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MULTIPLICITY THEOREMS FOR SEMILINEAR ELLIPTIC PROBLEMS DEPENDING ON A PARAMETER

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Abstract We consider semilinear elliptic problems in which the right-hand-side nonlinearity depends on a parameter $\lambda > 0$. Two multiplicity results are presented, guaranteeing the existence of at least three non-trivial solutions for this kind of problem, when the parameter λ belongs to an interval $(0, \lambda^*)$. Our approach is based on variational techniques, truncation methods and critical groups. The first result incorporates as a special case problems with concave–convex nonlinearities, while the second one involves concave nonlinearities perturbed by an asymptotically linear nonlinearity at infinity.

Keywords: critical point; local minimizer; truncated functional; critical groups

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1. Introduction

Let $Z \subset \mathbb{R}^N$ be a bounded domain with C^2-boundary ∂Z. We consider the following semilinear elliptic problem depending on a parameter $\lambda > 0$:

$$\left.\begin{aligned} -\Delta x(z) &= f(z, x(z), \lambda), \quad z \in Z, \\ x|_{\partial Z} &= 0. \end{aligned}\right\} \qquad (P)_\lambda$$

This paper deals with the existence of multiple solutions for problem $(P)_\lambda$. More precisely, we prove the existence of at least *three* non-trivial solutions of $(P)_\lambda$ for sufficiently small values of $\lambda > 0$, whenever the nonlinearity $f : Z \times \mathbb{R} \times (0, \bar{\lambda}) \to \mathbb{R}$, with $\bar{\lambda} > 0$, involves either

- *concave–convex type nonlinearities* (see Theorem 2.1), or
- *concave terms perturbed by an asymptotically linear nonlinearity at infinity* (see Theorem 2.3).

We emphasize that we do not impose any symmetry hypothesis on the nonlinearity f. As we know, if $f(z, x, \cdot)$ is odd, under further assumptions, one can usually produce infinitely many solutions for problem $(P)_\lambda$.

Problems with concave–convex nonlinearities (i.e. $f(z, x, \lambda) = \lambda|x|^{p-2}x + |x|^{r-2}x$, with $1 < p < 2 < r < 2^*$) were studied by Ambrosetti and coauthors [**2**, **3**], Bartsch and Willem [**4**], Adimurthi *et al.* [**1**], Tang [**20**] and others. In particular, Bartsch and Willem [**4**] proved, for every $\lambda > 0$, the existence of a sequence of solutions for $(P)_\lambda$ for which the energy level tends to 0. Wang [**22**] considered the nonlinearity $\lambda|x|^{p-2}x + f(z, x)$ with $f(\cdot\,, \cdot)$ on $Z \times \mathbb{R}$, odd in $x \in \mathbb{R}$ for small $|x|$, and $f(z, x) = o(|x|^p)$ at $x = 0$ and no condition on $f(z, \cdot)$ for large $x \in \mathbb{R}$; Wang still guarantees, for every $\lambda > 0$, a whole sequence of solutions of $(P)_\lambda$ with the same properties as in [**4**]. In [**17**], Perera deals with problems where the nonlinearity has the form $f(z, x, \lambda) = \lambda|x|^{p-2}x + ax + |x|^{r-2}x + o(|x|^{r-1})$ at $x = 0$ and at $x = \infty$, with $1 < p < 2 < r < 2^*$. When Z is the unit ball and $f(z, x, \lambda) = \lambda|x|^{p-2}x + |x|^{r-2}x$, with $1 < p < 2 < r < 2^*$, Adimurthi *et al.* [**1**] and Tang [**20**] proved that $(P)_\lambda$ has exactly two positive solutions whenever $\lambda > 0$ is sufficiently small. Taking into account these facts, our first result (see Theorem 2.1, below) can be fitted into the works [**1**, **20**] and [**2**–**4**, **22**], respectively.

Semilinear problems involving concave terms perturbed by an asymptotically linear nonlinearity at infinity has been investigated by several authors (see [**10**, **11**] and references therein). To the best of our knowledge, there is no multiplicity result that guarantees at least three non-trivial solutions in this context. The aim of our second result (see Theorem 2.3, below) is to give a contribution in this direction.

