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A variational inequality on the half line

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ABSTRACT

Multiple solutions are obtained for a variational inequality defined on the half line $(0, \infty)$. Our approach is based on a key embedding result as well as on the non-smooth critical point theory for Szulkin-type functionals.

1. Introduction

Variational inequalities either on bounded or unbounded domains describe real life phenomena from Mechanics and Mathematical Physics. A comprehensive monograph dealing with various forms of variational inequalities is due to Duvaut–Lions [1]. Motivated also by some mechanical problems where certain non-differentiable term perturbs the classical function, Panagiotopoulos [2] developed the so-called theory of hemivariational inequalities; see also Motreanu–Rădulescu [3].

The aim of the present paper is to study a variational inequality which is defined on the half line $(0, \infty)$ by exploiting variational arguments described below. The natural functional space we are dealing with is the well-known Sobolev space $W^{1,p}(0, \infty)$, $p > 1$. Since the domain is not bounded, the continuous embedding $W^{1,p}(0, \infty) \hookrightarrow L^\infty(0, \infty)$ is not compact. Moreover, since the domain is one-dimensional, the compactness cannot be regained from a symmetrization argument as in Esteban [4], Esteban–Lions [5], Kobayashi–Ôtani [6], Kristály [7]. However, bearing in mind a specific construction from [5], it is convenient to introduce the closed, convex cone

$$K = \{u \in W^{1,p}(0, \infty) : u \geq 0, u \text{ is nonincreasing on } (0, \infty)\}.$$

The main result of Section 2 is to prove that the embedding $W^{1,p}(0, \infty) \hookrightarrow L^\infty(0, \infty)$ transforms the closed bounded sets from K into compact sets, $p \in (1, \infty)$. This fact will be exploited (particularly, for $p = 2$) to obtain nontrivial solutions for a variational inequality defined on $(0, \infty)$, involving concave–convex nonlinearities. To be more precise, we consider the problem, denoted by (P_λ): *Find* $(u, \lambda) \in K \times (0, \infty)$ *such that*

$$Au(v - u) - \lambda \int_0^\infty a(x)|u(x)|^{q-2}u(x)(v(x) - u(x))\mathrm{d}x - \int_0^\infty b(x)f(u(x))(v(x) - u(x))\mathrm{d}x \geq 0, \quad \forall v \in K,$$

where

$$Au(v - u) = \int_0^\infty u'(x)(v'(x) - u'(x))\mathrm{d}x + \int_0^\infty u(x)(v(x) - u(x))\mathrm{d}x,$$

and $q \in (1, 2)$, $a, b \in L^1(0, \infty)$ and $f : \mathbb{R} \to \mathbb{R}$ has a suitable growth.

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By using the Ekeland variational principle and a non-smooth version of the Mountain Pass theorem for Szulkin-type functionals, we are able to guarantee the existence of $\lambda_0 > 0$ such that (P_λ) has two nontrivial solutions whenever $\lambda \in (0, \lambda_0)$.

The structure of the paper is as follows. In the next section we prove a compactness result; in Section 3 we recall some elements from the non-smooth critical point theory for Szulkin-type functionals; in Section 4 we state our main theorem and we prove some auxiliary results; and, in Section 5 we prove our main theorem.

2. A compactness result on $(0, \infty)$

We endow the space $W^{1,p}(0,\infty)$ by its natural norm

$$\|u\| = \left[\int_0^\infty |u|^p + \int_0^\infty |u'|^p\right]^{1/p},$$

and the space $L^\infty(0,\infty)$ by the standard sup-norm. The main result of this section is as follows.

Proposition 2.1. *Let $p \in (1,\infty)$. The embedding $W^{1,p}(0,\infty) \hookrightarrow L^\infty(0,\infty)$ transforms the closed bounded sets from K into compact sets.*

Proof. We notice that every function $u \in W^{1,p}(0,\infty)$ $(p > 1)$ admits a continuous representation, see Brézis [8]; in what follows, we will replace u by this element. It is enough to consider a bounded sequence $\{u_n\}$ in K and prove that there is a subsequence of it which converges strongly in $L^\infty(0,\infty)$. Taking a subsequence if necessary we may assume that $u_n \to u$ weakly in $W^{1,p}(0,\infty)$ for some $u \in W^{1,p}(0,\infty)$. Moreover, since K is strongly closed and convex, then it is weakly closed; in particular $u \in K$.

