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Multiple solutions for a degenerate elliptic equation involving sublinear terms at infinity ☆

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Abstract

Some multiplicity results are presented for the eigenvalue problem

$$\begin{cases} -\operatorname{div}\big(|x|^{-2a}\nabla u\big) = \lambda |x|^{-2b} f(u) + \mu |x|^{-2c} g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (\mathcal{P}_{\lambda,\mu})$$

where $\Omega \subset \mathbb{R}^n$ $(n \geqslant 3)$ is an open bounded domain with smooth boundary, $0 \in \Omega$, $0 < a < \frac{n-2}{2}$, $a \leqslant b$, $c < a+1$, and $f : \mathbb{R} \to \mathbb{R}$ is sublinear at infinity and superlinear at the origin. Various cases are treated depending on the behaviour of the nonlinear term g.

Keywords: Degenerate elliptic problem; Caffarelli–Kohn–Nirenberg inequality; Sublinearity at infinity

1. Introduction and main results

We consider the eigenvalue problem

$$\begin{cases} -\operatorname{div}\big(|x|^{-2a}\nabla u\big) = \lambda |x|^{-2b} f\big(u(x)\big) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (\mathcal{P}_{\lambda})$$

where $\Omega \subset \mathbb{R}^n$ $(n \geqslant 3)$ is an open bounded domain with smooth boundary, $0 \in \Omega$, $0 < a < \frac{n-2}{2}$, $a \leqslant b < a+1$, and $\lambda \in \mathbb{R}$ is a parameter.

Equations like $(\mathcal{P}_{\lambda})$ are introduced as model for several physical phenomena related to equilibrium of anisotropic media, see [6]. Due to this fact, problem $(\mathcal{P}_{\lambda})$ has been widely studied by several authors, see [1–3,7,13] and references therein. Usually, the nonlinear term $f : \mathbb{R} \to \mathbb{R}$ is considered to be *superlinear at infinity* or simply, $f(s) = |s|^{\theta-2}s$

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with $\theta > 2$. The common assumption in these papers is the well-known Ambrosetti–Rabinowitz condition: denoting by $F(s) = \int_0^s f(t)\,dt$, there exist $s_0 > 0$ and $\theta > 2$ such that

$$0 < \theta F(s) \leqslant sf(s), \quad \forall s \in \mathbb{R},\ |s| \geqslant s_0. \tag{AR}$$

A simple computation shows that (AR) implies

$$\big|f(s)\big| \geqslant c|s|^{\theta-1}, \quad \forall s \in \mathbb{R},\ |s| \geqslant s_0, \tag{AR$'$}$$

with $c > 0$, i.e., f is superlinear at infinity.

Our aim is to handle the counterpart of the above case, i.e., when $f : \mathbb{R} \to \mathbb{R}$ is *sublinear at infinity.* More precisely, we assume:

(f_1) $\lim_{|s|\to\infty} \frac{f(s)}{s} = 0$.

The presence of the parameter $\lambda \in \mathbb{R}$ is essential in our problem; indeed, if beside of (f_1), the nonlinear term f is uniformly Lipschitz (with Lipschitz constant $L > 0$), then problem $(\mathcal{P}_\lambda)$ has only the trivial solution whenever $|\lambda| < (LC_{2,2b}^2)^{-1}$; the constant $C_{2,2b} > 0$ is introduced after relation (3).

In order to state our main results, we introduce the weighted Sobolev space $\mathcal{D}_a^{1,2}(\Omega)$ where the solutions of $(\mathcal{P}_\lambda)$ are going to be sought, which is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_a = \left(\int_\Omega |x|^{-2a} |\nabla u|^2\, dx \right)^{1/2}.$$

Beside of (f_1), we assume

(f_2) $\lim_{s\to 0} \frac{f(s)}{s} = 0$ (superlinearity at zero);
(f_3) $\sup_{s\in\mathbb{R}} F(s) > 0$.

Theorem 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies (f_1), (f_2) and (f_3). Then, there exist an open interval $\Lambda \subset (0,\infty)$ and a constant $\nu > 0$ such that for every $\lambda \in \Lambda$ problem $(\mathcal{P}_\lambda)$ has at least two nontrivial weak solutions in $\mathcal{D}_a^{1,2}(\Omega)$ whose $\|\cdot\|_a$-norms are less than ν.*

Now, we consider the perturbation of the problem $(\mathcal{P}_\lambda)$ in the form

$$\begin{cases} -\operatorname{div}\big(|x|^{-2a}\nabla u\big) = \lambda |x|^{-2b} f(u) + \mu |x|^{-2c} g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{P}_{\lambda,\mu}$}$$

where $0 < a < \frac{n-2}{2}$, $a \leqslant b$, $c < a + 1$ and for the continuous function $g : \mathbb{R} \to \mathbb{R}$ we introduce the hypotheses:

(g_1) there exist $p \in (2, 2_{a,c}^\star)$ with $2_{a,c}^\star = \min\{\frac{2n}{n-2}, \frac{2(n-2c)}{n-2(a+1)}\}$ and $c_g > 0$ such that $|g(s)| \leqslant c_g(1 + |s|^{p-1})$ for every $s \in \mathbb{R}$;
(g_2) $\lim_{|s|\to\infty} \frac{|g(s)|}{|s|} = l < +\infty$ (asymptotically linear at infinity).

It is clear that (g_2) implies (g_1).