Our approach uses variational techniques based on critical point theory, suitable truncation methods and Morse theory. The strategy of the proofs can be described as follows. First, we construct two local minimizers of the energy functional associated with problem $(P)_\lambda$: one of them is positive, the other is negative. In order to find these elements, we exploit the result of Brézis and Nirenberg [**5**] and the strong maximum principle of Vázquez [**21**], as well as a result of Guedda and Véron [**8**]. Next, we may construct a mountain-pass-type solution for $(P)_\lambda$. An argument based on critical groups shows that this last element cannot be zero (see [**9**, **14**]).

Multiplicity results for elliptic problems depending on a parameter $\lambda > 0$ were also established by Delgado and Suárez [**7**], Maya and Shivaji [**13**], Motreanu *et al.* [**15**], Mugnai [**16**], Ricceri [**18**] and Shi [**19**], using different hypotheses and methods.

2. Multiplicity theorems

In the rest of the paper, let

$$C_0^1(\bar{Z}) = \{x \in C^1(\bar{Z}) : x|_{\partial Z} = 0\}.$$

This is an ordered Banach space with order the (positive) cone

$$K_+ = \{x \in C_0^1(\bar{Z}) : x(z) \geqslant 0 \text{ for all } z \in \bar{Z}\}.$$

It is well known that

$$\operatorname{Int} K_+ = \left\{ x \in K_+ : x(z) > 0 \text{ for all } z \in Z \text{ and } \frac{\partial x}{\partial n}(z) < 0 \text{ for all } z \in \partial Z \right\}.$$

Here, we denote by $n(z)$ the unit outward normal at $z \in \partial Z$. If $u_1 \in H_0^1(Z)$ is the L^2-normalized principal eigenfunction of $(-\Delta, H_0^1(Z))$, then we know that $u_1 \in \operatorname{Int} K_+$.

For the first multiplicity theorem, the hypotheses on the nonlinearity $f(z,x,\lambda)$ are as follows.

(H1) $f : Z \times \mathbb{R} \times (0, \bar{\lambda}) \to \mathbb{R}$, with $\bar{\lambda} > 0$, is a function such that

(i) for all $x \in \mathbb{R}$ and all $\lambda \in (0, \bar{\lambda})$, the function $z \mapsto f(z,x,\lambda)$ is measurable;

(ii) for almost all $z \in Z$ and all $\lambda \in (0, \bar{\lambda})$, the function $x \mapsto f(z,x,\lambda)$ is continuous and $f(z,0,\lambda) = 0$;

(iii) for almost all $z \in Z$ and all $(x,\lambda) \in \mathbb{R} \times (0,\bar{\lambda})$, we have

$$|f(z,x,\lambda)| \leqslant a(z,\lambda) + c|x|^{r-1}$$

with $a(\cdot\,,\lambda) \in L^\infty(Z)_+ = \{\eta \in L^\infty(Z) : \eta(x) \geqslant 0 \text{ a.e. } x \in Z\}$, $\|a(\cdot\,,\lambda)\|_\infty \to 0$ as $\lambda \to 0^+$, $c > 0$, $2 < r < 2^*$;

(iv) for every $\lambda \in (0,\bar{\lambda})$ there exist $M = M(\lambda) > 0$ and $\theta = \theta(\lambda) > 2$ such that

$$0 < \theta F(z,x,\lambda) \leqslant f(z,x,\lambda)x \quad \text{for a.a. } z \in Z \text{ and all } |x| \geqslant M,$$

with $F(z,x,\lambda) = \int_0^x f(z,s,\lambda)\,\mathrm{d}s$;

(v) for every $\lambda \in (0,\bar{\lambda})$ there exist $\mu = \mu(\lambda) \in (0,2)$, $\delta = \delta(\lambda) > 0$ and $\eta = \eta(\lambda) \in L^\infty(Z)_+$, with $\eta(z) \geqslant \lambda_1$ a.e. on Z (λ_1 denotes the principal eigenvalue of $(-\Delta, H_0^1(Z))$), $\eta \neq \lambda_1$ such that $\mu F(z,x,\lambda) \geqslant f(z,x,\lambda)x$ and $F(z,x,\lambda) \geqslant \frac{1}{2}\eta(z)x^2$ for a.a. $z \in Z$ and all $|x| \leqslant \delta$;

(vi) for every $\lambda \in (0,\bar{\lambda})$ we have $f(z,x,\lambda)x > 0$ for a.a. $z \in Z$, all $x \neq 0$ (sign condition).