Let us fix $y > 0$. Then

$$\begin{aligned}|u_n(y) - u(y)|^p y &\le 2^p[u_n^p(y) + u^p(y)]y \\ &\le 2^p \int_0^y [u_n^p(x) + u^p(x)]\mathrm{d}x \\ &< 2^p[\|u_n\|^p_{W^{1,p}} + \|u\|^p_{W^{1,p}}].\end{aligned}$$

Since $\{u_n\}$ is bounded in $W^{1,p}(0,\infty)$, dividing by $y > 0$ the above inequality, then for every $\varepsilon > 0$ there exits $R_\varepsilon > 0$ such that

$$|u_n(y) - u(y)| < 2[\|u_n\|^p_{W^{1,p}} + \|u\|^p_{W^{1,p}}]^{1/p} y^{-1/p} < \varepsilon/2$$

for every $y > R_\varepsilon$ and for every $n \in \mathbb{N}$. Thus

$$\|u_n - u\|_{L^\infty(R_\varepsilon,\infty)} < \varepsilon, \quad \forall n \in \mathbb{N}. \tag{2.1}$$

On the other hand, by Rellich theorem, $W^{1,p}(0,R_\varepsilon) \hookrightarrow C^0[0,R_\varepsilon] (p > 1)$ is compact. Since $u_n \rightharpoonup u$ in $W^{1,p}(0,\infty)$, in particular, $u_n \to u$ (strongly) in $C^0[0,R_\varepsilon]$, up to a subsequence, i.e., there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\|u_n - u\|_{C^0[0,R_\varepsilon]} < \varepsilon, \quad \forall n \ge n_\varepsilon.$$

Combining this fact with (2.1), we obtain

$$\|u_n - u\|_{L^\infty(0,\infty)} < \varepsilon, \quad \forall n \ge n_\varepsilon,$$

and thus the claim is proven. □

3. Szulkin-type functionals

Let X be a real Banach space and X^* its dual. Let $E : X \to \mathbb{R}$ be a functional of class C^1 and let $\psi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper (i.e. $\not\equiv +\infty$), convex, lower semicontinuous function. Then, $I = E + \psi$ is a *Szulkin-type functional*, see [9]. An element $u \in X$ is called a *critical point of* $I = E + \psi$ if

$$E'(u)(v - u) + \psi(v) - \psi(u) \ge 0 \quad \text{for all } v \in X, \tag{3.1}$$

or equivalently,

$$0 \in E'(u) + \partial\psi(u) \quad \text{in } X^*,$$

where $\partial\psi(u)$ stands for the subdifferential of the convex functional ψ at $u \in X$.

Proposition 3.1 ([9, p. 80]). *Every local minimum point of $I = E + \psi$ is a critical point of I in the sense of (3.1).*

Definition 3.1. The functional $I = E + \psi$ satisfies the *Palais–Smale condition at level* $c \in \mathbb{R}$, (shortly, $(PSZ)_c$-condition) if every sequence $\{u_n\} \subset X$ such that $\lim_n I(u_n) = c$ and

$$\langle E'(u_n), v - u_n\rangle_X + \psi(v) - \psi(u_n) \ge -\varepsilon_n\|v - u_n\| \quad \text{for all } v \in X,$$

where $\varepsilon_n \to 0$, possesses a convergent subsequence.

The following version of the Mountain Pass theorem will be used in Section 5.1.

Theorem 3.1. *Let X be a Banach space, $I = E + \psi : X \to \mathbb{R} \cup \{+\infty\}$ a Szulkin-type functional and we assume that*
(i) *$I(x) \geq \alpha$ for all $\|x\| = \rho$ with $\alpha, \rho > 0$, and $I(0) = 0$;*
(ii) *there is $e \in X$ with $\|e\| > \rho$ and $I(e) \leq 0$.*

If I satisfies the $(PSZ)_c$-condition for

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),$$
$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\},$$

then c is a critical value of I and $c \geq \alpha$.

4. Main theorem and related results

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. We denote by $F(s) = \int_0^s f(t)\mathrm{d}t$. We assume that
(**f1**): There exists $p > 2$ such that $f(s) = O(|s|^{p-1})$ as $s \to 0$.
(**f2**): There exists $\nu > p$ such that

$$\nu F(s) - f(s)s \leq 0, \quad \forall s \in \mathbb{R}.$$

(**f3**): There exists $R > 0$ such that

$$\max_{s \in [0,R]} F(s) > 0.$$

We shall prove the following theorem which represents the main result of this paper.