Theorem 1.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies the conditions (f_1), (f_2), (f_3). Then, there exists a nondegenerate compact interval $A \subset [0,\infty)$ with the following properties*:

(i) *there exists a number $\nu_1 > 0$ such that for every $\lambda \in A$ and every continuous function $g : \mathbb{R} \to \mathbb{R}$ verifying (g_1), there exists $\delta_1 > 0$ with the property that for each $\mu \in (0, \delta_1)$ the problem $(\mathcal{P}_{\lambda,\mu})$ has at least two distinct weak solutions whose $\|\cdot\|_a$-norms are less than ν_1;*
(ii) *there exists a number $\nu_2 > 0$ such that for every $\lambda \in A$ and every continuous function $g : \mathbb{R} \to \mathbb{R}$ verifying (g_2), there exists $\delta_2 > 0$ with the property that for each $\mu \in (0, \delta_2)$ the problem $(\mathcal{P}_{\lambda,\mu})$ has at least three distinct weak solutions whose $\|\cdot\|_a$-norms are less than ν_2.*

It is worth to notice that problem $(\mathcal{P}_{\lambda,\mu})$ may be viewed in particular as a degenerate elliptic problem involving concave–convex nonlinearities whenever (g_1) holds; indeed, f has a sublinear growth at infinity, while g can be superlinear (and subcritical) at infinity.

The main ingredient for the proof of Theorem 1.1 is a recent critical point result due to Bonanno [4] which is actually a refinement of a result of Ricceri [9,10]. In the proof of Theorem 1.2 we use a recent result of Ricceri [11] and a version of the mountain pass theorem due to Pucci and Serrin [8].

2. Preliminaries

The starting point of the variational approach to problems $(\mathcal{P}_\lambda)$ and $(\mathcal{P}_{\lambda,\mu})$ is the weighted Sobolev–Hardy inequality due to Caffarelli, Kohn, Nirenberg [5]: for all $u \in C_0^\infty(\mathbb{R}^n)$, there is a constant $K_{a,b} > 0$ such that

$$\left(\int_{\mathbb{R}^n} |x|^{-bq}|u|^q\,dx\right)^{2/q} \leqslant K_{a,b}\int_{\mathbb{R}^n} |x|^{-2a}|\nabla u|^2\,dx, \tag{1}$$

where

$$-\infty < a < \frac{n-2}{2}, \qquad a \leqslant b < a+1, \qquad q = 2^\star(a,b) = \frac{2n}{n-2d}, \qquad d = 1+a-b.$$

From the boundedness of Ω and standard approximations argument, it is easy to see that (1) holds on $\mathcal{D}_a^{1,2}(\Omega)$; more precisely, for every

$$1 \leqslant r \leqslant \frac{2n}{n-2} \quad \text{and} \quad \frac{\alpha}{r} \leqslant (1+a) + n\left(\frac{1}{r} - \frac{1}{2}\right), \tag{2}$$

we have

$$\left(\int_\Omega |x|^{-\alpha}|u|^r\,dx\right)^{2/r} \leqslant C\int_\Omega |x|^{-2a}|\nabla u|^2\,dx, \quad u \in \mathcal{D}_a^{1,2}(\Omega),$$

that is, the embedding $\mathcal{D}_a^{1,2}(\Omega) \hookrightarrow L^r(\Omega; |x|^{-\alpha})$ is continuous, where $L^r(\Omega; |x|^{-\alpha})$ is the weighted L^r-space with the norm

$$\|u\|_{r,\alpha} = \|u\|_{L^r(\Omega;|x|^{-\alpha})} = \left(\int_\Omega |x|^{-\alpha}|u|^r\,dx\right)^{1/r}. \tag{3}$$

We denote by $C_{r,\alpha} > 0$ the best Sobolev constant of the embedding $\mathcal{D}_a^{1,2}(\Omega) \hookrightarrow L^r(\Omega; |x|^{-\alpha})$.

The following version of the Rellich–Kondrachov compactness theorem can be stated, see Xuan [12,13].

Lemma A. *Suppose that $\Omega \subset \mathbb{R}^n$ is an open bounded domain with C^1 boundary and $0 \in \Omega$, $n \geqslant 3$, $-\infty < a < \frac{n-2}{2}$. The embedding $\mathcal{D}_a^{1,2}(\Omega) \hookrightarrow L^r(\Omega; |x|^{-\alpha})$ is compact if $1 \leqslant r < \frac{2n}{n-2}$ and $\alpha < (1+a)r + n(1-\frac{r}{2})$.*

First, we associate the energy functional $\mathcal{E}_\lambda : \mathcal{D}_a^{1,2}(\Omega) \to \mathbb{R}$ to problem $(\mathcal{P}_\lambda)$, given by

$$\mathcal{E}_\lambda(u) = \frac{1}{2}\|u\|_a^2 - \lambda\mathcal{F}(u), \quad u \in \mathcal{D}_a^{1,2}(\Omega),$$

where $\mathcal{F}(u) = \int_\Omega |x|^{-2b}F(u(x))\,dx$ and $F(s) = \int_0^s f(t)\,dt$.

Proposition 2.1. *Assume (f_1) and (f_2) hold. Then, for every $\lambda \in \mathbb{R}$ the functional $\mathcal{E}_\lambda$ is well defined, of class C^1 on $\mathcal{D}_a^{1,2}(\Omega)$, sequentially weakly lower semicontinuous, and coercive. Moreover, every critical point of $\mathcal{E}_\lambda$ is a weak solution of $(\mathcal{P}_\lambda)$.*

Proof. Fix $\lambda \in \mathbb{R}$. Combining (f_1) and (f_2), there exists $M > 0$ such that

$$\left|f(s)\right| \leqslant M\left(1+|s|\right) \quad \text{for all } s \in \mathbb{R}. \tag{4}$$

Then, for every $u \in \mathcal{D}_a^{1,2}(\Omega)$, we have

$$\left|\mathcal{F}(u)\right| \leqslant M\left(C_{1,2b}\|u\|_a + C_{2,2b}^2\|u\|_a^2\right) < \infty. \tag{5}$$

Note that the pairs $r=1$, $\alpha=2b$ and $r=2$, $\alpha=2b$ verify relation (2). Consequently, $\mathcal{E}_\lambda$ is well defined.

