Multiple solutions for a degenerate elliptic equation involving sublinear terms at infinity ∗

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Abstract

Some multiplicity results are presented for the eigenvalue problem

\[
\begin{cases}
-\text{div}(|x|^{-2a}\nabla u) = \lambda |x|^{-2b} f(u) + \mu |x|^{-2c} g(u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \(\Omega \subset \mathbb{R}^n (n \geq 3)\) is an open bounded domain with smooth boundary, \(0 \in \Omega, 0 < a < \frac{n-2}{2}, a \leq b, c < a + 1,\) and \(f: \mathbb{R} \to \mathbb{R}\) is sublinear at infinity and superlinear at the origin. Various cases are treated depending on the behaviour of the nonlinear term \(g\).

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1. Introduction and main results

We consider the eigenvalue problem

\[
\begin{cases}
-\text{div}(|x|^{-2a}\nabla u) = \lambda |x|^{-2b} f(u(x)) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \(\Omega \subset \mathbb{R}^n (n \geq 3)\) is an open bounded domain with smooth boundary, \(0 \in \Omega, 0 < a < \frac{n-2}{2}, a \leq b < b + a + 1,\) and \(\lambda \in \mathbb{R}\) is a parameter.

Equations like \((P_\lambda)\) are introduced as model for several physical phenomena related to equilibrium of anisotropic media, see [6]. Due to this fact, problem \((P_\lambda)\) has been widely studied by several authors, see [1–3,7,13] and references therein. Usually, the nonlinear term \(f: \mathbb{R} \to \mathbb{R}\) is considered to be superlinear at infinity or simply, \(f(s) = |s|^{\theta-2}s\)

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with $\theta > 2$. The common assumption in these papers is the well-known Ambrosetti–Rabinowitz condition: denoting by $F(s) = \int_0^s f(t) \, dt$, there exist $s_0 > 0$ and $\theta > 2$ such that

$$0 < \theta F(s) \leq sf(s), \quad \forall s \in \mathbb{R}, \ |s| \geq s_0.$$  \hfill (AR)

A simple computation shows that (AR) implies

$$|f(s)| \geq c|s|^\theta - 1, \quad \forall s \in \mathbb{R}, \ |s| \geq s_0,$$  \hfill (AR')

with $c > 0$, i.e., $f$ is superlinear at infinity.

Our aim is to handle the counterpart of the above case, i.e., when $f : \mathbb{R} \to \mathbb{R}$ is sublinear at infinity. More precisely, we assume:

\begin{itemize}
  \item[(f_1)] $\lim_{|s| \to \infty} \frac{f(s)}{s} = 0$.
\end{itemize}

The presence of the parameter $\lambda \in \mathbb{R}$ is essential in our problem; indeed, if beside of (f$_1$), the nonlinear term $f$ is uniformly Lipschitz (with Lipschitz constant $L > 0$), then problem $(P_\lambda)$ has only the trivial solution whenever $|\lambda| < (LC_2^2 2b)^{-1}$; the constant $C_2 2b > 0$ is introduced after relation (3).

In order to state our main results, we introduce the weighted Sobolev space $\mathcal{D}_{a,2}^1(\Omega)$ where the solutions of $(P_\lambda)$ are going to be sought, which is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_a = \left( \int_\Omega |x|^{-2a} |\nabla u|^2 \, dx \right)^{1/2}.$$  \hfill (1)\hfill (1)

Beside of (f$_1$), we assume

\begin{itemize}
  \item[(f$_2$)] $\lim_{s \to 0} \frac{f(s)}{s} = 0$ (superlinearity at zero);
  \item[(f$_3$)] $\sup_{s \in \mathbb{R}} F(s) > 0$.
\end{itemize}

**Theorem 1.1.** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies (f$_1$), (f$_2$) and (f$_3$). Then, there exist an open interval $\Lambda \subset (0, \infty)$ and a constant $\nu > 0$ such that for every $\lambda \in \Lambda$ problem $(P_\lambda)$ has at least two nontrivial weak solutions in $\mathcal{D}_{a,2}^1(\Omega)$ whose $\| \cdot \|_a$-norms are less than $\nu$.

Now, we consider the perturbation of the problem $(P_\lambda)$ in the form

$$\begin{cases}
  -\text{div}(|x|^{-2a} \nabla u) = \lambda |x|^{-2b} f(u) + \mu |x|^{-2c} g(u) & \text{in } \Omega, \\
  u = 0 & \text{on } \partial \Omega,
\end{cases}$$  \hfill (P{\lambda, \mu})

where $0 < a < \frac{2-2c}{2}$, $a \leq b, c < a + 1$ and for the continuous function $g : \mathbb{R} \to \mathbb{R}$ we introduce the hypotheses:

\begin{itemize}
  \item[(g$_1$)] there exist $p \in (2, 2a, c) \cap \mathbb{N}$ with $2a,c = \min\{\frac{2a}{a-2}, \frac{2(a-2c)}{a-2(a+c)}\}$ and $c_8 > 0$ such that $|g(s)| \leq c_8 (1 + |s|^{p-1})$ for every $s \in \mathbb{R}$;
  \item[(g$_2$)] $\lim_{|s| \to \infty} \frac{|g(s)|}{|s|} = l < +\infty$ (asymptotically linear at infinity).
\end{itemize}

It is clear that (g$_2$) implies (g$_1$).