Theorem 4.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies (**f1**)– (**f3**), $q \in (1, 2)$, and $a, b \in L^1(0, \infty)$ with $a, b > 0$. Then there exists $\lambda_0 > 0$ such that (P_λ) has at least two nontrivial, distinct solutions $u_\lambda^1, u_\lambda^2 \in K$ whenever $\lambda \in (0, \lambda_0)$.*

For every $\lambda > 0$, we define the functional $E_\lambda : W^{1,2}(0, \infty) \to \mathbb{R}$ by

$$E_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{q}\int_0^\infty a(x)|u|^q \mathrm{d}x - \mathcal{F}(u),$$

where

$$\mathcal{F}(u) = \int_0^\infty b(x)F(u(x))\mathrm{d}x.$$

Due to the continuous embedding $W^{1,2}(0, \infty) \hookrightarrow L^\infty(0, \infty)$, and $a, b \in L^1(0, \infty)$, the functional E_λ is well defined and of class C^1 on $W^{1,2}(0, \infty)$.

We define the indicator function of the set K, i.e.

$$\psi_K(u) = \begin{cases} 0, & \text{if } u \in K, \\ +\infty, & \text{if } u \notin K. \end{cases}$$

The function ψ_K is convex, proper, and lower semicontinuous. In conclusion, $I_\lambda = E_\lambda + \psi_K$ is a Szulkin-type functional. Moreover, one easily obtains the following

Proposition 4.1. *Fix $\lambda > 0$ arbitrarily. Every critical point $u \in W^{1,2}(0, \infty)$ of $I_\lambda = E_\lambda + \psi_K$ is a solution of (P_λ).*

Proof. Since $u \in W^{1,2}(0, \infty)$ is a critical point of $I_\lambda = E_\lambda + \psi_K$, one has

$$E'_\lambda(u)(v - u) + \psi_K(v) - \psi_K(u) \geq 0, \quad \forall v \in W^{1,2}(0, \infty).$$

In particular, u necessarily belongs to K. In case u does not belong to K we get $\psi_K(u) = +\infty$. Taking then, for instance $v = 0 \in K$ in the above inequality, we reach a contradiction. Now, we fix $v \in K$ arbitrarily. Since

$$E'_\lambda(u)(v - u) = Au(v - u) - \lambda \int_0^\infty a(x)|u(x)|^{q-2}u(x)(v(x) - u(x))\mathrm{d}x - \int_0^\infty b(x)f(u(x))(v(x) - u(x))\mathrm{d}x,$$

the desired inequality follows. □

We shall show next that $I_\lambda = E_\lambda + \psi_K$ fulfills the $(PSZ)_c$-condition for every $c \in \mathbb{R}$.

Proposition 4.2. *If the continuous function $f : \mathbb{R} \to \mathbb{R}$ verifies (**f2**) then $I_\lambda = E_\lambda + \psi_K$ satisfies $(PSZ)_c$ for every $\lambda > 0$ and $c \in \mathbb{R}$.*

Proof. Let $\lambda > 0$ and $c \in \mathbb{R}$ be some fixed numbers. Let $\{u_n\}$ be a sequence in $W^{1,2}(0, \infty)$ such that

$$I_\lambda(u_n) = E_\lambda(u_n) + \psi_K(u_n) \to c; \tag{4.1}$$

$$E'_\lambda(u_n)(v - u_n) + \psi_K(v) - \psi_K(u_n) \geq -\varepsilon_n\|v - u_n\|, \quad \forall v \in W^{1,2}(0, \infty), \tag{4.2}$$

$\{\varepsilon_n\}$ being a sequence in $[0, \infty)$ with $\varepsilon_n \to 0$. By (4.1) one concludes that the sequence $\{u_n\}$ belongs entirely to K. Setting $v = 2u_n$ in (4.2), we obtain

$$E'_\lambda(u_n)(u_n) \geq -\varepsilon_n \|u_n\|.$$

Therefore, we derive

$$\|u_n\|^2 - \lambda \int_0^\infty a(x)|u_n|^q \mathrm{d}x - \int_0^\infty b(x) f(u_n(x)) u_n(x) \mathrm{d}x \geq -\varepsilon_n \|u_n\|. \tag{4.3}$$