Our first result can be read as follows.

Theorem 2.1. *If the hypotheses of (H1) hold, then there exists $\lambda^* \in (0,\bar{\lambda})$ such that for all $\lambda \in (0,\lambda^*)$ problem $(P)_\lambda$ has three distinct solutions: $x_0 \in \operatorname{Int} K_+$, $v_0 \in -\operatorname{Int} K_+$, and $y_0 \in C_0^1(\bar{Z})$, $y_0 \neq 0$.*

Remark 2.2. The classical concave–convex nonlinearity $f(z,x,\lambda) = \lambda|x|^{p-2}x + |x|^{r-2}x$, $1 < p < 2 < r < 2^*$, satisfies Hypotheses (H1). Or, let $f(z,x,\lambda) = (2+\operatorname{sgn}(x))(\lambda|x|^{p-2}x+|x|^{r-2}x)$, which also verifies the hypotheses; in addition, it is not an odd function. Another possibility is the function $f(z,x,\lambda) = \lambda|x|^{p-2}x+\xi(z)x+|x|^{r-2}x+g(z,x)$, with $\xi \in L^\infty(Z)_+$, $\xi(z) \geqslant \lambda_1$ a.e. on Z, $\xi \neq \lambda_1$ and $g(z,x)$ a Carathéodory function such that there exist $\mu \in (0,2)$ and $\delta > 0$ for which we have $\mu G(z,x) \geqslant g(z,x)x$ for a.a. $z \in Z$ and all $|x| \leqslant \delta$, $g(z,x)x \geqslant 0$ for a.a. $z \in Z$, all $x \in \mathbb{R}$ and $g(z,0) = 0$ a.e. on Z. (Note that $g(z,\cdot)$ can be a non-odd function.)

In what follows, $\{\lambda_m\}_{m\geqslant 1}$ are the distinct eigenvalues of $(-\Delta, H_0^1(Z))$. For the second multiplicity theorem, we impose the following conditions on the nonlinearity $f(z,x,\lambda)$.

(H2) $f : Z \times \mathbb{R} \times (0,\bar{\lambda}) \to \mathbb{R}$, with $\bar{\lambda} > 0$, is a function such that (i), (ii), (v), (vi) are the same as those in (H1), and

(iii) for almost all $z \in Z$ and all $(x,\lambda) \in \mathbb{R} \times (0,\bar{\lambda})$, we have

$$|f(z,x,\lambda)| \leqslant a(z,\lambda) + c\min\{|x|, |x|^{r-1}\}$$

with $a(\cdot\,,\lambda) \in L^\infty(Z)_+$, $\|a(\cdot\,,\lambda)\|_\infty \to 0$ as $\lambda \to 0^+$, $c > 0$, $2 < r < 2^*$;

(iv) for every $\lambda \in (0,\bar{\lambda})$, there exist functions $\theta = \theta(\lambda)$, $\hat{\theta} = \hat{\theta}(\lambda) \in L^\infty(Z)_+$ such that for some $m \in \mathbb{N}$ we have

$$\lambda_m \leqslant \theta(z) \leqslant \hat{\theta}(z) \leqslant \lambda_{m+1} \quad \text{for a.a. } z \in Z,\ \theta \neq \lambda_m,\ \hat{\theta} \neq \lambda_{m+1},$$

and

$$\theta(z) \leqslant \liminf_{x\to\pm\infty} \frac{f(z,x,\lambda)}{x} \leqslant \limsup_{x\to\pm\infty} \frac{f(z,x,\lambda)}{x} \leqslant \hat{\theta}(z)$$

uniformly for a.a. $z \in Z$.

Theorem 2.3. *If the hypotheses of (H2) hold, then there exists $\lambda^* \in (0,\bar{\lambda})$ such that for all $\lambda \in (0,\lambda^*)$ problem $(P)_\lambda$ has three distinct solutions $x_0 \in \operatorname{Int} K_+$, $v_0 \in -\operatorname{Int} K_+$, and $y_0 \in C_0^1(\bar{Z})$, $y_0 \neq 0$.*

Remark 2.4. Hypotheses (iv) and (v) of (H2) imply that our setting incorporates problems with a concave nonlinearity $\lambda|x|^{p-2}x$ perturbed by an asymptotically linear term at infinity. A concrete (odd) function verifying the hypotheses of (H2) is $f(x,\lambda) = \lambda|x|^{p-2}x + \theta\min\{|x|^{r-2}, 1\}x$, where $1 < p < 2 < r < 2^*$, and $\lambda_m < \theta < \lambda_{m+1}$. Or we may consider $f(x,\lambda) = \lambda(2+\operatorname{sgn} x)|x|^{p-2}x + \theta\min\{|x|^{r-2}, 1\}x$, which is not odd but also verifies (H2).