**Theorem 1.2.** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies the conditions (f$_1$), (f$_2$), (f$_3$). Then, there exists a nondegenerate compact interval $A \subset [0, \infty)$ with the following properties:

\begin{itemize}
  \item[(i)] there exists a number $v_1 > 0$ such that for every $\lambda \in A$ and every continuous function $g : \mathbb{R} \to \mathbb{R}$ verifying (g$_1$), there exists $\delta_1 > 0$ with the property that for each $\mu \in (0, \delta_1)$ the problem $(P{\lambda, \mu})$ has at least two distinct weak solutions whose $\| \cdot \|_a$-norms are less than $v_1$;
  \item[(ii)] there exists a number $v_2 > 0$ such that for every $\lambda \in A$ and every continuous function $g : \mathbb{R} \to \mathbb{R}$ verifying (g$_2$), there exists $\delta_2 > 0$ with the property that for each $\mu \in (0, \delta_2)$ the problem $(P{\lambda, \mu})$ has at least three distinct weak solutions whose $\| \cdot \|_a$-norms are less than $v_2$.
\end{itemize}
It is worth to notice that problem \((P_{\lambda,\mu})\) may be viewed in particular as a degenerate elliptic problem involving concave–convex nonlinearities whenever \((g_1)\) holds; indeed, \(f\) has a sublinear growth at infinity, while \(g\) can be superlinear (and subcritical) at infinity.

The main ingredient for the proof of Theorem 1.1 is a recent critical point result due to Bonanno [4] which is actually a refinement of a result of Ricceri [9,10]. In the proof of Theorem 1.2 we use a recent result of Ricceri [11] and a version of the mountain pass theorem due to Pucci and Serrin [8].

\section{Preliminaries}

The starting point of the variational approach to problems \((P_{\lambda})\) and \((P_{\lambda,\mu})\) is the weighted Sobolev–Hardy inequality due to Caffarelli, Kohn, Nirenberg [5]: for all \(u \in C_0^\infty(\mathbb{R}^n)\), there is a constant \(K_{a,b} > 0\) such that

\[
\left( \int_{\mathbb{R}^n} |x|^{-bq} |u|^q \, dx \right)^{2/q} \leq K_{a,b} \int_{\mathbb{R}^n} |x|^{-2a} |\nabla u|^2 \, dx,
\]

where

\[-\infty < a < \frac{n-2}{2}, \quad a \leq b < a + 1, \quad q = 2^*(a,b) = \frac{2n}{n-2d}, \quad d = 1 + a - b.\]

From the boundedness of \(\Omega\) and standard approximations argument, it is easy to see that (1) holds on \(D^{1,2}_a(\Omega)\); more precisely, for every

\[1 \leq r \leq \frac{2n}{n-2} \quad \text{and} \quad \frac{\alpha}{r} \leq (1+a) + n\left(\frac{1}{r} - \frac{1}{2}\right),\]

we have

\[
\left( \int_{\Omega} |x|^{-\alpha} |u|^r \, dx \right)^{2/r} \leq C \int_{\Omega} |x|^{-2a} |\nabla u|^2 \, dx, \quad u \in D^{1,2}_a(\Omega),
\]

that is, the embedding \(D^{1,2}_a(\Omega) \hookrightarrow L^r(\Omega; |x|^{-\alpha})\) is continuous, where \(L^r(\Omega; |x|^{-\alpha})\) is the weighted \(L^r\)-space with the norm

\[\|u\|_{r,\alpha} = \|u\|_{L^r(\Omega; |x|^{-\alpha})} = \left( \int_{\Omega} |x|^{-\alpha} |u|^r \, dx \right)^{1/r}.\]

We denote by \(C_{r,\alpha} > 0\) the best Sobolev constant of the embedding \(D^{1,2}_a(\Omega) \hookrightarrow L^r(\Omega; |x|^{-\alpha})\). The following version of the Rellich–Kondrachov compactness theorem can be stated, see Xuan [12,13].

\begin{lemma}
Suppose that \(\Omega \subset \mathbb{R}^n\) is an open bounded domain with \(C^1\) boundary and \(0 \in \Omega, \; n \geq 3, \; -\infty < a < \frac{n-2}{2}\). The embedding \(D^{1,2}_a(\Omega) \hookrightarrow L^r(\Omega; |x|^{-\alpha})\) is compact if \(1 \leq r < \frac{2n}{n-2}\) and \(\alpha < (1+a)r + n\left(1 - \frac{r}{2}\right)\).
\end{lemma}

First, we associate the energy functional \(E_\lambda : D^{1,2}_a(\Omega) \to \mathbb{R}\) to problem \((P_{\lambda})\), given by

\[E_\lambda(u) = \frac{1}{2} \|u\|_a^2 - \lambda \mathcal{F}(u), \quad u \in D^{1,2}_a(\Omega),\]

where \(\mathcal{F}(u) = \int_{\Omega} |x|^{-2b} F(u(x)) \, dx\) and \(F(s) = \int_0^s f(t) \, dt\).