By (4.1) for large $n \in \mathbb{N}$ we get

$$c + 1 \geq \frac{1}{2}\|u_n\|^2 - \frac{\lambda}{q}\int_0^\infty a(x)|u_n|^q \mathrm{d}x - \int_0^\infty b(x) F(u_n(x)) \mathrm{d}x. \tag{4.4}$$

Multiplying (4.3) by ν^{-1}, adding this one to (4.4) and applying the Hölder inequality, for large $n \in \mathbb{N}$ we obtain

$$\begin{aligned} c + 1 + \frac{1}{\nu}\|u_n\| &\geq \left(\frac{1}{2} - \frac{1}{\nu}\right)\|u_n\|^2 - \lambda\left(\frac{1}{q} - \frac{1}{\nu}\right)\int_0^\infty a(x)|u_n|^q \mathrm{d}x \\ &\quad - \frac{1}{\nu}\int_0^\infty b(x)[-f(u_n(x))u_n(x) + \nu F(u_n(x))]\mathrm{d}x \\ &\overset{(\mathbf{f2})}{\geq} \left(\frac{1}{2} - \frac{1}{\nu}\right)\|u_n\|^2 - \lambda\left(\frac{1}{q} - \frac{1}{\nu}\right)\|a\|_{L^1}\|u_n\|_{L^\infty}^q \\ &\geq \left(\frac{1}{2} - \frac{1}{\nu}\right)\|u_n\|^2 - \lambda\left(\frac{1}{q} - \frac{1}{\nu}\right)\|a\|_{L^1} k_\infty^q \|u_n\|^q, \end{aligned}$$

where $k_\infty > 0$ is the best Sobolev constant of the embedding $W^{1,2}(0, \infty) \hookrightarrow L^\infty(0, \infty)$. Since $q < 2 < \nu$, from the above estimation we derive that the sequence $\{u_n\}$ is bounded in K. Therefore, due to Proposition 2.1, up to a subsequence, we can suppose that

$$u_n \to u \quad \text{weakly in } W^{1,2}(0, \infty); \tag{4.5}$$

$$u_n \to u \quad \text{strongly in } L^\infty(0, \infty). \tag{4.6}$$

As K is (weakly) closed, $u \in K$. Setting $v = u$ in (4.2), we obtain

$$Au_n(u - u_n) + \int_0^\infty b(x) f(u_n(x))(u_n(x) - u(x))\mathrm{d}x - \lambda\int_0^\infty a(x)|u_n|^{q-2}u_n(u - u_n)\mathrm{d}x \geq -\varepsilon_n\|u - u_n\|.$$

Therefore, for large $n \in \mathbb{N}$, we have

$$\begin{aligned} \|u - u_n\|^2 &\leq Au(u - u_n) + \int_0^\infty b(x) f(u_n(x))(u_n(x) - u(x))\mathrm{d}x - \lambda\int_0^\infty a(x)|u_n|^{q-2}u_n(u - u_n)\mathrm{d}x + \varepsilon_n\|u - u_n\| \\ &\leq Au(u - u_n) + \|b\|_{L^1}\max_{s\in[-M,M]}|f(s)| \cdot \|u - u_n\|_{L^\infty} + \lambda\|a\|_{L^1}M^{q-1}\|u - u_n\|_{L^\infty} + \varepsilon_n\|u - u_n\|, \end{aligned}$$

where $M = \|u\|_{L^\infty} + 1$. Due to (4.5), we have

$$\lim_n Au(u - u_n) = 0.$$

Taking into account (4.6), the second and the third term in the last expression also tend to 0. Finally, since $\varepsilon_n \to 0^+$, $\{u_n\}$ converges strongly to u in $W^{1,2}(0, \infty)$. This completes the proof. □

5. Proof of Theorem 4.1

We assume throughout this section that all the hypotheses of Theorem 4.1 are fulfilled. The present section is divided into two parts; in the first subsection we guarantee the existence of a solution for problem (P_λ) by using the Mountain Pass theorem (see Theorem 3.1); the second subsection proves the existence of a second solution for the problem (P_λ) by applying a local minimization argument based on the Ekeland variational principle.