One can see in a standard way that $\mathcal{E}_\lambda$ is of class C^1 on $\mathcal{D}_a^{1,2}(\Omega)$ and every critical point of $\mathcal{E}_\lambda$ is a weak solution of $(\mathcal{P}_\lambda)$.

We prove that $\mathcal{F}$ is sequential weak continuous which clearly implies the sequential weak lower semicontinuity of $\mathcal{E}_\lambda$. To do this, we argue by contradiction; let $\{u_k\} \subset \mathcal{D}_a^{1,2}(\Omega)$ be a sequence which converges weakly to $u \in \mathcal{D}_a^{1,2}(\Omega)$ but $\{\mathcal{F}(u_k)\}$ does not converge to $\mathcal{F}(u)$ as $k \to \infty$. Therefore, up to a subsequence, one can find a number $\varepsilon_0 > 0$ such that

$$0 < \varepsilon_0 \leqslant \left|\mathcal{F}(u_k) - \mathcal{F}(u)\right| \quad \text{for every } k \in \mathbb{N},$$

and $\{u_k\}$ converges strongly to u in $L^1(\Omega; |x|^{-2b}) \cap L^2(\Omega; |x|^{-2b})$. Here, the pairs $r=1$, $\alpha=2b$, and $r=2$, $\alpha=2b$ verify relations from Lemma A. Using Hölder inequality and (4), for every $k \in \mathbb{N}$ one has $0 < \theta_k < 1$ such that

$$\begin{aligned} 0 < \varepsilon_0 \leqslant \left|\mathcal{F}(u_k) - \mathcal{F}(u)\right| &\leqslant \int_\Omega |x|^{-2b}\left|f\left(u + \theta_k(u_k - u)\right)\right||u_k - u|\,dx \\ &\leqslant M\left(\|u_k - u\|_{1,2b} + \left\|u_k + \theta_k(u_k - u)\right\|_{2,2b}\|u_k - u\|_{2,2b}\right). \end{aligned}$$

Since $\{u_k\}$ converges strongly to u in $L^1(\Omega; |x|^{-2b}) \cap L^2(\Omega; |x|^{-2b})$, both terms in the right-hand side tend to 0 as $k \to \infty$, contradicting $\varepsilon_0 > 0$.

We prove now that $\mathcal{E}_\lambda$ is coercive. By (f_1) there exists $\delta_0 = \delta(\lambda) > 0$ such that

$$\left|f(s)\right| \leqslant C_{2,2b}^{-2}\left(1 + |\lambda|\right)^{-1}|s| \quad \text{for every } |s| \geqslant \delta_0.$$

Integrating the above inequality we get that

$$\left|F(s)\right| \leqslant \frac{1}{2}C_{2,2b}^{-2}\left(1 + |\lambda|\right)^{-1}|s|^2 + \max_{|t| \leqslant \delta_0}\left|f(t)\right||s| \quad \text{for every } s \in \mathbb{R}.$$

Thus, for every $u \in \mathcal{D}_a^{1,2}(\Omega)$, we have

$$\left|\mathcal{F}(u)\right| \leqslant \frac{1}{2}\left(1 + |\lambda|\right)^{-1}\|u\|_a^2 + C_{1,2b}\max_{|t| \leqslant \delta_0}\left|f(t)\right|\|u\|_a. \tag{6}$$

Using (6), we obtain the inequality

$$\mathcal{E}_\lambda(u) \geqslant \frac{1}{2}\|u\|_a^2 - |\lambda|\left|\mathcal{F}(u)\right| \geqslant \frac{1}{2(1 + |\lambda|)}\|u\|_a^2 - |\lambda|C_{1,2b}\max_{|t| \leqslant \delta_0}\left|f(t)\right|\|u\|_a.$$

Consequently, if $\|u\|_a \to \infty$ then $\mathcal{E}_\lambda(u) \to \infty$ as well, i.e., $\mathcal{E}_\lambda$ is coercive. □

3. Proof of Theorem 1.1

Throughout of this section, we assume that the assumptions of Theorem 1.1 are fulfilled. First, we prove two lemmas.

Lemma 3.1. $\lim_{\rho \to 0^+} \frac{\sup\{\mathcal{F}(u):\ \|u\|_a^2 < 2\rho\}}{\rho} = 0.$

Proof. Due to (f_2), for an arbitrary small $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\left|f(s)\right| < \frac{\varepsilon}{2}C_{2,2b}^{-2}|s| \quad \text{for every } |s| < \delta.$$

Combining the above inequality with (4), we obtain

$$\left|F(s)\right| \leqslant \varepsilon C_{2,2b}^{-2}|s|^2 + K(\delta)|s|^q \quad \text{for every } s \in \mathbb{R}, \tag{7}$$

where $q \in (2, \min\{\frac{2n}{n-2}, \frac{2(n-2b)}{n-2(a+1)}\})$ is fixed and $K(\delta) > 0$ does not depend on s.