\begin{proposition}
Assume \((f_1)\) and \((f_2)\) hold. Then, for every \(\lambda \in \mathbb{R}\) the functional \(E_\lambda\) is well defined, of class \(C^1\) on \(D^{1,2}_a(\Omega)\), sequentially weakly lower semicontinuous, and coercive. Moreover, every critical point of \(E_\lambda\) is a weak solution of \((P_{\lambda})\).
\end{proposition}

\begin{proof}
Fix \(\lambda \in \mathbb{R}\). Combining \((f_1)\) and \((f_2)\), there exists \(M > 0\) such that

\[|f(s)| \leq M(1 + |s|) \quad \text{for all} \; s \in \mathbb{R}.\]

\end{proof}
Then, for every $u \in D^{1,2}_a(\Omega)$, we have
\[
|\mathcal{F}(u)| \leq M(C_{1,2b} \|u\|_a + C_{2,2b}^2 \|u\|_a^2) < \infty.
\] (5)

Note that the pairs $r = 1$, $\alpha = 2b$ and $r = 2$, $\alpha = 2b$ verify relation (2). Consequently, $\mathcal{E}_\lambda$ is well defined.

One can see in a standard way that $\mathcal{E}_\lambda$ is of class $C^1$ on $D^{1,2}_a(\Omega)$ and every critical point of $\mathcal{E}_\lambda$ is a weak solution of $(P_\lambda)$.

We prove that $\mathcal{F}$ is sequential weak continuous which clearly implies the sequential weak lower semicontinuity of $\mathcal{E}_\lambda$. To do this, we argue by contradiction; let $\{u_k\} \subset D^{1,2}_a(\Omega)$ be a sequence which converges weakly to $u \in D^{1,2}_a(\Omega)$ but $|\mathcal{F}(u_k)|$ does not converge to $\mathcal{F}(u)$ as $k \to \infty$. Therefore, up to a subsequence, one can find a number $\varepsilon_0 > 0$ such that
\[
0 < \varepsilon_0 \leq |\mathcal{F}(u_k) - \mathcal{F}(u)| \quad \text{for every } k \in \mathbb{N},
\]
and $\{u_k\}$ converges strongly to $u$ in $L^1(\Omega; |x|^{-2b}) \cap L^2(\Omega; |x|^{-2b})$. Here, the pairs $r = 1$, $\alpha = 2b$, and $r = 2$, $\alpha = 2b$ verify relations from Lemma A. Using Hölder inequality and (4), for every $k \in \mathbb{N}$ one has $0 < \theta_k < 1$ such that
\[
0 < \varepsilon_0 \leq |\mathcal{F}(u_k) - \mathcal{F}(u)| \leq \int_\Omega |x|^{-2b} |f(u + \theta_k(u_k - u))| |u_k - u| \, dx
\]
\[
\leq M \left( \|u_k - u\|_{1,2b} + \|u_k + \theta_k(u_k - u)\|_{2,2b} \|u_k - u\|_{2,2b} \right).
\]

Since $\{u_k\}$ converges strongly to $u$ in $L^1(\Omega; |x|^{-2b}) \cap L^2(\Omega; |x|^{-2b})$, both terms in the right-hand side tend to 0 as $k \to \infty$, contradicting $\varepsilon_0 > 0$.

We prove now that $\mathcal{E}_\lambda$ is coercive. By $(f_1)$ there exists $\delta_0 = \delta(\lambda) > 0$ such that
\[
|f(s)| \leq C_{2,2b}^{-2}(1 + |\lambda|)^{-1} |s| \quad \text{for every } |s| \geq \delta_0.
\]

Integrating the above inequality we get that
\[
|F(s)| \leq \frac{1}{2} C_{2,2b}^{-2}(1 + |\lambda|)^{-1} |s|^2 + \max_{|t| \leq \delta_0} |f(t)| |s| \quad \text{for every } s \in \mathbb{R}.
\]

Thus, for every $u \in D^{1,2}_a(\Omega)$, we have
\[
|\mathcal{F}(u)| \leq \frac{1}{2} (1 + |\lambda|) \|u\|_a^2 + C_{1,2b} \max_{|t| \leq \delta_0} |f(t)| \|u\|_a.
\] (6)

Using (6), we obtain the inequality
\[
\mathcal{E}_\lambda(u) \geq \frac{1}{2} \|u\|_a^2 - |\lambda| \|\mathcal{F}(u)\| \geq \frac{1}{2(1 + |\lambda|)} \|u\|_a^2 - |\lambda| C_{1,2b} \max_{|t| \leq \delta_0} |f(t)| \|u\|_a.
\]

Consequently, if $\|u\|_a \to \infty$ then $\mathcal{E}_\lambda(u) \to \infty$ as well, i.e., $\mathcal{E}_\lambda$ is coercive. \[\square\]

3. Proof of Theorem 1.1

Throughout of this section, we assume that the assumptions of Theorem 1.1 are fulfilled. First, we prove two lemmas.