5.1. MP geometry of $I_\lambda = E_\lambda + \psi_K$; the first solution of (P_λ)

Lemma 5.1. *There exist $c_1, c_2 > 0$ such that*

$$F(s) \geq c_1 s^\nu - c_2 s^p, \quad \forall s \geq 0.$$

Proof. Due to (**f3**), there exists $\rho_0 \in [0, R]$ such that $F(\rho_0) > 0$. Clearly, $\rho_0 \neq 0$, since $F(0) = 0$. We consider the function $g : (0, \infty) \to \mathbb{R}$ defined by $g(t) = t^{-\nu}F(t\rho_0)$. Let $t > 1$. By using a mean value theorem, there exists $\tau \in (1, t)$ such that $g(t) - g(1) = [-\nu\tau^{-\nu-1}F(\tau\rho_0) + \tau^{-\nu}\rho_0 f(\tau\rho_0)](t-1)$. By (**f2**), one has $g(t) \geq g(1)$, i.e., $F(t\rho_0) \geq t^\nu F(\rho_0)$ for every $t \geq 1$. Therefore, we have

$$F(s) \geq \frac{F(\rho_0)}{\rho_0^\nu} s^\nu, \quad \forall s \geq \rho_0.$$

On the other hand, by (**f1**), there exist $\delta, L > 0$ such that $|F(s)| \leq L|s|^p$ for $|s| \leq \delta$. In particular, we have that

$$F(s) \geq -Ls^p, \quad \forall s \in [0, \delta].$$

It remains to combine these two relations in order to obtain our claim. □

Proposition 5.1. *There exists $\lambda_0 > 0$ such that for every $\lambda \in (0, \lambda_0)$ the following assertions are true:*

(i) *there exist constants $\alpha_\lambda > 0$ and $\rho_\lambda > 0$ such that $I_\lambda(u) \geq \alpha_\lambda$ for all $\|u\| = \rho_\lambda$;*
(ii) *there exists $e_\lambda \in W^{1,2}(0, \infty)$ with $\|e_\lambda\| > \rho_\lambda$ and $I_\lambda(e_\lambda) \leq 0$.*

Proof. (i) Let $\delta, L > 0$ from the proof of the previous lemma. For $u \in W^{1,2}(0, \infty)$ complying with $\|u\|_{L^\infty} \leq \delta$, we have

$$\mathcal{F}(u) \leq L\|b\|_{L^1}\|u\|_{L^\infty}^p \leq L\|b\|_{L^1}k_\infty^p\|u\|^p.$$

It suffices to restrict our attention to elements u which belong to K; otherwise $I_\lambda(u)$ would be $+\infty$, i.e. (i) holds trivially. Due to the above inequality, for every $\lambda > 0$ and $u \in K$ with $\|u\|_{L^\infty} \leq \delta$, we have

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{2}\|u\|^2 - \lambda\|a\|_{L^1}k_\infty^q\|u\|^q - L\|b\|_{L^1}k_\infty^p\|u\|^p \\ &= \left(\frac{1}{2} - \lambda A\|u\|^{q-2} - B\|u\|^{p-2}\right)\|u\|^2, \end{aligned}$$

where $A = \|a\|_{L^1}k_\infty^q > 0$, and $B = L\|b\|_{L^1}k_\infty^p > 0$.

For every $0 < \lambda < \frac{\delta^{p-q}B(p-2)}{A(2-q)}$, we define the function $g_\lambda : (0, \delta) \to \mathbb{R}$ by

$$g_\lambda(t) = \frac{1}{2} - \lambda At^{q-2} - Bt^{p-2}.$$

Clearly, $g'_\lambda(t_\lambda) = 0$ if and only if $t_\lambda = (\lambda\frac{2-q}{p-2}\frac{A}{B})^{\frac{1}{p-q}}$. Moreover, $g_\lambda(t_\lambda) = \frac{1}{2} - D\lambda^{\frac{p-2}{p-q}}$, where $D = D(p, q, A, B) > 0$. Choosing $0 < \lambda_0 < \frac{\delta^{p-q}B(p-2)}{A(2-q)}$ so small that $g_{\lambda_0}(t_{\lambda_0}) > 0$, one clearly has for every $\lambda \in (0, \lambda_0)$ that $g_\lambda(t_\lambda) > 0$. Therefore, for every $\lambda \in (0, \lambda_0)$, setting $\rho_\lambda = t_\lambda/k_\infty$ and $\alpha_\lambda = g_\lambda(t_\lambda)t_\lambda^2/k_\infty^2$, the assertion from (i) holds true.

