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ASYMPTOTICALLY CRITICAL PROBLEMS ON HIGHER-DIMENSIONAL SPHERES

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Abstract. The paper is concerned with the equation $-\Delta_h u = f(u)$ on S^d where Δ_h denotes the Laplace-Beltrami operator on the standard unit sphere (S^d, h), while the continuous nonlinearity $f : \mathbb{R} \to \mathbb{R}$ oscillates either at zero or at infinity having an asymptotically critical growth in the Sobolev sense. In both cases, by using a group-theoretical argument and an appropriate variational approach, we establish the existence of $[d/2] + (-1)^{d+1} - 1$ sequences of sign-changing weak solutions in $H_1^2(S^d)$ whose elements in different sequences are mutually symmetrically distinct whenever f has certain symmetry and $d \geq 5$. Although we are dealing with a smooth problem, we are forced to use a non-smooth version of the principle of symmetric criticality (see Kobayashi-Ôtani, J. Funct. Anal. 214 (2004), 428-449). The L^∞– and H_1^2–asymptotic behaviour of the sequences of solutions are also fully characterized.

1. **Introduction.** We consider the nonlinear elliptic problem

$$-\Delta_h u = f(u) \ \text{ on } S^d, \tag{P}$$

where $\Delta_h u = \mathrm{div}_h(\nabla u)$ denotes the Laplace-Beltrami operator acting on $u : S^d \to \mathbb{R}$, (S^d, h) is the unit sphere, h being the canonical metric induced from $\mathbb{R}^{d+1}$.

Denoting formally 0^+ or $+\infty$ by the common symbol L (standing for a limit point), we assume on the continuous nonlinearity $f : \mathbb{R} \to \mathbb{R}$ that

(f_1^L) $-\infty < \liminf_{s\to L} \frac{F(s)}{s^2} \leq \limsup_{s\to L} \frac{F(s)}{s^2} = +\infty$;

(f_2^L) $\liminf_{s\to L} \frac{f(s)}{s} < 0$,

where $F(s) = \int_0^s f(t)dt$. One can easily observe that f has an *oscillatory behaviour at* L. In particular, a whole sequence of distinct, *constant* solutions for (P) appears as zeros of the function $s \mapsto f(s)$, $s > 0$.

The purpose of the present paper is to investigate the existence of *non*-constant solutions for (P) under the assumptions (f_1^L) and (f_2^L). This problem will be achieved by constructing sign-changing solutions for (P). We prove two multiplicity results corresponding to $L = 0^+$ and $L = +\infty$, respectively; the 'piquancy' is that not only infinitely many sign-changing solutions for (P) are guaranteed but we also

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give a lower estimate of the number of those sequences of solutions for (P) whose elements in different sequences are mutually symmetrically distinct.

In order to handle this problem, solutions for (P) are being sought in the standard Sobolev space $H_1^2(S^d)$ which is the completion of $C^\infty(S^d)$ with respect to the usual norm

$$\|u\|_{H_1^2} = \left(\int_{S^d} |\nabla u|^2 d\sigma_h + \int_{S^d} u^2 d\sigma_h\right)^{1/2}.$$

We say that $u \in H_1^2(S^d)$ is a weak solution for (P) if

$$\int_{S^d} \langle \nabla u, \nabla v\rangle d\sigma_h = \int_{S^d} f(u) v d\sigma_h \quad \text{for all } v \in H_1^2(S^d),$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product associated with the Riemannian metric h for 1-forms, and $d\sigma_h$ is the Riemannian measure.