**Lemma 3.1.** \(\lim_{\rho \to 0^+} \frac{\sup_{\|u\| < 2\rho} \|\mathcal{F}(u)\|}{\rho} = 0.\)

**Proof.** Due to $(f_2)$, for an arbitrary small $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that
\[
|f(s)| \leq \frac{\varepsilon}{2} C_{2,2b}^{-2} |s| \quad \text{for every } |s| < \delta.
\]

Combining the above inequality with (4), we obtain
\[
|F(s)| \leq \varepsilon C_{2,2b}^{-2} |s|^2 + K(\delta)|s|^q \quad \text{for every } s \in \mathbb{R},
\] (7)
where $q \in (2, \min\left\{\frac{2n}{n-2}, \frac{2(n-2b)}{n-2(a+1)}\right\})$ is fixed and $K(\delta) > 0$ does not depend on $s$. \[\square\]
From (7) we get
\[ \mathcal{F}(u) \leq \varepsilon C_{q,2b}^2 \int_{\Omega} |x|^{-2b} |u|^2 \, dx + K(\delta) \int_{\Omega} |x|^{-2b} |u|^q \, dx \leq \varepsilon \|u\|_a^2 + K(\delta) C_{q,2b}^q \|u\|_a^q. \]

From the above relation we obtain that
\[ \sup \{ \mathcal{F}(u) : \|u\|_a^2 < 2\rho \} \leq 2\varepsilon \rho + K(\delta) C_{q,2b}^q (2\rho)^{\frac{q}{2}}. \]

Because \( q > 2 \) and \( \varepsilon > 0 \) is arbitrarily, we obtain
\[ \lim_{\rho \to 0^+} \frac{\sup \{ \mathcal{F}(u) : \|u\|_a^2 < 2\rho \}}{\rho} = 0. \]

**Lemma 3.2.** For every \( \lambda \in \mathbb{R} \) the functional \( \mathcal{E}_\lambda \) satisfies the usual (PS)-condition.

**Proof.** Let \( \{u_k\} \subset D_1^{1,2}(\Omega) \) be a (PS)-sequence, i.e., \( \{\mathcal{E}_\lambda(u_k)\} \) is bounded and \( \mathcal{E}_\lambda'(u_k) \to 0 \) in \( (D_1^{1,2}(\Omega))^* \) as \( k \to \infty \).

Since the function \( \mathcal{E}_\lambda \) is coercive, it follows that the sequence \( \{u_k\} \) is bounded in \( D_1^{1,2}(\Omega) \). Up to a subsequence, we may suppose that \( u_k \to u \) weakly in \( D_1^{1,2}(\Omega) \), and \( u_k \to u \) strongly in \( L^1(\Omega; |x|^{-2b}) \cap L^2(\Omega; |x|^{-2b}) \) for some \( u \in D_1^{1,2}(\Omega) \), see Lemma A. On the other hand, we have
\[ \|u_k - u\|_a^2 = \mathcal{E}_\lambda'(u_k)(u_k - u) + \mathcal{E}_\lambda'(u)(u - u_k) + \lambda \int_{\Omega} |x|^{-2b} \left| f(u_k(x)) - f(u(x)) \right| (u_k(x) - u(x)) \, dx. \]

It is clear the first two terms from the last expression tend to 0, while by means of (4) and Hölder’s inequality, one has
\[ \int_{\Omega} |x|^{-2b} \left| f(u_k(x)) - f(u(x)) \right| |u_k(x) - u(x)| \, dx \]
\[ \leq M \left[ 2\|u_k - u\|_{1,2b} + (\|u_k\|_{2,2b} + \|u\|_{2,2b}) \|u_k - u\|_{2,2b} \right] \to 0 \]
as \( k \to \infty \). Thus, we have \( \|u_k - u\|_a \to 0 \) as \( k \to \infty \).

Let \( s_0 \in \mathbb{R} \) such that \( F(s_0) > 0 \), see (f3). Here and in the sequel, let \( x_0 \in \Omega \) and \( r_0 > 0 \) so small such that \( |x_0| > r_0 \) and \( B(x_0, r_0) \subset \Omega \). Then, clearly, \( B(x_0, r_0) \subset \Omega \setminus \{0\} \). As usual \( B(x_0, r_0) \) denotes the \( n \)-dimensional open ball with center in \( x_0 \) and radius \( r_0 > 0 \).

For \( \sigma \in (0, 1) \) we define
\[ u_\sigma(x) = \begin{cases} 0, & \text{if } x \in \Omega \setminus B(x_0, r_0); \\ \frac{s_0}{1-\sigma} - \frac{s_0}{r_0(1-\sigma)} |x - x_0|, & \text{if } x \in B(x_0, r_0) \setminus B(x_0, \sigma r_0); \\ \frac{s_0}{1-\sigma}, & \text{if } x \in B(x_0, \sigma r_0). \end{cases} \]

It is clear that \( u_\sigma \in D_1^{1,2}(\Omega) \). Moreover, one has
\[ \|u_\sigma\|_a^2 \geq s_0^2 (|x_0| + r_0)^{-2a} (1 - \sigma^{-2}) (1 - \sigma^n) \omega_n r_0^{n-2} \]
and
\[ \mathcal{F}(u_\sigma) \geq K_{s_0, x_0, r_0}(\sigma), \]
where
\[ K_{s_0, x_0, r_0}(\sigma) = \left[ F(s_0)(|x_0| + r_0)^{-2b} \sigma^n - \max_{|t| \leq |s_0|} |F(t)(|t_0| - r_0)^{-2b}(1 - \sigma^n)| \right] \omega_n r_0^n \]
and \( \omega_n \) denotes the volume of the \( n \)-dimensional unit ball. For \( \sigma \) close enough to 1, the right-hand side of (10) becomes strictly positive; choose such a number, say \( \sigma_0 \).