From (7) we get

$$\mathcal{F}(u) \leqslant \varepsilon C_{2,2b}^{-2} \int_\Omega |x|^{-2b}|u|^2\,dx + K(\delta)\int_\Omega |x|^{-2b}|u|^q\,dx \leqslant \varepsilon\|u\|_a^2 + K(\delta)C_{q,2b}^q\|u\|_a^q.$$

From the above relation we obtain that

$$\sup\{\mathcal{F}(u):\ \|u\|_a^2 < 2\rho\} \leqslant 2\varepsilon\rho + K(\delta)C_{q,2b}^q(2\rho)^{\frac{q}{2}}.$$

Because $q > 2$ and $\varepsilon > 0$ is arbitrarily, we obtain

$$\lim_{\rho\to 0^+} \frac{\sup\{\mathcal{F}(u):\ \|u\|_a^2 < 2\rho\}}{\rho} = 0. \qquad \square$$

Lemma 3.2. *For every $\lambda \in \mathbb{R}$ the functional $\mathcal{E}_\lambda$ satisfies the usual* (PS)*-condition.*

Proof. Let $\{u_k\} \subset \mathcal{D}_a^{1,2}(\Omega)$ be a (PS)-sequence, i.e., $\{\mathcal{E}_\lambda(u_k)\}$ is bounded and $\mathcal{E}_\lambda'(u_k) \to 0$ in $(\mathcal{D}_a^{1,2}(\Omega))^*$ as $k \to \infty$. Since the function $\mathcal{E}_\lambda$ is coercive, it follows that the sequence $\{u_k\}$ is bounded in $\mathcal{D}_a^{1,2}(\Omega)$. Up to a subsequence, we may suppose that $u_k \to u$ weakly in $\mathcal{D}_a^{1,2}(\Omega)$, and $u_k \to u$ strongly in $L^1(\Omega; |x|^{-2b}) \cap L^2(\Omega; |x|^{-2b})$ for some $u \in \mathcal{D}_a^{1,2}(\Omega)$, see Lemma A. On the other hand, we have

$$\|u_k - u\|_a^2 = \mathcal{E}_\lambda'(u_k)(u_k - u) + \mathcal{E}_\lambda'(u)(u - u_k) + \lambda\int_\Omega |x|^{-2b}\big[f\big(u_k(x)\big) - f\big(u(x)\big)\big]\big(u_k(x) - u(x)\big)\,dx.$$

It is clear the first two terms from the last expression tend to 0, while by means of (4) and Hölder's inequality, one has

$$\int_\Omega |x|^{-2b}\big|f\big(u_k(x)\big) - f\big(u(x)\big)\big|\big|u_k(x) - u(x)\big|\,dx$$
$$\leqslant M\big[2\|u_k - u\|_{1,2b} + \big(\|u_k\|_{2,2b} + \|u\|_{2,2b}\big)\|u_k - u\|_{2,2b}\big] \to 0$$

as $k \to \infty$. Thus, we have $\|u_k - u\|_a \to 0$ as $k \to \infty$.

Let $s_0 \in \mathbb{R}$ such that $F(s_0) > 0$, see (f_3). Here and in the sequel, let $x_0 \in \Omega$ and $r_0 > 0$ so small such that $|x_0| > r_0$ and $B(x_0, r_0) \subset \Omega$. Then, clearly, $B(x_0, r_0) \subset \Omega \setminus \{0\}$. As usual $B(x_0, r_0)$ denotes the n-dimensional open ball with center in x_0 and radius $r_0 > 0$.

For $\sigma \in (0, 1)$ we define

$$u_\sigma(x) = \begin{cases} 0, & \text{if } x \in \Omega \setminus B(x_0, r_0);\\ \frac{s_0}{1-\sigma} - \frac{s_0}{r_0(1-\sigma)}|x - x_0|, & \text{if } x \in B(x_0, r_0) \setminus B(x_0, \sigma r_0);\\ s_0, & \text{if } x \in B(x_0, \sigma r_0). \end{cases} \tag{8}$$

It is clear that $u_\sigma \in \mathcal{D}_a^{1,2}(\Omega)$. Moreover, one has

$$\|u_\sigma\|_a^2 \geqslant s_0^2\big(|x_0| + r_0\big)^{-2a}(1 - \sigma)^{-2}\big(1 - \sigma^n\big)\omega_n r_0^{n-2} \tag{9}$$

and

$$\mathcal{F}(u_\sigma) \geqslant K_{s_0,x_0,r_0}(\sigma), \tag{10}$$

where

$$K_{s_0,x_0,r_0}(\sigma) = \Big[F(s_0)\big(|x_0| + r_0\big)^{-2b}\sigma^n - \max_{|t|\leqslant|s_0|}\big|F(t)\big|\big(|x_0| - r_0\big)^{-2b}\big(1 - \sigma^n\big)\Big]\omega_n r_0^n$$

and ω_n denotes the volume of the n-dimensional unit ball. For σ close enough to 1, the right-hand side of (10) becomes strictly positive; choose such a number, say σ_0.

Now, we recall a recent result from critical point theory, due to Ricceri [9,10], and Bonanno [4].

Theorem R1. *(See [4, Theorem 2.1].) Let X be a separable and reflexive real Banach space, and let $\mathcal{A}, \mathcal{F} : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals. Assume that there exists $x_0 \in X$ such that $\mathcal{A}(x_0) = \mathcal{F}(x_0) = 0$ and $\mathcal{A}(x) \geqslant 0$ for every $x \in X$ and that there exist $x_1 \in X$, $\rho > 0$ such that*

(i) $\rho < \mathcal{A}(x_1)$;
(ii) $\sup_{\mathcal{A}(x)<\rho} \mathcal{F}(x) < \rho \frac{\mathcal{F}(x_1)}{\mathcal{A}(x_1)}$.