(ii) By Lemma 5.1 we have $\mathcal{F}(u) \geq \int_0^\infty b(x)[c_1u^\nu - c_2u^p]\mathrm{d}x$ for every $u \in K$. Then, for every $u \in K$ we have

$$I_\lambda(u) \leq \frac{1}{2}\|u\|^2 - \frac{\lambda}{q}\int_0^\infty a(x)u^q\mathrm{d}x - \int_0^\infty b(x)[c_1u^\nu - c_2u^p]\mathrm{d}x. \tag{5.1}$$

Fix $u_0(x) = \max(1 - x, 0)$, $x > 0$; it is clear that $u_0 \in K$. Letting $u = su_0 (s > 0)$ in (5.1), we have that $I_\lambda(su_0) \to -\infty$ as $s \to +\infty$, since $\nu > p > 2 > q$ and $b > 0$. Thus, for every $\lambda \in (0, \lambda_0)$, it is possible to set $s = s_\lambda$ so large that for $e_\lambda = s_\lambda u_0$, we have $\|e_\lambda\| > \rho_\lambda$ and $I_\lambda(e_\lambda) \leq 0$. This concludes the proof of the proposition. □

By Proposition 4.2, the functional I_λ satisfies the $(PSZ)_c$-condition ($c \in \mathbb{R}$), and $I_\lambda(0) = 0$ for every $\lambda > 0$. Let us fix $\lambda \in (0, \lambda_0)$. By Proposition 5.1 it follows that there exist constants $\alpha_\lambda, \rho_\lambda > 0$ and $e_\lambda \in W^{1,2}(0, \infty)$ such that I_λ fulfills the properties (i) and (ii) from Theorem 3.1. Therefore, the number

$$c_\lambda^1 = \inf_{\gamma\in\Gamma}\sup_{t\in[0,1]} I_\lambda(\gamma(t))$$

is a critical value of I_λ with $c_\lambda^1 \geq \alpha_\lambda > 0$, where

$$\Gamma = \{\gamma \in C([0, 1], W^{1,2}(0, \infty)) : \gamma(0) = 0, \gamma(1) = e_\lambda\}.$$

It is clear that the critical point $u_\lambda^1 \in W^{1,2}(0, \infty)$ which corresponds to c_λ^1 cannot be trivial since $I_\lambda(u_\lambda^1) = c_\lambda^1 > 0 = I_\lambda(0)$. It remains to apply Proposition 4.1 which concludes that u_λ^1 is actually an element of K and a solution of (P_λ).

5.2. Local minimization; the second solution of (P_λ)

Let us fix $\lambda \in (0, \lambda_0)$ arbitrarily; λ_0 was defined in the previous subsection. By Proposition 5.1, there exists $\rho_\lambda > 0$ such that

$$\inf_{\|u\|=\rho_\lambda} I_\lambda(u) > 0. \tag{5.2}$$

Since $a > 0$, for $u_0(x) = \max(1-x, 0)$, $x > 0$, we have $\int_0^\infty a(x)u_0^q \mathrm{d}x > 0$. Taking into account that $\nu > p > 2 > q$, for $t > 0$ small enough one has

$$I_\lambda(tu_0) \le \frac{t^2}{2}\|u_0\|^2 - \frac{\lambda t^q}{q}\int_0^\infty a(x)u_0^q \mathrm{d}x - \int_0^\infty b(x)[c_1 t^\nu u_0^\nu - c_2 t^p u_0^p]\mathrm{d}x < 0.$$

For $r > 0$, we denote by $B_r = \{u \in W^{1,2}(0,\infty) : \|u\| \le r\}$ and by $S_r = \{u \in W^{1,2}(0,\infty) : \|u\| = r\}$. Using these notations, relation (5.2) and the above inequality can be summarized as

$$c_\lambda^2 = \inf_{u \in B_{\rho_\lambda}} I_\lambda(u) < 0 < \inf_{u \in S_{\rho_\lambda}} I_\lambda(u). \tag{5.3}$$