3. Proof of Theorem 2.1

Before starting the proof, we recall the notion of critical groups. Let X be a Hilbert space and let $\varphi \in C^1(X)$ be a functional which satisfies the Palais–Smale (PS) condition. Let $x_0 \in X$ be an isolated critical point of φ with $\varphi(x_0) = c_0$. For any $c \in \mathbb{R}$, let $\varphi^c = \{x \in X : \varphi(x) \leqslant c\}$ and let U be a neighbourhood of x_0. The nth-order critical group (over $\mathbb{Z}$) of φ at x_0 is defined by

$$C_n(\varphi,x_0) = H_n(\varphi^{c_0} \cap U, \varphi^{c_0} \cap U \setminus \{x_0\}),$$

where $H_n(\cdot\,,\cdot)$ is the nth singular relative homology group with integer coefficients, $n \in \mathbb{N}$.

Proof of Theorem 2.1. Let $e \in \operatorname{Int} K_+$ be the unique solution of the Dirichlet problem

$$-\Delta x(z) = 1 \quad \text{a.e. on } Z, \qquad x|_{\partial Z} = 0.$$

We claim that we can find $\lambda^* \in (0,\bar{\lambda})$ such that, if $\lambda \in (0,\lambda^*)$, then we can choose $\gamma_1 = \gamma_1(\lambda) > 0$ such that

$$\|a(\cdot\,,\lambda)\|_\infty + c(\gamma_1\|e\|_\infty)^{r-1} < \gamma_1. \tag{3.1}$$

We argue indirectly. Suppose that (3.1) is not true. Then we can find a sequence $\lambda_n \to 0^+$ such that

$$\gamma \leqslant \|a(\cdot\,,\lambda_n)\|_\infty + c(\gamma\|e\|_\infty)^{r-1} \quad \text{for all } n \in \mathbb{N} \text{ and all } \gamma > 0.$$

We let $n \to \infty$ and, on account of (H1) (iii), we obtain

$$1 \leqslant c\gamma^{r-2}\|e\|_\infty^{r-1} \quad \text{for all } \gamma > 0.$$

Since $r > 2$, letting $\gamma \to 0^+$, we have a contradiction. This shows that (3.1) is true.

Fix $\lambda \in (0,\lambda^*)$ and set $\bar{x} = \gamma_1 e \in \operatorname{Int} K_+$, with $\gamma_1 > 0$ given by (3.1). We define the truncated Carathéodory nonlinearity as follows:

$$f^+(z,x,\lambda) = \begin{cases} 0 & \text{if } x < 0, \\ f(z,x,\lambda) & \text{if } 0 \leqslant x \leqslant \bar{x}(z), \\ f(z,\bar{x}(z),\lambda) & \text{if } \bar{x}(z) < x. \end{cases} \tag{3.2}$$

We introduce the functional $\varphi_\lambda^+ : H_0^1(Z) \to \mathbb{R}$ defined by

$$\varphi_\lambda^+(x) = \tfrac{1}{2}\|Dx\|_2^2 - \int_Z F^+(z,x(z),\lambda)\,\mathrm{d}z \quad \text{with } F^+(z,x,\lambda) = \int_0^x f^+(z,s,\lambda)\,\mathrm{d}s.$$

Clearly, $\varphi_\lambda^+ \in C^1(H_0^1(Z))$ and, exploiting the compact embedding of $H_0^1(Z)$ into $L^2(Z)$, we can easily check that φ_λ^+ is sequentially weakly lower semicontinuous. Moreover, because of (3.2) we have

$$\varphi_\lambda^+(x) \geqslant \tfrac{1}{2}\|Dx\|_2^2 - c_1(\lambda)\|Dx\|_2$$

for some $c_1(\lambda) > 0$, i.e. φ_λ^+ is coercive and bounded from below. Therefore, we can find $x_0 = x_0(\lambda) \in H_0^1(Z)$ such that

$$\varphi_\lambda^+(x_0) = \inf_{H_0^1(Z)} \varphi_\lambda^+.$$

Consequently, $(\varphi_\lambda^+)'(x_0) = 0$ and

$$-\Delta x_0(z) = f^+(z,x_0(z),\lambda) \quad \text{a.e. on } Z, \qquad x_0|_{\partial Z} = 0. \tag{3.3}$$