Now, we recall a recent result from critical point theory, due to Ricceri [9,10], and Bonanno [4].
**Theorem R1.** (See [4, Theorem 2.1].) Let $X$ be a separable and reflexive real Banach space, and let $A, F : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals. Assume that there exists $x_0 \in X$ such that $A(x_0) = F(x_0) = 0$ and $A(x) \geq 0$ for every $x \in X$ and that there exist $x_1 \in X$, $\rho > 0$ such that

(i) $\rho < A(x_1)$;
(ii) $\sup_{A(x) < \rho} F(x) < \rho F(x_1)$.

Further, put

$$\tilde{\alpha} = \frac{\zeta \rho}{A(x_1)} - \sup_{A(x) < \rho} F(x),$$

with $\zeta > 1$, assume that the functional $A - \lambda F$ is sequentially weakly lower semicontinuous, satisfies the Palais–Smale condition and

(iii) $\lim_{\|x\| \to \infty} (A(x) - \lambda F(x)) = \infty$ for every $\lambda \in [0, \tilde{\alpha}]$.

Then there is an open interval $\Lambda \subset [0, \tilde{\alpha}]$ and a number $\nu > 0$ such that for each $\lambda \in \Lambda$, the equation $A'(x) - \lambda F'(x) = 0$ admits at least three distinct solutions in $X$ having norm less than $\nu$.

**Proof of Theorem 1.1 completed.** On account of Lemma 3.1, (9) and (10), we may choose $\rho_0 > 0$ so small such that

$$2\rho_0 < \|u_{\sigma_0}\|^2_a, \quad \sup \{ F(u) : \|u\|^2_a < 2\rho_0 \} < \frac{2K_{\sigma_0, x_0, r_0}(\sigma_0)}{\|u_{\sigma_0}\|^2_a \rho_0}. $$

By choosing $X = D^{1,2}_a(\Omega)$, $A = \frac{1}{2} \| \cdot \|^2_a$, $x_0 = 0$, $x_1 = u_{\sigma_0}$, and

$$\tilde{\alpha} = \frac{2F(u_{\sigma_0})}{\|u_{\sigma_0}\|^2_a} - \frac{1 + \rho_0}{\rho_0 \|u_{\sigma_0}\|^2_a \sup \{ F(u) : \|u\|^2_a < 2\rho_0 \}},$$

all the hypotheses of Theorem R1 are verified, see also Proposition 2.1 and Lemma 3.2.

Thus there exist an open interval $\Lambda \subset [0, \tilde{\alpha}]$ and a number $\nu > 0$ such that for each $\lambda \in \Lambda$, the equation $E'(u) \equiv A'(u) - \lambda F'(u) = 0$ admits at least three distinct solutions in $D^{1,2}_a(\Omega)$ having $D^{1,2}_a(\Omega)$-norm less than $\nu$. Since one of them may be the trivial one ($f(0) = 0$, see (f_2)), we still have at least two nontrivial solutions of $(P_{\lambda})$ with the required properties. \( \square \)

**4. Proof of Theorems 1.2**

Throughout of this section, we assume that the assumptions of Theorem 1.2 are fulfilled.

Let us define the function

$$\beta(t) = \sup \{ F(u) : \|u\|^2_a < 2t \}, \quad t > 0.$$

Then, Lemma 3.1 yields that

$$\lim_{t \to 0^+} \frac{\beta(t)}{t} = 0. \quad (11)$$

Take the function from (8) for $\sigma_0 > 0$ fixed in the previous section; thus, $u_{\sigma_0} \in D^{1,2}_a(\Omega) \setminus \{0\}$ and $F(u_{\sigma_0}) > 0$. Therefore it is possible to choose a number $\eta > 0$ such that

$$0 < \eta < \frac{2}{\|u_{\sigma_0}\|^2_a}.$$
From (11) we get the existence of a number $t_0 \in (0, \|u_{\sigma_0}\|_a^2/2)$ such that $\beta(t_0) < \eta t_0$. Thus
\[ \beta(t_0) < \frac{2}{\|u_{\sigma_0}\|_a^2} F(u_{\sigma_0}) t_0. \] (12)

Due to the choice of $t_0$ and using (12), we conclude that there exists $\rho_0 > 0$ such that
\[ \beta(t_0) < \rho_0 < F(u_{\sigma_0}) \frac{2}{\|u_{\sigma_0}\|_a^2} t_0 < F(u_{\sigma_0}). \] (13)

Define now the function $H : D_{a}^{1,2}(\Omega) \times \mathbb{R} \to \mathbb{R}$ by
\[ H(u, \lambda) = E_{\lambda}(u) + \lambda \rho_0, \]
where $\mathbb{R} = [0, \infty)$. We prove that the following inequality holds:
\[ \sup_{\lambda \in \mathbb{R}} \inf_{u \in D_{a}^{1,2}(\Omega)} H(u, \lambda) < \inf_{u \in D_{a}^{1,2}(\Omega)} \sup_{\lambda \in \mathbb{R}} H(u, \lambda). \] (14)