A simple argument shows that c_λ^2 is finite. Moreover, we will show that c_λ^2 is another critical value of I_λ. To this end, let $n \in \mathbb{N} \setminus \{0\}$ such that

$$\frac{1}{n} < \inf_{u \in S_{\rho_\lambda}} I_\lambda(u) - \inf_{u \in B_{\rho_\lambda}} I_\lambda(u). \tag{5.4}$$

By the Ekeland variational principle, applied to the lower semicontinuous functional $I_{\lambda|_{B_{\rho_\lambda}}}$, which is bounded below (see (5.3)), there exists $u_{\lambda,n} \in B_{\rho_\lambda}$ such that

$$I_\lambda(u_{\lambda,n}) \le \inf_{u \in B_{\rho_\lambda}} I_\lambda(u) + \frac{1}{n}; \tag{5.5}$$

$$I_\lambda(w) \ge I_\lambda(u_{\lambda,n}) - \frac{1}{n}\|w - u_{\lambda,n}\|, \quad \forall w \in B_{\rho_\lambda}. \tag{5.6}$$

By (5.4) and (5.5) we have that $I_\lambda(u_{\lambda,n}) < \inf_{u \in S_{\rho_\lambda}} I_\lambda(u)$; therefore $\|u_{\lambda,n}\| < \rho_\lambda$.

Fix an element $v \in W^{1,2}(0,\infty)$. It is possible to choose $t > 0$ small enough such that $w = u_{\lambda,n} + t(v - u_{\lambda,n}) \in B_{\rho_\lambda}$. Applying (5.6) to this element, using the convexity of ψ_K and dividing by $t > 0$, one concludes

$$\frac{E_\lambda(u_{\lambda,n} + t(v - u_{\lambda,n})) - E_\lambda(u_{\lambda,n})}{t} + \psi_K(v) - \psi_K(u_{\lambda,n}) \ge -\frac{1}{n}\|v - u_{\lambda,n}\|.$$

Letting $t \to 0^+$, we obtain

$$E_\lambda'(u_{\lambda,n})(v - u_{\lambda,n}) + \psi_K(v) - \psi_K(u_{\lambda,n}) \ge -\frac{1}{n}\|v - u_{\lambda,n}\|. \tag{5.7}$$

On the other hand, by (5.3) and (5.5) it follows

$$I_\lambda(u_{\lambda,n}) = E_\lambda(u_{\lambda,n}) + \psi_K(u_{\lambda,n}) \to c_\lambda^2 \tag{5.8}$$

as $n \to \infty$. Since v is arbitrarily fixed in (5.7), the sequence $\{u_{\lambda,n}\}$ fulfills (4.1) and (4.2), respectively. Therefore, in a similar manner as in Proposition 4.2, we may prove that $\{u_{\lambda,n}\}$ contains a convergent subsequence; we denote it again by $\{u_{\lambda,n}\}$, its limit point being u_λ^2. It is clear that u_λ^2 belongs to B_{ρ_λ}. By the lower semicontinuity of ψ_K we have

$$\psi_K(u_\lambda^2) \le \liminf_n \psi_K(u_{\lambda,n}),$$

and due to the fact that E_λ is of class C^1 on $W^{1,2}(0,\infty)$, we have

$$\lim_n E_\lambda'(u_{\lambda,n})(v - u_{\lambda,n}) = E_\lambda'(u_\lambda^2)(v - u_\lambda^2).$$

Combining these relations with (5.7) we obtain

$$E_\lambda'(u_\lambda^2)(v - u_\lambda^2) + \psi_{\mathcal{K}}(v) - \psi_K(u_\lambda^2) \ge 0, \quad \forall v \in W^{1,2}(0,\infty),$$

i.e. u_λ^2 is a critical point of I_λ. Moreover,

$$c_\lambda^2 \overset{(5.3)}{=} \inf_{u \in B_{\rho_\lambda}} I_\lambda(u) \le I_\lambda(u_\lambda^2) \le \liminf_n I_\lambda(u_{\lambda,n}) \overset{(5.8)}{=} c_\lambda^2,$$

i.e. $I_\lambda(u_\lambda^2) = c_\lambda^2$. Since $c_\lambda^2 < 0$ (see (5.3)), it follows that u_λ^2 is not trivial. We apply again Proposition 4.1, concluding that u_λ^2 is another solution of (P_λ) different from u_λ^1. This concludes the proof of Theorem 4.1. □

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