Further, put

$$\bar{a} = \frac{\zeta\rho}{\rho\frac{\mathcal{F}(x_1)}{\mathcal{A}(x_1)} - \sup_{\mathcal{A}(x)<\rho} \mathcal{F}(x)},$$

with $\zeta > 1$, assume that the functional $\mathcal{A} - \lambda\mathcal{F}$ is sequentially weakly lower semicontinuous, satisfies the Palais–Smale condition and

(iii) $\lim_{\|x\|\to\infty}(\mathcal{A}(x) - \lambda\mathcal{F}(x)) = \infty$ *for every* $\lambda \in [0, \bar{a}]$.

Then there is an open interval $\Lambda \subset [0, \bar{a}]$ and a number $\nu > 0$ such that for each $\lambda \in \Lambda$, the equation $\mathcal{A}'(x) - \lambda\mathcal{F}'(x) = 0$ admits at least three distinct solutions in X having norm less than ν.

Proof of Theorem 1.1 completed. On account of Lemma 3.1, (9) and (10), we may choose $\rho_0 > 0$ so small such that

$$2\rho_0 < \|u_{\sigma_0}\|_a^2,$$
$$\frac{\sup\{\mathcal{F}(u)\colon\ \|u\|_a^2 < 2\rho_0\}}{\rho_0} < \frac{2K_{s_0,x_0,r_0}(\sigma_0)}{\|u_{\sigma_0}\|_a^2}.$$

By choosing $X = \mathcal{D}_a^{1,2}(\Omega)$, $\mathcal{A} = \frac{1}{2}\|\cdot\|_a^2$, $x_0 = 0$, $x_1 = u_{\sigma_0}$, and

$$\bar{a} = \frac{1+\rho_0}{\frac{2\mathcal{F}(u_{\sigma_0})}{\|u_{\sigma_0}\|_a^2} - \frac{\sup\{\mathcal{F}(u)\colon\ \|u\|_a^2<2\rho_0\}}{\rho_0}},$$

all the hypotheses of Theorem R1 are verified, see also Proposition 2.1 and Lemma 3.2.

Thus there exist an open interval $\Lambda \subset [0, \bar{a}]$ and a number $\nu > 0$ such that for each $\lambda \in \Lambda$, the equation $\mathcal{E}_\lambda'(u) \equiv \mathcal{A}'(u) - \lambda\mathcal{F}'(u) = 0$ admits at least three distinct solutions in $\mathcal{D}_a^{1,2}(\Omega)$ having $\mathcal{D}_a^{1,2}(\Omega)$-norm less than ν. Since one of them may be the trivial one ($f(0) = 0$, see (f_2)), we still have at least two nontrivial solutions of $(\mathcal{P}_\lambda)$ with the required properties. □

4. Proof of Theorems 1.2

Throughout of this section, we assume that the assumptions of Theorem 1.2 are fulfilled.

Let us define the function

$$\beta(t) = \sup\{\mathcal{F}(u)\colon\ \|u\|_a^2 < 2t\}, \quad t > 0.$$

Then, Lemma 3.1 yields that

$$\lim_{t\to 0^+} \frac{\beta(t)}{t} = 0. \tag{11}$$

Take the function from (8) for $\sigma_0 > 0$ fixed in the previous section; thus, $u_{\sigma_0} \in \mathcal{D}_a^{1,2}(\Omega) \setminus \{0\}$ and $\mathcal{F}(u_{\sigma_0}) > 0$. Therefore it is possible to choose a number $\eta > 0$ such that

$$0 < \eta < \mathcal{F}(u_{\sigma_0})\frac{2}{\|u_{\sigma_0}\|_a^2}.$$

From (11) we get the existence of a number $t_0 \in (0, \|u_{\sigma_0}\|_a^2/2)$ such that $\beta(t_0) < \eta t_0$. Thus

$$\beta(t_0) < \frac{2}{\|u_{\sigma_0}\|_a^2}\mathcal{F}(u_{\sigma_0})t_0. \tag{12}$$

Due to the choice of t_0 and using (12), we conclude that there exists $\rho_0 > 0$ such that

$$\beta(t_0) < \rho_0 < \mathcal{F}(u_{\sigma_0})\frac{2}{\|u_{\sigma_0}\|_a^2}t_0 < \mathcal{F}(u_{\sigma_0}). \tag{13}$$

Define now the function $\mathcal{H} : \mathcal{D}_a^{1,2}(\Omega) \times \mathbb{I} \to \mathbb{R}$ by

$$\mathcal{H}(u,\lambda) = \mathcal{E}_\lambda(u) + \lambda\rho_0,$$

where $\mathbb{I} = [0,\infty)$. We prove that the following inequality holds:

$$\sup_{\lambda\in\mathbb{I}} \inf_{u\in\mathcal{D}_a^{1,2}(\Omega)} \mathcal{H}(u,\lambda) < \inf_{u\in\mathcal{D}_a^{1,2}(\Omega)} \sup_{\lambda\in\mathbb{I}} \mathcal{H}(u,\lambda). \tag{14}$$

