x))(u_n(x) - u(x))\,dx - \lambda \int_0^\infty a(x)|u_n|^{q-2}u_n(u_n - u)\,dx \geq -\epsilon_n \| u - u_n \|.
\]
Therefore, for large \( n \in \mathbb{N} \), we have
\[
\begin{align*}
    \| u - u_n \|^2 &\leq Au(u-u_n) + \int_0^\infty b(x)f(u_n(x))(u_n(x) - u(x))\,dx - \lambda \int_0^\infty a(x)|u_n|^{q-2}u_n(u_n - u)\,dx + \epsilon_n \| u - u_n \|
    \leq Au(u-u_n) + \| b \|_{L^1} \max_{s \in [-M, M]} \| f(s) \| \| u - u_n \|_{L^\infty} + \lambda \| a \|_{L^q} M^{q-1} \| u - u_n \|_{L^\infty} + \epsilon_n \| u - u_n \|,
\end{align*}
\]
where \( M = \| u \|_{L^\infty} + 1 \). Due to (4.5), we have
\[
\lim_n Au(u-u_n) = 0.
\]
Taking into account (4.6), the second and the third term in the last expression also tend to 0. Finally, since \( \epsilon_n \to 0^+ \), \( \{u_n\} \) converges strongly to \( u \) in \( W^{1,2}(0, \infty) \). This completes the proof. \( \square \)

5. Proof of Theorem 4.1

We assume throughout this section that all the hypotheses of Theorem 4.1 are fulfilled. The present section is divided into two parts; in the first subsection we guarantee the existence of a solution for problem (P\(_\lambda\)) by using the Mountain Pass theorem (see Theorem 3.1); the second subsection proves the existence of a second solution for the problem (P\(_\lambda\)) by applying a local minimization argument based on the Ekeland variational principle.

5.1. MP geometry of \( I_\lambda = E_\lambda + \psi_K \): the first solution of (P\(_\lambda\))

Lemma 5.1. There exist \( c_1, c_2 > 0 \) such that
\[
F(s) \geq c_1 s^p - c_2 s^q, \quad \forall s \geq 0.
\]
Due to (f3), there exists \( \rho_0 \in (0, R] \) such that \( F(\rho_0) > 0 \). Clearly, \( \rho_0 \neq 0 \), since \( F(0) = 0 \). We consider the function \( g : (0, \infty) \to \mathbb{R} \) defined by \( g(t) = t^{-1}F(t\rho_0) \). Let \( t > 1 \). By using a mean value theorem, there exists \( \tau \in (1, t) \) such that \( g(t) - g(1) = -t^{-1}F'(\tau t\rho_0) + t^{-1}F'(\tau_0 \rho_0)(t - 1) \). By (f2), one has \( g(t) \geq g(1) \), i.e., \( F(t\rho_0) \geq t^\alpha F(\rho_0) \) for every \( t \geq 1 \). Therefore, we have
\[
F(s) \geq \frac{F(\rho_0)}{\rho_0^\alpha} s^\alpha, \quad \forall s \geq \rho_0.
\]
On the other hand, by (f1), there exist \( \delta, L > 0 \) such that \( |F(s)| \leq L|s|^p \) for \( |s| \leq \delta \). In particular, we have that
\[
F(s) \geq -Ls^p, \quad \forall s \in [0, \delta).
\]
It remains to combine these two relations in order to obtain our claim.

**Proposition 5.1.** There exists \( \lambda_0 > 0 \) such that for every \( \lambda \in (0, \lambda_0) \) the following assertions are true:

(i) there exist constants \( \alpha_\lambda > 0 \) and \( \rho_\lambda > 0 \) such that \( I_\lambda(u) \geq \alpha_\lambda \) for all \( \|u\| = \rho_\lambda \);

(ii) there exist \( \varepsilon_\delta \in W^{1,2}(0, \infty) \) with \( \|\varepsilon_\delta\| > \rho_\lambda \) and \( I_\lambda(\varepsilon_\delta) \leq 0 \).

**Proof.** (i) Let \( \delta, L > 0 \) from the proof of the previous lemma. For \( u \in W^{1,2}(0, \infty) \) complying with \( \|u\|_\infty \leq \delta \), we have
\[
F(u) \leq L\|u\|_1\|u\|_\infty^p \leq L\|u\|_1 K_\delta^p \|u\|^p.
\]
It suffices to restrict our attention to elements \( u \) which belong to \( K \); otherwise \( I_\lambda(u) \) would be \( +\infty \), i.e. (i) holds trivially. Due to the above inequality, for every \( \lambda > 0 \) and \( u \in K \) with \( \|u\|_\infty \leq \delta \), we have
\[
I_\lambda(u) \geq \frac{1}{2} \|u\|^2 - \lambda A\|u\|_1\|u\|_\infty^q - L\|u\|_1 K_\delta^p \|u\|^p.
\]
where \( A = \|a\|_1 K_\delta^p > 0 \), and \( B = L\|u\|_1 K_\delta^p > 0 \).