The function
\[ \lambda \to \inf_{u \in D_{a}^{1,2}(\Omega)} \left[ \frac{1}{2}\|u\|_{a}^2 + \lambda (\rho_0 - F(u)) \right] \]
is obviously upper semicontinuous on $\mathbb{R}$. It follows from (13) that
\[ \lim_{\lambda \to +\infty} \inf_{u \in D_{a}^{1,2}(\Omega)} H(u, \lambda) \leq \lim_{\lambda \to +\infty} \left[ \frac{1}{2}\|u_{\sigma_0}\|_{a}^2 + \lambda (\rho_0 - F(u_{\sigma_0})) \right] = -\infty. \]

Thus we find an element $\bar{\lambda} \in \mathbb{R}$ such that
\[ \sup_{\lambda \in \mathbb{R}} \inf_{u \in D_{a}^{1,2}(\Omega)} H(u, \lambda) = \inf_{u \in D_{a}^{1,2}(\Omega)} \sup_{\lambda \in \mathbb{R}} H(u, \lambda). \] (15)

Since $\beta(t_0) < \rho_0$, it follows that for all $u \in D_{a}^{1,2}(\Omega)$ with $\|u\|_{a}^2 < 2t_0$ we have $F(u) < \rho_0$. Hence
\[ t_0 \leq \inf \left\{ \frac{1}{2}\|u\|_{a}^2 : F(u) \geq \rho_0 \right\}. \] (16)

On the other hand,
\[ \inf_{u \in D_{a}^{1,2}(\Omega)} \sup_{\lambda \in \mathbb{R}} H(u, \lambda) = \inf_{u \in D_{a}^{1,2}(\Omega)} \left[ \frac{1}{2}\|u\|_{a}^2 + \sup_{\lambda \in \mathbb{R}} (\lambda (\rho_0 - F(u))) \right] = \inf_{u \in D_{a}^{1,2}(\Omega)} \left\{ \frac{1}{2}\|u\|_{a}^2 : F(u) \geq \rho_0 \right\}. \]

Thus inequality (16) is equivalent to
\[ t_0 \leq \inf_{u \in D_{a}^{1,2}(\Omega)} \sup_{\lambda \in \mathbb{R}} H(u, \lambda). \] (17)

We consider the following two cases:

(I) If $0 \leq \bar{\lambda} < \frac{t_0}{\rho_0}$, then we have that
\[ \inf_{u \in D_{a}^{1,2}(\Omega)} \left[ \frac{1}{2}\|u\|_{a}^2 + \bar{\lambda} (\rho_0 - F(u)) \right] \leq H(0, \bar{\lambda}) = \bar{\lambda} \rho_0 < t_0. \]

Combining this inequality with (15) and (17) we obtain (14).

(II) If $\frac{t_0}{\rho_0} \leq \bar{\lambda}$, then from the fact that $\rho_0 < F(u_{\sigma_0})$ and from (13), it follows that
\[ \inf_{u \in D_{a}^{1,2}(\Omega)} \left[ \frac{1}{2}\|u\|_{a}^2 + \bar{\lambda} (\rho_0 - F(u)) \right] \leq \frac{1}{2}\|u_{\sigma_0}\|_{a}^2 + \bar{\lambda} (\rho_0 - F(u_{\sigma_0})) \leq \frac{1}{2}\|u_{\sigma_0}\|_{a}^2 + \frac{t_0}{\rho_0} (\rho_0 - F(u_{\sigma_0})) < t_0, \]
which proves (14).

Now, we are in the position to apply the following result of Ricceri:
Theorem R2. (See [11, Theorem 5].) Let $X$ be a reflexive real Banach space, $I \subset \mathbb{R}$ an interval, and let $\Psi : X \times I \to \mathbb{R}$ be a function such that $\Psi(x, \cdot)$ is concave on $I$ for all $x \in X$, and $\Psi(\cdot, \lambda)$ is continuous, coercive and sequentially weakly lower semicontinuous on $X$ for all $\lambda \in I$. Further, assume that

$$
\sup_{x \in X} \inf_{\lambda \in I} \Psi(x, \lambda) < \inf_{x \in X} \sup_{\lambda \in I} \Psi(x, \lambda).
$$

Then, for each $\gamma > \sup_{\lambda \in I} \inf_{x \in X} \Psi(x, \lambda)$, there exists a nonempty open set $C \subset I$ with the following property: For every $\lambda \in C$ and every sequentially weakly lower semicontinuous functional $\Phi : X \to \mathbb{R}$ there exists $\delta > 0$ such that, for each $\mu \in (0, \delta)$, the functional $\Psi(\cdot, \lambda) + \mu \Phi(\cdot)$ has at least two distinct local minima lying in the set $\{x \in X : \Psi(x, \lambda) < \gamma\}$.