The function

$$\lambda \in \mathbb{I} \mapsto \inf_{u\in\mathcal{D}_a^{1,2}(\Omega)}\left[\frac{1}{2}\|u\|_a^2 + \lambda\big(\rho_0 - \mathcal{F}(u)\big)\right]$$

is obviously upper semicontinuous on $\mathbb{I}$. It follows from (13) that

$$\lim_{\lambda\to+\infty} \inf_{u\in\mathcal{D}_a^{1,2}(\Omega)} \mathcal{H}(u,\lambda) \leqslant \lim_{\lambda\to+\infty}\left[\frac{1}{2}\|u_{\sigma_0}\|_a^2 + \lambda\big(\rho_0 - \mathcal{F}(u_{\sigma_0})\big)\right] = -\infty.$$

Thus we find an element $\bar\lambda \in \mathbb{I}$ such that

$$\sup_{\lambda\in\mathbb{I}} \inf_{u\in\mathcal{D}_a^{1,2}(\Omega)} \mathcal{H}(u,\lambda) = \inf_{u\in\mathcal{D}_a^{1,2}(\Omega)}\left[\frac{1}{2}\|u\|_a^2 + \bar\lambda\big(\rho_0 - \mathcal{F}(u)\big)\right]. \tag{15}$$

Since $\beta(t_0) < \rho_0$, it follows that for all $u \in \mathcal{D}_a^{1,2}(\Omega)$ with $\|u\|_a^2 < 2t_0$ we have $\mathcal{F}(u) < \rho_0$. Hence

$$t_0 \leqslant \inf\left\{\frac{1}{2}\|u\|_a^2\colon\ \mathcal{F}(u) \geqslant \rho_0\right\}. \tag{16}$$

On the other hand,

$$\inf_{u\in\mathcal{D}_a^{1,2}(\Omega)} \sup_{\lambda\in\mathbb{I}} \mathcal{H}(u,\lambda) = \inf_{u\in\mathcal{D}_a^{1,2}(\Omega)}\left[\frac{1}{2}\|u\|_a^2 + \sup_{\lambda\in\mathbb{I}}\big(\lambda\big(\rho_0 - \mathcal{F}(u)\big)\big)\right] = \inf_{u\in\mathcal{D}_a^{1,2}(\Omega)}\left\{\frac{1}{2}\|u\|_a^2\colon\ \mathcal{F}(u) \geqslant \rho_0\right\}.$$

Thus inequality (16) is equivalent to

$$t_0 \leqslant \inf_{u\in\mathcal{D}_a^{1,2}(\Omega)} \sup_{\lambda\in\mathbb{I}} \mathcal{H}(u,\lambda). \tag{17}$$

We consider the following two cases:

(I) If $0 \leqslant \bar\lambda < \frac{t_0}{\rho_0}$, then we have that

$$\inf_{u\in\mathcal{D}_a^{1,2}(\Omega)}\left[\frac{1}{2}\|u\|_a^2 + \bar\lambda\big(\rho_0 - \mathcal{F}(u)\big)\right] \leqslant \mathcal{H}(0,\bar\lambda) = \bar\lambda\rho_0 < t_0.$$

Combining this inequality with (15) and (17) we obtain (14).

(II) If $\frac{t_0}{\rho_0} \leqslant \bar\lambda$, then from the fact that $\rho_0 < \mathcal{F}(u_{\sigma_0})$ and from (13), it follows that

$$\inf_{u\in\mathcal{D}_a^{1,2}(\Omega)}\left[\frac{1}{2}\|u\|_a^2 + \bar\lambda\big(\rho_0 - \mathcal{F}(u)\big)\right] \leqslant \frac{1}{2}\|u_{\sigma_0}\|_a^2 + \bar\lambda\big(\rho_0 - \mathcal{F}(u_{\sigma_0})\big) \leqslant \frac{1}{2}\|u_{\sigma_0}\|_a^2 + \frac{t_0}{\rho_0}\big(\rho_0 - \mathcal{F}(u_{\sigma_0})\big) < t_0,$$

which proves (14).

Now, we are in the position to apply the following result of Ricceri:

Theorem R2. *(See [11, Theorem 5].) Let X be a reflexive real Banach space, $\mathbb{I} \subset \mathbb{R}$ an interval, and let $\Psi : X \times \mathbb{I} \to \mathbb{R}$ be a function such that $\Psi(x, \cdot)$ is concave on $\mathbb{I}$ for all $x \in X$, and $\Psi(\cdot, \lambda)$ is continuous, coercive and sequentially weakly lower semicontinuous on X for all $\lambda \in \mathbb{I}$. Further, assume that*

$$\sup_{\lambda \in \mathbb{I}} \inf_{x \in X} \Psi(x, \lambda) < \inf_{x \in X} \sup_{\lambda \in \mathbb{I}} \Psi(x, \lambda).$$

Then, for each $\gamma > \sup_{\lambda \in \mathbb{I}} \inf_{x \in X} \Psi(x, \lambda)$, there exists a nonempty open set $C \subset \mathbb{I}$ with the following property: For every $\lambda \in C$ and every sequentially weakly lower semicontinuous functional $\Phi : X \to \mathbb{R}$ there exists $\delta > 0$ such that, for each $\mu \in (0, \delta)$, the functional $\Psi(\cdot, \lambda) + \mu\Phi(\cdot)$ has at least two distinct local minima lying in the set $\{x \in X\colon \Psi(x, \lambda) < \gamma\}$.

Proof of Theorem 1.2(i) completed. We choose in Theorem R2: $X = \mathcal{D}_a^{1,2}(\Omega)$, $\mathbb{I} = [0, \infty)$ and $\Psi = \mathcal{H}$. It is clear that for each $u \in \mathcal{D}_a^{1,2}(\Omega)$ the functional $\mathcal{H}(u, \cdot)$ is concave on $\mathbb{I}$. Obviously $\mathcal{H}(\cdot, \lambda)$ is continuous, and it follows from Proposition 2.1 that $\mathcal{H}(\cdot, \lambda)$ is coercive and sequentially weakly lower semicontinuous on $\mathcal{D}_a^{1,2}(\Omega)$. The minimax inequality is precisely relation (14).