For every \( 0 < \lambda < \frac{\beta p q \rho_\lambda - 2}{A (2 - q)} \), we define the function \( g_\lambda : (0, \delta) \to \mathbb{R} \) by
\[
g_\lambda(t) = \frac{1}{2} - \lambda A t^q - B t^{p-2}.
\]
Clearly, \( g_\lambda'(t_\lambda(\lambda)) = 0 \) if and only if \( t_\lambda(\lambda) = \left( \frac{\beta q \lambda - 2}{p - 2} \right)^{1/q} \). Moreover, \( g_\lambda(t_\lambda(\lambda)) = \frac{1}{2} - D\lambda \frac{q^{p-2}}{2} \), where \( D = D(p, q, A, B) > 0 \). Choosing \( 0 < \lambda_0 < \frac{\beta p q \rho_\lambda - 2}{A (2 - q)} \) so small that \( g_\lambda(t_\lambda(\lambda)) > 0 \), one clearly has for every \( \lambda \in (0, \lambda_0) \) that \( g_\lambda(t_\lambda(\lambda)) > 0 \). Therefore, for every \( \lambda \in (0, \lambda_0) \), setting \( \rho_\lambda = t_\lambda/\kappa_\infty \) and \( \alpha_\lambda = g_\lambda(t_\lambda)/\kappa_\infty^p \), the assertion from (i) holds true.

(ii) By Lemma 5.1 we have \( F(u) \geq \int_0^\infty b(x)[c_1 u^p - c_2 u^q]dx \) for every \( u \in K \). Then, for every \( u \in K \) we have
\[
I_\lambda(u) \geq \int_0^\infty b(x)[c_1 u^p - c_2 u^q]dx.
\]

Fix \( u_0(x) = \max(1 - x, 0), x > 0 \); it is clear that \( u_0 \in K \). Letting \( u = su_0 \in (5.1) \), we have that \( I_\lambda(su_0) \to -\infty \) as \( s \to +\infty \), since \( q > p > 2 > q \) and \( b > 0 \). Thus, for every \( \lambda \in (0, \lambda_0) \), it is possible to set \( s = s_\lambda \) so large that for \( e_\lambda = s_\lambda u_0 \), we have \( \|e_\lambda\| > \rho_\lambda \) and \( I_\lambda(e_\delta) \leq 0 \). This concludes the proof of the proposition.

By Proposition 4.2, the functional \( I_\lambda \) satisfies the (PS)\(_z\)-condition \( (c \in \mathbb{R}) \), and \( I_\lambda(0) = 0 \) for every \( \lambda > 0 \). Let us fix \( \lambda \in (0, \lambda_0) \). By Proposition 5.1 it follows that there exist constants \( \alpha_\lambda, \rho_\lambda > 0 \) and \( e_\delta \in W^{1,2}(0, \infty) \) such that \( I_\lambda \) fulfills the properties (i) and (ii) from Theorem 3.1. Therefore, the number
\[
c_1^\lambda = \inf_{\gamma \in \Gamma} \sup_{t \in [0, 1]} I_\lambda(\gamma(t))
\]
is a critical value of \( I_\lambda \) with \( c_1^\lambda \geq \alpha_\lambda > 0 \), where
\[
\Gamma = \{ \gamma \in C([0, 1], W^{1,2}(0, \infty)): \gamma(0) = 0, \gamma(1) = e_\delta \}.
\]
It is clear that the critical point \( u_\lambda^1 \in W^{1,2}(0, \infty) \) which corresponds to \( c_1^\lambda \) cannot be trivial since \( I_\lambda(u_\lambda^1) = c_1^\lambda > 0 = I_\lambda(0) \).

It remains to apply Proposition 4.1 which concludes that \( u_\lambda^1 \) is actually an element of \( K \) and a solution of (P\(_\lambda\)).