From regularity theory, we have $x_0 \in C_0^1(\bar{Z})$. Moreover, using $-x_0^- \in H_0^1(Z)$ as a test function in (3.3), we obtain

$$\|Dx_0^-\|_2^2 = -\int_Z f^+(z,x_0(z),\lambda)x_0^-(z)\,\mathrm{d}z = 0.$$

Therefore, $x_0^- = 0$, i.e. $x_0 \geqslant 0$.

Next, we show that $x_0 \neq 0$. To this end, let $\delta > 0$ be as in Hypothesis (H1) (v). Due to the fact that $\bar{x} \in \operatorname{Int} K_+$, one may choose $t > 0$ small enough such that $tu_1 \leqslant \bar{x}$ and $tu_1(z) \in [0, \delta]$ for all $z \in \bar{Z}$. Then

$$\begin{aligned}
\varphi_\lambda^+(tu_1) &= \tfrac{1}{2}t^2\|Du_1\|_2^2 - \int_Z F^+(z, tu_1(z), \lambda)\,\mathrm{d}z \\
&= \tfrac{1}{2}t^2\|Du_1\|_2^2 - \int_Z F(z, tu_1(z), \lambda)\,\mathrm{d}z \\
&\leqslant \tfrac{1}{2}t^2\|Du_1\|_2^2 - \tfrac{1}{2}t^2 \int_Z \eta u_1^2(z)\,\mathrm{d}z \quad \text{(see Hypothesis (H1) (v))} \\
&= \tfrac{1}{2}t^2 \int_Z (\lambda_1 - \eta(z))u_1^2(z)\,\mathrm{d}z \quad (\text{since } \|Du_1\|_2^2 = \lambda_1\|u_1\|_2^2) \\
&< 0 \quad (\text{recall the hypothesis on } \eta \text{ and that } u_1 \in \operatorname{Int} K_+).
\end{aligned}$$

Therefore, we have $\varphi_\lambda^+(x_0) \leqslant \varphi_\lambda^+(tu_1) < 0 = \varphi_\lambda^+(0)$. Consequently, $x_0 \neq 0$, and $x_0 \geqslant 0$. Due to (H1) (vi), we have

$$\Delta x_0(z) = -f^+(z, x_0(z), \lambda) \leqslant 0 \quad \text{a.e. on } Z.$$

Invoking the strong maximum principle of Vázquez [**21**], we conclude that

$$x_0 \in \operatorname{Int} K_+. \tag{3.4}$$

On the other hand, if $\langle \cdot\,, \cdot \rangle$ denotes the duality bracket for the pair $(H^{-1}(Z), H_0^1(Z))$, then, from (3.3), (H1) (iii) and (3.1), we have

$$\langle -\Delta x_0 + \Delta \bar{x}, (x_0 - \bar{x})^+ \rangle = \int_Z (f^+(z, x_0, \lambda) - \gamma_1)(x_0 - \bar{x})^+\,\mathrm{d}z \leqslant 0.$$

Therefore, $\|D(x_0 - \bar{x})^+\|_2^2 \leqslant 0$, i.e. $(x_0 - \bar{x})^+ = 0$. Consequently, $x_0 \leqslant \bar{x}$ and

$$\begin{aligned}
-\Delta x_0(z) &= f^+(z, x_0(z), \lambda) \\
&= f(z, x_0(z), \lambda) \\
&\leqslant \|a(\cdot\,, \lambda)\|_\infty + c\|\bar{x}\|_\infty^{r-1} \quad \text{(see Hypothesis (H1) (iii))} \\
&< \gamma_1 \quad \text{(see (3.1))} \\
&= -\Delta \bar{x}(z) \quad \text{a.e. on } Z.
\end{aligned}$$