Assume that g satisfies (g_1), and $a \leqslant c < a + 1$. We denote by

$$\mathcal{G}(u) = -\int_\Omega |x|^{-2c} G\big(u(x)\big)\, dx, \quad u \in \mathcal{D}_a^{1,2}(\Omega),$$

where $G(s) = \int_0^s g(t)\, dt$. The functional $\mathcal{G}$ is well defined, of class C^1, and sequentially weakly continuous on $\mathcal{D}_a^{1,2}(\Omega)$. The first two facts follow in a standard way; we deal only with the sequential weak continuity of $\mathcal{G}$. We suppose that there exists a sequence $\{u_k\} \subset \mathcal{D}_a^{1,2}(\Omega)$ which converges weakly to $u \in \mathcal{D}_a^{1,2}(\Omega)$ but $\{\mathcal{G}(u_k)\}$ does not converge to $\mathcal{G}(u)$ as $k \to \infty$. So, up to a subsequence, we can find a number $\varepsilon_0 > 0$ such that

$$0 < \varepsilon_0 \leqslant \big|\mathcal{G}(u_k) - \mathcal{G}(u)\big| \quad \text{for every } k \in \mathbb{N},$$

and $\{u_k\}$ converges strongly to u in $L^1(\Omega; |x|^{-2c}) \cap L^p(\Omega; |x|^{-2c})$, where $p \in (2, 2^*_{a,c})$ is from (g_1). Note that the pairs $r = 1$, $\alpha = 2c$, and $r = p$, $\alpha = 2c$ verify relations from Lemma A. Using Hölder inequality and (g_1), for every $k \in \mathbb{N}$ one has $0 < \theta_k < 1$ such that

$$\begin{aligned} 0 < \varepsilon_0 &\leqslant \big|\mathcal{G}(u_k) - \mathcal{G}(u)\big| \\ &\leqslant \int_\Omega |x|^{-2c} \big|g\big(u + \theta_k(u_k - u)\big)\big| |u_k - u|\, dx \\ &\leqslant c_g\big(\|u_k - u\|_{1,2c} + \big\|u + \theta_k(u_k - u)\big\|_{p,2c}^{p-1} \|u_k - u\|_{p,2c}\big). \end{aligned}$$

Since u_k converges strongly to u in $L^1(\Omega; |x|^{-2c}) \cap L^p(\Omega; |x|^{-2c})$, both terms in the right-hand side tend to 0 as $k \to \infty$, contradicting $\varepsilon_0 > 0$. Therefore, the functional $\mathcal{G}$ is sequential weak continuous.

Now, for a fixed $\gamma > \sup_{\lambda \in \mathbb{I}} \inf_{u \in \mathcal{D}_a^{1,2}(\Omega)} \mathcal{H}(u, \lambda)$, Theorem R2 assures that there exists a nonempty open set $C \subset \mathbb{I}$ with the property that for every $\lambda \in C$ there exists $\delta_1 > 0$ such that for each $\mu \in (0, \delta_1)$ the function $u \mapsto \mathcal{H}(u, \lambda) + \mu\mathcal{G}(u)$ has at least two local minima $u^1_{\lambda,\mu}$ and $u^2_{\lambda,\mu}$ belonging to the set $\{u \in \mathcal{D}_a^{1,2}(\Omega)\colon \mathcal{H}(u, \lambda) < \gamma\}$. Therefore, the energy functional $\mathcal{E}_{\lambda,\mu}$ associated to the problem $(\mathcal{P}_{\lambda,\mu})$, which is nothing but

$$\mathcal{E}_{\lambda,\mu}(u) = \mathcal{H}(u, \lambda) + \mu\mathcal{G}(u) - \lambda\rho_0$$

has two local minima in the set $\{u \in \mathcal{D}_a^{1,2}(\Omega)\colon \mathcal{H}(u, \lambda) < \gamma\}$. Consequently, $u^1_{\lambda,\mu}$ and $u^2_{\lambda,\mu}$ are weak solutions for the problem $(\mathcal{P}_{\lambda,\mu})$.

Finally let $A = [c_0, c_1] \subset C$ be any non-degenerate compact interval with $c_0 > 0$. It is clear that

$$\bigcup_{\lambda \in [c_0, c_1]} \big\{u \in \mathcal{D}_a^{1,2}(\Omega)\colon \mathcal{H}(u, \lambda) \leqslant \gamma\big\} \subseteq \big\{u \in \mathcal{D}_a^{1,2}(\Omega)\colon \mathcal{H}(u, c_0) \leqslant \gamma\big\} \cup \big\{u \in \mathcal{D}_a^{1,2}(\Omega)\colon \mathcal{H}(u, c_1) \leqslant \gamma\big\}.$$

Since $\mathcal{H}(\cdot, \lambda) = \mathcal{E}_\lambda + \lambda\rho_0$ is coercive it follows that the set

$$S := \bigcup_{\lambda \in [c_0, c_1]} \{u \in \mathcal{D}_a^{1,2}(\Omega): \mathcal{H}(u, \lambda) \leqslant \gamma\}$$

is bounded. Hence the $\|\cdot\|_a$-norms of the local minima of $\mathcal{E}_{\lambda,\mu}$ are less or equal than ν_1, where $\nu_1 = \sup_{u \in S} \|u\|_a$.