### 5.2. Local minimization; the second solution of (P\(_\lambda\))

Let us fix \( \lambda \in (0, \lambda_0) \) arbitrarily; \( \lambda_0 \) was defined in the previous subsection. By Proposition 5.1, there exists \( \rho_\lambda > 0 \) such that
\[
\inf_{\|u\| = \rho_\lambda} I_\lambda(u) > 0.
\]
Since \( a > 0 \), for \( u_0(x) = \max(1 - x, 0), x > 0 \), we have \( \int_0^\infty a(x)u_0^2 \, dx > 0 \). Taking into account that \( v > p > 2 > q \), for \( t > 0 \) small enough one has
\[
I_\lambda(tu_0) \leq \frac{t^2}{2} \|u_0\|^2 - \frac{\lambda t^q}{q} \int_0^\infty a(x)u_0^q \, dx - \int_0^\infty b(x)[c_1 t^r u_0^p - c_2 t^q u_0^p] \, dx < 0.
\]
For \( r > 0 \), we denote by \( B_r = \{u \in W^{1,2}(0, \infty) : \|u\| \leq r\} \) and by \( S_r = \{u \in W^{1,2}(0, \infty) : \|u\| = r\} \). Using these notations, relation (5.2) and the above inequality can be summarized as
\[
c^2_\lambda = \inf_{\lambda \in B_r} I_\lambda(u) < 0 < \inf_{u \in S_r} I_\lambda(u). \tag{5.3}
\]
A simple argument shows that \( c^2_\lambda \) is finite. Moreover, we will show that \( c^2_\lambda \) is another critical value of \( I_\lambda \). To this end, let \( n \in \mathbb{N} \setminus \{0\} \) such that
\[
\frac{1}{n} < \inf_{\lambda \in B_r} I_\lambda(u) - \inf_{\lambda \in B_r} I_\lambda(u).
\tag{5.4}
\]
By the Ekeland variational principle, applied to the lower semicontinuous functional \( I_{\lambda \in B_r} \), which is bounded below (see (5.3)), there exists \( u_{\lambda,n} \in B_{r_n} \) such that
\[
I_\lambda(u_{\lambda,n}) \leq \inf_{\lambda \in B_r} I_\lambda(u) + \frac{1}{n}; \tag{5.5}
\]
\[
I_\lambda(w) \geq I_\lambda(u_{\lambda,n}) - \frac{1}{n} \|w - u_{\lambda,n}\|, \quad \forall w \in B_{r_n}. \tag{5.6}
\]
By (5.4) and (5.5) we have that \( I_\lambda(u_{\lambda,n}) < \inf_{\lambda \in B_r} I_\lambda(u) \); therefore \( \|u_{\lambda,n}\| < r_n \).

Fix an element \( v \in W^{1,2}(0, \infty) \). It is possible to choose \( t > 0 \) small enough such that \( w = u_{\lambda,n} + t(v - u_{\lambda,n}) \in B_{r_n} \).

Applying (5.6) to this element, using the convexity of \( \psi_{\lambda} \) and dividing by \( t > 0 \), one concludes
\[
\frac{E_\lambda(u_{\lambda,n} + t(v - u_{\lambda,n})) - E_\lambda(u_{\lambda,n})}{t} = \psi_{\lambda}(v) - \psi_{\lambda}(u_{\lambda,n}) \geq -\frac{1}{n} \|v - u_{\lambda,n}\|.
\]
Letting \( t \to 0^+ \), we obtain
\[
E'_\lambda(u_{\lambda,n})(v - u_{\lambda,n}) + \psi_{\lambda}(v) - \psi_{\lambda}(u_{\lambda,n}) \geq -\frac{1}{n} \|v - u_{\lambda,n}\|. \tag{5.7}
\]

On the other hand, by (5.3) and (5.5) it follows
\[
I_\lambda(u_{\lambda,n}) = E'_\lambda(u_{\lambda,n}) + \psi_{\lambda}(u_{\lambda,n}) \to c^2_\lambda \tag{5.8}
\]
as \( n \to \infty \). Since \( v \) is arbitrarily fixed in (5.7), the sequence \( \{u_{\lambda,n}\} \) fulfills (4.1) and (4.2), respectively. Therefore, in a similar manner as in Proposition 4.2, we may prove that \( \{u_{\lambda,n}\} \) contains a convergent subsequence; we denote it again by \( \{u_{\lambda,n}\} \), its limit point being \( u^2_\lambda \). It is clear that \( u^2_\lambda \) belongs to \( B_{r_\lambda} \). By the lower semicontinuity of \( \psi_{\lambda} \) we have
\[
\psi_{\lambda}(u^2_\lambda) \leq \lim\inf_{n} \psi_{\lambda}(u_{\lambda,n}),
\]
and due to the fact that \( E_\lambda \) is of class \( C^1 \) on \( W^{1,2}(0, \infty) \), we have
\[
\lim_{n} E'_\lambda(u_{\lambda,n})(v - u_{\lambda,n}) = E'_\lambda(u^2_\lambda)(v - u^2_\lambda).
\]

Combining these relations with (5.7) we obtain
\[
E'_\lambda(u^2_\lambda)(v - u^2_\lambda) + \psi_{\lambda}(v) - \psi_{\lambda}(u^2_\lambda) \geq 0, \quad \forall v \in W^{1,2}(0, \infty),
\]
i.e. \( u^2_\lambda \) is a critical point of \( I_\lambda \). Moreover,
\[
c^2_\lambda = \inf_{\lambda \in B_r} I_\lambda(u) \leq I_\lambda(u^2_\lambda) \leq \lim\inf_{n} I_\lambda(u_{\lambda,n}) \to c^2_\lambda, \tag{5.8}
\]
i.e. \( I_\lambda(u^2_\lambda) = c^2_\lambda \). Since \( c^2_\lambda < 0 \) (see (5.3)), it follows that \( u^2_\lambda \) is not trivial. We apply again Proposition 4.1, concluding that \( u^2_\lambda \) is another solution of \( (P_\lambda) \) different from \( u^1_\lambda \). This concludes the proof of Theorem 4.1. \( \square \)

Acknowledgments

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References