Proof of Theorem 1.2(i) completed. We choose in Theorem R2: $X = D^{1,2}_a(\Omega)$, $I = [0, \infty)$ and $\Psi = \mathcal{H}$. It is clear that for each $u \in D^{1,2}_a(\Omega)$ the functional $\mathcal{H}(u, \cdot)$ is concave on $I$. Obviously $\mathcal{H}(\cdot, \lambda)$ is continuous, and it follows from Proposition 2.1 that $\mathcal{H}(\cdot, \lambda)$ is coercive and sequentially weakly lower semicontinuous on $D^{1,2}_a(\Omega)$. The minimax inequality is precisely relation (14).

Assume that $g$ satisfies (g1), and $a \leq c < a + 1$. We denote by

$$
\mathcal{G}(u) = - \int_\Omega |x|^{-2c} G(u(x)) \, dx, \quad u \in D^{1,2}_a(\Omega),
$$

where $G(s) = \int_0^s g(t) \, dt$. The functional $\mathcal{G}$ is well defined, of class $C^1$, and sequentially weakly continuous on $D^{1,2}_a(\Omega)$. The first two facts follow in a standard way; we deal only with the sequential weak continuity of $\mathcal{G}$. We suppose that there exists a sequence $\{u_k\} \subset D^{1,2}_a(\Omega)$ which converges weakly to $u \in D^{1,2}_a(\Omega)$ but $\{\mathcal{G}(u_k)\}$ does not converge to $\mathcal{G}(u)$ as $k \to \infty$. So, up to a subsequence, we can find a number $\varepsilon_0 > 0$ such that

$$
0 < \varepsilon_0 \leq |\mathcal{G}(u_k) - \mathcal{G}(u)| \quad \text{for every } k \in \mathbb{N},
$$

and $\{u_k\}$ converges strongly to $u$ in $L^1(\Omega; |x|^{-2c}) \cap L^p(\Omega; |x|^{-2c})$, where $p \in (2, 2^*_a, c)$ is from (g1). Note that the pairs $r = 1, a = 2c$, and $r = p, a = 2c$ verify relations from Lemma A. Using Hölder inequality and (g1), for every $k \in \mathbb{N}$ one has $0 < \theta_k < 1$ such that

$$
0 < \varepsilon_0 \leq |\mathcal{G}(u_k) - \mathcal{G}(u)| \leq \int_\Omega |x|^{-2c} |g(u + \theta_k(u_k - u))| |u_k - u| \, dx
$$

$$
\leq c_k (\|u_k - u\|_{1,2c} + \|u + \theta_k(u_k - u)\|_{p,2c} \|u_k - u\|_{p,2c}).
$$

Since $u_k$ converges strongly to $u$ in $L^1(\Omega; |x|^{-2c}) \cap L^p(\Omega; |x|^{-2c})$, both terms in the right-hand side tend to 0 as $k \to \infty$, contradicting $\varepsilon_0 > 0$. Therefore, the functional $\mathcal{G}$ is sequential weakly continuous.

Now, for a fixed $\gamma > \sup_{\lambda \in I} \inf_{u \in D^{1,2}_a(\Omega)} \mathcal{H}(u, \lambda)$, Theorem R2 assures that there exists a nonempty open set $C \subset I$ with the property that for every $\lambda \in C$ there exists $\delta_1 > 0$ such that for each $\mu \in (0, \delta_1)$ the function $u \mapsto \mathcal{H}(u, \lambda) + \mu \mathcal{G}(u)$ has at least two local minima $u^{1, \lambda}_{\mu}$ and $u^{2, \lambda}_{\mu}$ belonging to the set $\{u \in D^{1,2}_a(\Omega) : \mathcal{H}(u, \lambda) < \gamma\}$. Therefore, the energy functional $\mathcal{E}_{\lambda, \mu}$ associated to the problem $(P_{\lambda, \mu})$, which is nothing but

$$
\mathcal{E}_{\lambda, \mu}(u) = \mathcal{H}(u, \lambda) + \mu \mathcal{G}(u) - \lambda \rho_0
$$

has two local minima in the set $\{u \in D^{1,2}_a(\Omega) : \mathcal{H}(u, \lambda) < \gamma\}$. Consequently, $u^{1, \lambda}_{\mu}$ and $u^{2, \lambda}_{\mu}$ are weak solutions for the problem $(P_{\lambda, \mu})$.

Finally let $A = [c_0, c_1] \subset C$ be any non-degenerate compact interval with $c_0 > 0$. It is clear that

$$
\bigcup_{\lambda \in [c_0, c_1]} \{u \in D^{1,2}_a(\Omega) : \mathcal{H}(u, \lambda) \leq \gamma\} \subseteq \{u \in D^{1,2}_a(\Omega) : \mathcal{H}(u, c_0) \leq \gamma\} \cup \{u \in D^{1,2}_a(\Omega) : \mathcal{H}(u, c_1) \leq \gamma\}.
$$

Since $\mathcal{H}(\cdot, \lambda) = \mathcal{E}_{\lambda} + \lambda \rho_0$ is coercive it follows that the set
On the other hand, the coercivity of $E$ is bounded. Hence the $\| \cdot \|_a$-norms of the local minima of $E_{\lambda, \mu}$ are less or equal than $v_1$, where $v_1 = \sup_{u \in S} \| u \|_a$.