Proof of Theorem 1.2(ii) completed. Since (g_2) implies (g_1), we may consider $\lambda \in A = [c_0, c_1]$ and $\mu \in (0, \delta_1)$ from (i), i.e., the functional $\mathcal{E}_{\lambda,\mu}$ has at least two local minima $u^1_{\lambda,\mu}, u^2_{\lambda,\mu} \in S$.

In order to establish the existence of the third solution, we prove that $\mathcal{E}_{\lambda,\mu}$ is still coercive for $\lambda \in A$ and μ small enough. Condition (g_2) implies the existence of a constant $m > 0$ such that

$$|G(s)| \leqslant m|s|^2 + m|s| \quad \text{for every } s \in \mathbb{R}.$$

This inequality yields that

$$|\mathcal{G}(u)| \leqslant mC^2_{2,2c}\|u\|^2_a + mC_{1,2c}\|u\|_a. \tag{18}$$

Let $\delta_2 = \min\{\delta_1, 2^{-1}m^{-1}(1+\lambda)^{-1}C^{-2}_{2,2c}\}$ and fix $\mu \in (0, \delta_2)$. Using (6) and (18) we get that

$$\mathcal{E}_{\lambda,\mu}(u) \geqslant \left(\frac{1}{2(1+\lambda)} - \mu m C^2_{2,2c}\right)\|u\|^2_a - \left(\lambda C_{1,2b} \max_{|t| \leqslant \delta_0} |f(t)| + \mu m C^2_{1,2c}\right)\|u\|_a.$$

Due to the choice of δ_2, it follows that the functional $\mathcal{E}_{\lambda,\mu}$ is coercive. Thus, as in Lemma 3.2, $\mathcal{E}_{\lambda,\mu}$ satisfies the (PS)-condition whenever $\lambda \in A$ and $\mu \in (0, \delta_2)$.

For $\lambda \in A$ and $\mu \in (0, \delta_2)$ fixed, let us consider the set $\Gamma_{\lambda,\mu}$ of continuous paths $w : [0,1] \to \mathcal{D}_a^{1,2}(\Omega)$ joining $u^1_{\lambda,\mu}$ and $u^2_{\lambda,\mu}$, and define the minimax value

$$c_{\lambda,\mu} = \inf_{w \in \Gamma_{\lambda,\mu}} \max_{t \in [0,1]} \mathcal{E}_{\lambda,\mu}(w(t)).$$

Combining [8, Theorem 1] and [8, Corollary 1], there exists a third critical point $u^3_{\lambda,\mu} \in \mathcal{D}_a^{1,2}(\Omega)$ for $\mathcal{E}_{\lambda,\mu}$ which is different from $u^1_{\lambda,\mu}$ and $u^2_{\lambda,\mu}$ and $\mathcal{E}_{\lambda,\mu}(u^3_{\lambda,\mu}) = c_{\lambda,\mu}$.

It remains to find a norm estimate for $u^3_{\lambda,\mu}$ as we did in (i) for $u^1_{\lambda,\mu}$ and $u^2_{\lambda,\mu}$, respectively. To complete this, let us fix the path $w_0 \in \Gamma_{\mu,\lambda}$ defined by

$$w_0(t) = (1-t)u^1_{\lambda,\mu} + tu^2_{\lambda,\mu} \quad \text{for all } t \in [0,1].$$

Note that for all $t \in [0,1]$ we have $\|w_0(t)\|_a < \nu_1$. Consequently, due to (5) and (18) we have

$$\begin{aligned}\mathcal{E}_{\lambda,\mu}(w_0(t)) &\leqslant \frac{1}{2\|w_0(t)\|^2_a} + \lambda|\mathcal{F}(w_0(t))| + \mu|\mathcal{G}(w_0(t))| \\ &\leqslant \frac{1}{2}\nu_1^2 + c_1 M C^2_{2,2b}\nu_1^2 + c_1 M C_{1,2b}\nu_1 + \delta_1 m C^2_{2,2c}\nu_1^2 + \delta_1 m C_{1,2c}\nu_1 =: K.\end{aligned}$$

Therefore,

$$\mathcal{E}_{\lambda,\mu}(u^3_{\lambda,\mu}) = c_{\lambda,\mu} \leqslant \sup_{t \in [0,1]} \mathcal{E}_{\lambda,\mu}(w_0(t)) \leqslant K.$$

Now, we introduce for every $\mu \in [0, \delta_2]$ the set

$$Z_\mu = \{u \in \mathcal{D}_a^{1,2}(\Omega): \mathcal{E}_{\lambda,\mu}(u) \leqslant K\}.$$

Then, for every $\lambda \in A$ and $\mu \in (0, \delta_2)$ we have

$$u^3_{\lambda,\mu} \in Z := \bigcup_{\mu \in (0,\delta_2)} Z_\mu \subset \bigcup_{\mu \in [0,\delta_2]} Z_\mu \subset Z_0 \cup Z_{\delta_2}.$$

On the other hand, the coercivity of $\mathcal{E}_{\lambda,\mu}$ implies the boundedness of the set $Z \subset Z_0 \cup Z_{\delta_2}$. Therefore, there exists $\tilde{\nu} > 0$ such that $\|u\|_a < \tilde{\nu}$ for all $u \in Z$. Thus $\|u^i_{\lambda,\mu}\|_a < \max\{\nu_1, \tilde{\nu}\} =: \nu_2$ $(i \in \{1,2,3\})$. This concludes the proof. □

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