Since we are interested in the existence of infinitely many sign-changing solutions, it seems some kind of symmetry hypothesis on the nonlinearity f is indispensable; namely, we assume that f is odd in an arbitrarily small neighborhood of the origin whenever $L = 0^+$, and f is odd on the whole $\mathbb{R}$ whenever $L = +\infty$. In the case $L = 0^+$ no further assumption on f is needed at infinity (neither symmetry nor growth of f; in particular, f may have even a *super*critical growth). However, when $L = +\infty$, we have to control the growth of f; we assume $f(s) = O(s^{\frac{d+2}{d-2}})$ as $s \to \infty$, i.e. f has an *asymptotically critical growth at infinity*. In both cases ($L = 0^+$ and $L = +\infty$), the energy functional $\mathcal{E} : H_1^2(S^d) \to \mathbb{R}$ associated with (P) is well-defined, which is the key tool in order to achieve our results.

The first task is to construct certain subspaces of $H_1^2(S^d)$ containing invariant functions under special actions defined by means of carefully chosen subgroups of the orthogonal group $O(d+1)$. A particular form of this construction has been first exploited by Ding [7]. In our case, every nontrivial element from these subspaces of $H_1^2(S^d)$ changes the sign. The main feature of these subspaces of $H_1^2(S^d)$ is based on the symmetry properties of their elements: no nontrivial element from one subspace can belong to another subspace, i.e., elements from distinct subspaces are distinguished by their symmetries. Consequently, guaranteeing nontrivial solutions for (P) in distinct subspaces of $H_1^2(S^d)$ of the above type, these elements cannot be compared with each other. We show by an explicit construction that the minimal number of these subspaces of $H_1^2(S^d)$ is $s_d = [d/2] + (-1)^{d+1} - 1$. Here, $[\cdot]$ denotes the integer function. For details, see Section 3.

We roughly describe the strategy to construct infinitely many distinct sign-changing solutions for (P) in a fixed subspace of $H_1^2(S^d)$ of the above type; for the sake of simplicity, we denote by W such a subspace of $H_1^2(S^d)$. Now, we restrict the energy functional $\mathcal{E}$ to W, denoting it by $\mathcal{E}_W$, and we fix certain L^∞-level sets in W, say $W_k \subset W$, $k \in \mathbb{N}$; the sequence $\{W_k\}_k$ is decreasing (resp. increasing) whenever $L = 0^+$ (resp. $L = +\infty$). Up to a subsequence of $\{W_k\}_k \subset W$, the relative minimizers of $\mathcal{E}_W$ to W_k have different energy levels; so their set is uncountable. The non-smooth principle of symmetric criticality for Szulkin-type functionals (see Kobayashi-Ôtani [11] and Akagi-Kobayashi-Ôtani [1]) and a careful truncation argument imply that the relative minimizers of $\mathcal{E}_W$ on the sets W_k are actually weak solutions of (P). In some respects, this approach is quite unusual: dealing with a problem in a pure smooth context we are forced to use a proper *non*-smooth principle. Moreover, the L^∞–norm and H_1^2–norm of the sequences of solutions for (P) tend to $L \in \{0^+, +\infty\}$ whenever f oscillates at L; this fact fully reflects the oscillatory behaviour of f at L.

Elliptic problems involving oscillatory nonlinearities have been studied in Omari-Zanolin [15], Ricceri [16], Saint Raymond [17], subjected to standard Neumann or Dirichlet boundary value conditions on bounded open domains of $\mathbb{R}^n$, or even on unbounded domains, see Faraci-Kristály [8], Kristály [12]. Results in finding sign-changing solutions for semilinear problems can be found in Li-Wang [13], Zou [19] and references therein. The strategy in these last papers is to construct suitable closed convex sets which contain all the positive and negative solutions in the interior, and are invariant with respect to some vector fields. Our approach is rather different than those of [13], [19] and is related to the works of Bartsch-Schneider-Weth [2] and Bartsch-Willem [3], where the existence of non-radial and sign-changing solutions are studied for Schrödinger and polyharmonic equations defined on $\mathbb{R}^n$. For further results concerning sign-changing solutions, see [4], [6], [14] and references therein.