**Proof of Theorem 1.2(ii) completed.** Since $(g_2)$ implies $(g_1)$, we may consider $\lambda \in A = [c_0, c_1]$ and $\mu \in (0, \delta_1)$ from (i), i.e., the functional $E_{\lambda, \mu}$ has at least two local minima $u_{1, \lambda, \mu}, u_{2, \lambda, \mu} \in S$. In order to establish the existence of the third solution, we prove that $E_{\lambda, \mu}$ is still coercive for $\lambda \in A$ and $\mu$ small enough. Condition $(g_2)$ implies the existence of a constant $m > 0$ such that

$$|G(s)| \leq m|s|^2 + m|s| \quad \text{for every } s \in \mathbb{R}.$$ 

This inequality yields that

$$|G(u)| \leq mC_{2,2c}^2 \| u \|_a^2 + mC_{1,2c} \| u \|_a.$$  \hspace{1cm} (18)

Let $\delta_2 = \min\{\delta_1, 2^{-1}m^{-1}(1 + \lambda)^{-1}C_{2,2c}^{-2}\}$ and fix $\mu \in (0, \delta_2)$. Using (6) and (18) we get that

$$E_{\lambda, \mu}(u) \geq \left(\frac{1}{2(1 + \lambda)} - \mu mC_{2,2c}^2\right) \| u \|_a^2 - \left(\lambda C_{1,2b} \max_{|t| \leq \delta_0} |f(t)| + \mu mC_{1,2c}\right) \| u \|_a.$$ 

Due to the choice of $\delta_2$, it follows that the functional $E_{\lambda, \mu}$ is coercive. Thus, as in Lemma 3.2, $E_{\lambda, \mu}$ satisfies the (PS)-condition whenever $\lambda \in A$ and $\mu \in (0, \delta_2)$.

For $\lambda \in A$ and $\mu \in (0, \delta_2)$ fixed, let us consider the set $\Gamma_{\lambda, \mu}$ of continuous paths $w : [0, 1] \to D_{a}^{1,2}(\Omega)$ joining $u_{1, \lambda, \mu}$ and $u_{2, \lambda, \mu}$, and define the minimax value

$$c_{\lambda, \mu} = \inf_{w \in \Gamma_{\lambda, \mu}} \max_{t \in [0, 1]} E_{\lambda, \mu}(w(t)).$$

Combining [8, Theorem 1] and [8, Corollary 1], there exists a third critical point $u_{3, \lambda, \mu} \in D_{a}^{1,2}(\Omega)$ for $E_{\lambda, \mu}$ which is different from $u_{1, \lambda, \mu}$ and $u_{2, \lambda, \mu}$ and $E_{\lambda, \mu}(u_{3, \lambda, \mu}) = c_{\lambda, \mu}$.

It remains to find a norm estimate for $u_{3, \lambda, \mu}$ as we did in (i) for $u_{1, \lambda, \mu}$ and $u_{2, \lambda, \mu}$, respectively. To complete this, let us fix the path $w_0 \in \Gamma_{\mu, \lambda}$ defined by

$$w_0(t) = (1 - t)u_{1, \lambda, \mu} + tu_{2, \lambda, \mu} \quad \text{for all } t \in [0, 1].$$

Note that for all $t \in [0, 1]$ we have $\| w_0(t) \|_a < v_1$. Consequently, due to (5) and (18) we have

$$E_{\lambda, \mu}(w_0(t)) \leq \frac{1}{2} \| w_0(t) \|_a^2 + \lambda |F(w_0(t))| + \mu |G(w_0(t))| \leq \frac{1}{2} v_1^2 + c_1 M C_{2,2b}^2 v_1^2 + c_1 M C_{1,2b} v_1 + \delta_1 m C_{2,2c}^2 v_2^2 + \delta_1 m C_{1,2c} v_1 =: K.$$ 

Therefore,

$$E_{\lambda, \mu}(u_{3, \lambda, \mu}) = c_{\lambda, \mu} \leq \sup_{t \in [0, 1]} E_{\lambda, \mu}(w_0(t)) \leq K.$$ 

Now, we introduce for every $\mu \in [0, \delta_2]$ the set

$$Z_\mu = \left\{ u \in D_{a}^{1,2}(\Omega) : E_{\lambda, \mu}(u) \leq K \right\}.$$ 

Then, for every $\lambda \in A$ and $\mu \in (0, \delta_2)$ we have

$$u_{3, \lambda, \mu} \in Z := \bigcup_{\mu \in (0, \delta_2)} Z_\mu \subset \bigcup_{\mu \in [0, \delta_2]} Z_\mu \subset Z_0 \cup Z_{\delta_2}.$$ 

On the other hand, the coercivity of $E_{\lambda, \mu}$ implies the boundedness of the set $Z \subset Z_0 \cup Z_{\delta_2}$. Therefore, there exists $\tilde{v} > 0$ such that $\| u \|_a < \tilde{v}$ for all $u \in Z$. Thus $u_{3, \lambda, \mu} \|_a < \max\{v_1, \tilde{v}\} =: v_2 \quad (i \in \{1, 2, 3\})$. This concludes the proof. \qed
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References