In particular, one can see that $x_0 \in \operatorname{Int} K_+$ is a solution of $(P)_\lambda$ (see the second equality). Moreover, by [**8**, Proposition 2.2], we infer that

$$\bar{x} - x_0 \in \operatorname{Int} K_+. \tag{3.5}$$

In addition, if $\varphi_\lambda : H_0^1(Z) \to \mathbb{R}$ denotes the Euler functional for the problem $(P)_\lambda$, defined by

$$\varphi_\lambda(x) = \tfrac{1}{2}\|Dx\|_2^2 - \int_Z F(z, x(z), \lambda)\,\mathrm{d}z \quad \text{for all } x \in H_0^1(Z),$$

then, using (3.4) and (3.5), the element x_0 is a local minimizer not only of φ_λ^+ but also of φ_λ in the $C_0^1(\bar{Z})$-topology. Hence, by [**5**], it is also a local minimizer of φ_λ in the $H_0^1(Z)$-topology.

Analogously to the previous case, we find an element $v_0 \in -\operatorname{Int} K_+$ that is a solution for the problem $(P)_\lambda$ and a local minimizer of φ_λ.

Because of Hypotheses (H1) (iii) and (iv), the functional φ_λ satisfies the (PS) condition (see, for example, [**12**, p. 100]). Since $x_0 \in \operatorname{Int} K_+$ and $v_0 \in -\operatorname{Int} K_+$ are local minimizers of φ_λ, we have

$$C_n(\varphi_\lambda, x_0) = C_n(\varphi_\lambda, v_0) = \delta_{n,0}\mathbb{Z} \quad \text{for all } n \in \mathbb{N} \tag{3.6}$$

(see [**6**, Example 1, p. 33]). On the other hand, by (H1) (iii), one can find $r = r(\lambda) > 0$ small enough such that $r < \|Dx_0\|_2$, and

$$\varphi_\lambda(x) > 0 \quad \text{for all } x \in H_0^1(Z), \qquad \|Dx\|_2 = r.$$

Moreover, by construction, $\varphi_\lambda(x_0) < \varphi_\lambda(0) = 0$. Since $\{0, x_0\} \subset H_0^1(Z)$ and $\{x \in H_0^1(Z) : \|Dx\|_2 = r\}$ are homologically linked, we have $H_1(\varphi_\lambda^b, \varphi_\lambda^0) \neq 0$, where $b > \max\{\varphi_\lambda(tx_0) : t \in [0,1]\}$ (see [**6**, Theorem 1.1′, p. 84]). Taking into account [**6**, Theorem 1.5, p. 89], we can find a mountain-pass-type critical point $y_0 \in H_0^1(Z)$ of φ_λ, and hence a solution of $(P)_\lambda$, such that

$$C_1(\varphi_\lambda, y_0) \neq 0. \tag{3.7}$$

By (3.6), $x_0 \neq y_0 \neq v_0$. Moreover, by Hypotheses (H1) (v) and (vi) and [**14**] or [**9**, Proposition 2.1], we have

$$C_n(\varphi_\lambda, 0) = 0 \quad \text{for all } n \in \mathbb{N}. \tag{3.8}$$

By (3.7) and (3.8), we conclude that $y_0 \neq 0$ and by regularity theory one has $y_0 \in C_0^1(\bar{Z})$. □

Remark 3.1. Theorem 2.1 extends [**3**, Theorem 2.1] and [**17**, Theorem 4.1.1].

4. Proof of Theorem 2.3

Proposition 4.1. *If the hypotheses of (H2) hold and $\lambda \in (0, \bar{\lambda})$, then φ_λ satisfies the (PS) condition.*

Proof. Let $\{x_n\}_{n\geqslant 1} \subset H_0^1(Z)$ be a sequence such that $|\varphi_\lambda(x_n)| \leqslant M_1$ for some $M_1 > 0$, all $n \geqslant 1$, and $\varphi_\lambda'(x_n) \to 0$ in $H^{-1}(Z)$ as $n \to \infty$. As is well known, it suffices to show that $\{x_n\}_{n\geqslant 1} \subset H_0^1(Z)$ is bounded. We proceed by contradiction. Suppose that $\|x_n\| \to \infty$. Let $y_n = x_n/\|x_n\|$, $n \geqslant 1$. We may assume that

$$y_n \xrightarrow{w} y \in H_0^1(Z), \quad y_n \to y \in L^2(Z), \quad y_n(z) \to y(z) \text{ a.e. on } Z$$

and

$$|y_n(z)| \leqslant k(z) \text{ a.e. on } Z \text{ for all } n \geqslant 1 \text{ with } k \in L^2(Z)_+.$$