The plan of the paper is as follows. In the sequel we state our main theorems. In Section 3, by using a group-theoretical argument, we explicitly construct $s_d = [d/2] + (-1)^{d+1} - 1$ subspaces of $H_1^2(S^d)$ with special symmetrical properties. In Sections 4 and 5 we prove our main Theorems 2.1 and 2.2, respectively, while Section 6 contains a list of concluding remarks.

2. **Main results.**

In the sequel, we denote by $\|\cdot\|_\infty$ the usual sup-norm on S^d. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $F(s) = \int_0^s f(t)dt$. We assume that

(f_1^0) $\quad -\infty < \liminf_{s\to 0^+} \frac{F(s)}{s^2} \leq \limsup_{s\to 0^+} \frac{F(s)}{s^2} = +\infty;$

(f_2^0) $\quad \liminf_{s\to 0^+} \frac{f(s)}{s} < 0.$

The first result can be formulated as follows:

Theorem 2.1. *Let $d \geq 5$ and $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which is odd in an arbitrarily small neighborhood of the origin, verifying (f_1^0) and (f_2^0). Then there exist at least $s_d = [d/2] + (-1)^{d+1} - 1$ sequences $\{u_k^i\}_k \subset H_1^2(S^d)$, $i \in \{1, ..., s_d\}$, of sign-changing weak solutions of* (P) *distinguished by their symmetry properties. In addition,*

$$\lim_{k\to\infty} \|u_k^i\|_\infty = \lim_{k\to\infty} \|u_k^i\|_{H_1^2} = 0 \quad \textit{for every} \quad i \in \{1, ..., s_d\}.$$

Example 1. Let $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\alpha + \beta > 1 > \alpha > 0$, and $\gamma \in (0, 1)$. Then, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(0) = 0$ and $f(s) = |s|^{\alpha-1}s(\gamma + \sin|s|^{-\beta})$ near the origin (but $s \neq 0$) and extended in an arbitrarily way to the whole $\mathbb{R}$, verifies both (f_1^0) and (f_2^0).

We have a counterpart of Theorem 2.1 when the nonlinear term oscillates at infinity. Instead of (f_1^0) and (f_2^0), respectively, we assume

(f_1^∞) $\quad -\infty < \liminf_{s\to\infty} \frac{F(s)}{s^2} \leq \limsup_{s\to\infty} \frac{F(s)}{s^2} = +\infty;$

(f_2^∞) $\quad \liminf_{s\to\infty} \frac{f(s)}{s} < 0.$

Unlike in Theorem 2.1 where no further assumption is needed at infinity, we have to control here the growth of f. We assume that f has an *asymptotically critical* growth at infinity, namely,

(f_3^∞) $\quad \limsup_{s\to\infty} \frac{|f(s)|}{1+s^{2^*-1}} < \infty$, where $2^* = \frac{2d}{d-2}$.

Our next result can be formulated as follows:

Theorem 2.2. *Let $d \geq 5$ and $f : \mathbb{R} \to \mathbb{R}$ be an odd, continuous function which verifies (f_1^∞), (f_2^∞) and (f_3^∞). Then there exist at least $s_d = [d/2] + (-1)^{d+1} - 1$ sequences $\{\tilde{u}_k^i\}_k \subset H_1^2(S^d)$, $i \in \{1, ..., s_d\}$, of sign-changing weak solutions of* (P) *distinguished by their symmetry properties. In addition,*

$$\lim_{k\to\infty} \|\tilde{u}_k^i\|_\infty = \lim_{k\to\infty} \|\tilde{u}_k^i\|_{H_1^2} = \infty \quad \text{for every} \quad i \in \{1, ..., s_d\}.$$

Example 2. Let $d \geq 5$ and $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\frac{d+2}{d-2} \geq \alpha > 1$, $|\alpha - \beta| < 1$, and $\gamma \in (0, 1)$. Then, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(s) = |s|^{\alpha-1} s(\gamma + \sin |s|^\beta)$ verifies the hypotheses (f_1^∞), (f_2^∞) and (f_3^∞), respectively.

3. **Subspaces of $H_1^2(S^d)$ with special symmetries: a group-theoretical argument.**

Let $d \geq 5$ and $s_d = [d/2] + (-1)^{d+1} - 1$. For every $i \in \{1, ..., s_d\}$, we define

$$G_{d,i} = \begin{cases} O(i+1) \times O(d-2i-1) \times O(i+1), & \text{if} \quad i \neq \frac{d-1}{2}, \\ O(\frac{d+1}{2}) \times O(\frac{d+1}{2}), & \text{if} \quad i = \frac{d-1}{2}. \end{cases}$$

Let us denote by $\langle G_{d,i}; G_{d,j} \rangle$ the group generated by $G_{d,i}$ and $G_{d,j}$. The key result of this section is

Proposition 3.1. *For every $i, j \in \{1, ..., s_d\}$ with $i \neq j$, the group $\langle G_{d,i}; G_{d,j} \rangle$ acts transitively on S^d.*

Proof. Without loosing the generality, we may assume that $i < j$. The proof is divided into three steps. For abbreviation, we introduce the notation $0_k = (0, ..., 0) \in \mathbb{R}^k$, $k \in \{1, ..., d+1\}$.

<u>Step 1.</u> *The group $\langle G_{d,i}; G_{d,j} \rangle$ acts transitively on $S^{d-j-1} \times \{0_{j+1}\}$.*

When $j = \frac{d-1}{2}$, the proof is trivial since $O(\frac{d+1}{2})$ acts transitively on $S^{\frac{d-1}{2}}$. Assume so that $j \neq \frac{d-1}{2}$. We show that for every $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in S^{d-j-1}$ with $\sigma_1 \in \mathbb{R}^{i+1}$, $\sigma_2 \in \mathbb{R}^{j-i}$, $\sigma_3 \in \mathbb{R}^{d-2j-1}$, and $\omega \in S^j$ fixed arbitrarily, there exists $g_{ij} \in \langle G_{d,i}; G_{d,j} \rangle$ such that

$$g_{ij}(\omega, 0_{d-j}) = (\sigma, 0_{j+1}). \tag{1}$$

Since $O(j+1)$ acts transitively on S^j, for every $\tilde{\sigma}_2 \in \mathbb{R}^{j-i}$ with the property that $(\sigma_1, \tilde{\sigma}_2) \in S^j$, there exists an element $g_j \in O(j+1)$ such that

$$g_j \omega = (\sigma_1, \tilde{\sigma}_2). \tag{2}$$

Note that $|\sigma_1|^2 + |\tilde{\sigma}_2|^2 = 1$ and $|\sigma_1|^2 + |\sigma_2|^2 + |\sigma_3|^2 = 1$; so $|\tilde{\sigma}_2|^2 = |\sigma_2|^2 + |\sigma_3|^2$.

If $\tilde{\sigma}_2 = 0_{j-i}$ then $\sigma_2 = 0_{j-i}$ and $\sigma_3 = 0_{d-2j-1}$; thus, $\sigma = (\sigma_1, 0_{d-j-i-1})$. Let $g_{ij} := g_j \times id_{\mathbb{R}^{d-j}} \in G_{d,j}$. Then, due to (2), we have

$$g_{ij}(\omega, 0_{d-j}) = (g_j\omega, 0_{d-j}) = (\sigma_1, 0_{j-i}, 0_{d-j}) = (\sigma_1, 0_{d-i}) = (\sigma, 0_{j+1}),$$

which proves (1).