Let $N_\lambda : L^2(Z) \to L^2(Z)$ be the Nemytskii operator corresponding to the function $(z,x) \mapsto f(z,x,\lambda)$, i.e.

$$N_\lambda(x)(\cdot) = f(\cdot\,, x(\cdot), \lambda) \quad \text{for all } x \in L^2(Z).$$

By virtue of Hypotheses (H2) (iii) and (iv) and Krasnosel'skii's theorem, N_λ is continuous and bounded. From the choice of the sequence $\{x_n\}_{n\geqslant 1} \subset H_0^1(Z)$, we have

$$|\langle -\Delta x_n - N_\lambda(x_n), v\rangle| \leqslant \varepsilon_n \|v\| \quad \text{for all } v \in H_0^1(Z) \text{ with } \varepsilon_n \to 0^+.$$

Consequently,

$$\left|\left\langle -\Delta y_n - \frac{N_\lambda(x_n)}{\|x_n\|}, v\right\rangle\right| \leqslant \frac{\varepsilon_n}{\|x_n\|}\|v\| \quad \text{for all } v \in H_0^1(Z). \tag{4.1}$$

Let $v := y_n - y$. By using (4.1) and the fact that $\{N_\lambda(x_n)/\|x_n\|\}_{n\geqslant 1} \subset L^2(Z)$ is bounded (see (H2) (iv)), we obtain

$$\langle -\Delta y_n, y_n - y\rangle \to 0 \quad \text{as } n \to \infty.$$

Therefore, $\|Dy_n\|_2 \to \|Dy\|_2$. Since $Dy_n \xrightarrow{w} Dy$ in $L^2(Z,\mathbb{R}^N)$, from the Kadec–Klee property of Hilbert spaces, we have $Dy_n \to Dy$ in $L^2(Z,\mathbb{R}^N)$; hence, $y_n \to y$ in $H_0^1(Z)$, and so $\|y\| = 1$.

Based on the above observation, we may assume that

$$\frac{N_\lambda(x_n)}{\|x_n\|} \xrightarrow{w} h \in L^2(Z) \quad \text{as } n \to \infty.$$

Clearly, we have that $x_n(z) \to +\infty$ a.e. on $\{y > 0\}$, $x_n(z) \to -\infty$ a.e. on $\{y < 0\}$ and, from the linear growth of $f(z,\cdot,\lambda)$ (see Hypotheses (H2) (iii) and (iv)), we see that $h(z) = 0$ a.e. on $\{y = 0\}$. Therefore, again using Hypothesis (H2) (iv) we can easily check that

$$h(z) = g(z)y(z) \quad \text{a.e. on } Z,$$

with $g \in L^\infty(Z)_+$, $\theta(z) \leqslant g(z) \leqslant \hat{\theta}(z)$ a.e. on Z.

If in (4.1) we pass to the limit as $n \to \infty$ and since $v \in H_0^1(Z)$ is arbitrary, we obtain

$$-\Delta y(z) = g(z)y(z) \quad \text{a.e. on } Z, \qquad y|_{\partial Z} = 0. \tag{4.2}$$

Exploiting the monotonicity of the eigenvalues of $(-\Delta, H_0^1(Z))$ on the weight function, we have

$$\hat{\lambda}_m(g) < \hat{\lambda}_m(\lambda_m) = 1 \tag{4.3}$$

and

$$\hat{\lambda}_{m+1}(g) > \hat{\lambda}_{m+1}(\lambda_{m+1}) = 1. \tag{4.4}$$

Comparing the expressions in (4.2)–(4.4), we conclude that $y = 0$, a contradiction to the fact that $\|y\| = 1$. This proves that $\{x_n\}_{n\geqslant 1} \subset H_0^1(Z)$ is bounded and so φ_λ satisfies the (PS) condition. □

Proof of Theorem 2.3. Arguing exactly as we did in the proof of Theorem 2.1, we may construct $x_0, v_0 \in H_0^1(Z)$, two local minima of φ_λ. (Note that Hypothesis (H2) (iii) implies (H1) (iii).) Since φ_λ verifies the (PS) condition (see Proposition 4.1), the mountain-pass-type element $y_0 \in H_0^1$ can be obtained as above. □

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