If $\tilde{\sigma}_2 \neq 0_{j-i}$, let $r = |\tilde{\sigma}_2| > 0$. Since $O(d-2i-1)$ acts transitively on S^{d-2i-2} (thus, also on the sphere rS^{d-2i-2}), then there exists $g_i \in O(d-2i-1)$ such that $g_i(\tilde{\sigma}_2, 0_{d-j-i-1}) = (\sigma_2, \sigma_3, 0_{j-i}) \in rS^{d-2i-2}$. Let

$$\tilde{g}_i = id_{\mathbb{R}^{i+1}} \times g_i \times id_{\mathbb{R}^{i+1}} \in G_{d,i} \quad \text{and} \quad \tilde{g}_j = g_j \times id_{\mathbb{R}^{d-j}} \in G_{d,j}.$$

Then $g_{ij} := \tilde{g}_i\tilde{g}_j \in \langle G_{d,i}; G_{d,j}\rangle$ and on account of (2) and $i+1 < d-j$ (since $i < j \le s_d$), we have

$$\begin{aligned}\tilde{g}_i\tilde{g}_j(\omega, 0_{d-j}) = \tilde{g}_i(g_j\omega, 0_{d-j}) = \tilde{g}_i(\sigma_1, \tilde{\sigma}_2, 0_{d-j}) &= (\sigma_1, g_i(\tilde{\sigma}_2, 0_{d-j-i-1}), 0_{i+1})\\ &= (\sigma_1, \sigma_2, \sigma_3, 0_{j-i}, 0_{i+1}) = (\sigma, 0_{j+1}),\end{aligned}$$

i.e., relation (1).

Now, let $\overline{\sigma}, \tilde{\sigma} \in S^{d-j-1}$. Then, fixing $\omega \in S^j$, on account of (1), there are $g_1, g_2 \in \langle G_{d,i}; G_{d,j}\rangle$ such that $g_1(\omega, 0_{d-j}) = (\overline{\sigma}, 0_{j+1})$ and $g_2(\omega, 0_{d-j}) = (\tilde{\sigma}, 0_{j+1})$. Consequently, $g_2g_1^{-1} \in \langle G_{d,i}; G_{d,j}\rangle$ and $g_2g_1^{-1}(\overline{\sigma}, 0_{j+1}) = (\tilde{\sigma}, 0_{j+1})$, i.e., the group $\langle G_{d,i}; G_{d,j}\rangle$ acts transitively on $S^{d-j-1} \times \{0_{j+1}\}$.

Step 2. *The group $\langle G_{d,i}; G_{d,j}\rangle$ acts transitively on $S^{d-i-1} \times \{0_{i+1}\}$.*
We can proceed in a similar way as in Step 1; however, for the reader's convenience, we sketch the proof. We show that for every $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in S^{d-i-1}$ with $\sigma_1 \in \mathbb{R}^{i+1}$, $\sigma_2 \in \mathbb{R}^{d-j-i-1}$, $\sigma_3 \in \mathbb{R}^{j-i}$, and $\omega \in S^{d-j-1}$ fixed arbitrarily, there is $g_{ij} \in \langle G_{d,i}; G_{d,j}\rangle$ such that

$$g_{ij}(\omega, 0_{j+1}) = (\sigma, 0_{i+1}). \tag{3}$$

Let $\tilde{\sigma}_2 \in \mathbb{R}^{d-j-i-1}$ be such that $|\sigma_1|^2 + |\tilde{\sigma}_2|^2 = 1$. Then, due to Step 1, there exists $\tilde{g}_{ij} \in \langle G_{d,i}; G_{d,j}\rangle$ such that $\tilde{g}_{ij}(\omega, 0_{j+1}) = (\sigma_1, \tilde{\sigma}_2, 0_{j+1})$.

If $\tilde{\sigma}_2 = 0_{d-j-i-1}$ then $\sigma_2 = 0_{d-j-i-1}$ and $\sigma_3 = 0_{j-i}$; thus, (3) is verified with the choice $g_{ij} := \tilde{g}_{ij} \in \langle G_{d,i}; G_{d,j}\rangle$.

If $\tilde{\sigma}_2 \ne 0_{d-j-i-1}$ then let $r = |\tilde{\sigma}_2| > 0$. Since $O(d-2i-1)$ acts transitively on S^{d-2i-2} (thus, also on the sphere rS^{d-2i-2}), then there exists $g_i \in O(d-2i-1)$ such that $g_i(\tilde{\sigma}_2, 0_{j-i}) = (\sigma_2, \sigma_3) \in rS^{d-2i-2}$. Let $\tilde{g}_i = id_{\mathbb{R}^{i+1}} \times g_i \times id_{\mathbb{R}^{i+1}} \in G_{d,i}$. Then

$$\tilde{g}_i\tilde{g}_{ij}(\omega, 0_{j+1}) = \tilde{g}_i(\sigma_1, \tilde{\sigma}_2, 0_{j+1}) = (\sigma_1, g_i(\tilde{\sigma}_2, 0_{j-i}), 0_{i+1}) = (\sigma, 0_{i+1}).$$

Consequently, $g_{ij} := \tilde{g}_i\tilde{g}_{ij} \in \langle G_{d,i}; G_{d,j}\rangle$ verifies (3). Now, following the last part of Step 1, our claim follows.

Step 3. (Proof concluded) *The group $\langle G_{d,i}; G_{d,j}\rangle$ acts transitively on S^d.*
We show that for every $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in S^d$ with $\sigma_1 \in \mathbb{R}^{i+1}$, $\sigma_2 \in \mathbb{R}^{d-j-i-1}$, $\sigma_3 \in \mathbb{R}^{j+1}$, and $\omega \in S^{d-i-1}$ fixed arbitrarily, there is $g_{ij} \in \langle G_{d,i}; G_{d,j}\rangle$ such that

$$g_{ij}(\omega, 0_{i+1}) = \sigma. \tag{4}$$

Let $\tilde{\sigma}_3 \in \mathbb{R}^{j-i}$ such that $|\tilde{\sigma}_3| = |\sigma_3|$. Then, due to Step 2, there exists $\tilde{g}_{ij} \in \langle G_{d,i}; G_{d,j}\rangle$ such that $\tilde{g}_{ij}(\omega, 0_{i+1}) = (\sigma_1, \sigma_2, \tilde{\sigma}_3, 0_{i+1})$.

If $\tilde{\sigma}_3 = 0_{j-i}$ then $\sigma_3 = 0_{j+1}$ and (4) is verified by choosing $g_{ij} := \tilde{g}_{ij} \in \langle G_{d,i}; G_{d,j}\rangle$.

If $\tilde{\sigma}_3 \ne 0_{j-i}$, let $r = |\tilde{\sigma}_3| = |\sigma_3| > 0$. Since $O(j+1)$ acts transitively on S^j, there exists $g_j \in O(j+1)$ such that $g_j(\tilde{\sigma}_3, 0_{i+1}) = \sigma_3 \in rS^j$. Let us fix the element $\tilde{g}_j = id_{\mathbb{R}^{d-j}} \times g_j \in G_{d,j}$. Then

$$\tilde{g}_j\tilde{g}_{ij}(\omega, 0_{i+1}) = \tilde{g}_j(\sigma_1, \sigma_2, \tilde{\sigma}_3, 0_{i+1}) = (\sigma_1, \sigma_2, g_j(\tilde{\sigma}_3, 0_{i+1})) = (\sigma_1, \sigma_2, \sigma_3) = \sigma.$$

Consequently, $g_{ij} := \tilde{g}_j\tilde{g}_{ij} \in \langle G_{d,i}; G_{d,j}\rangle$ verifies (4).

Now, let $\overline{\sigma}, \tilde{\sigma} \in S^d$. Then, fixing $\omega \in S^{d-i-1}$, on account of (4), there are $g_1, g_2 \in \langle G_{d,i}; G_{d,j}\rangle$ such that $g_1(\omega, 0_{i+1}) = \overline{\sigma}$ and $g_2(\omega, 0_{i+1}) = \tilde{\sigma}$. Consequently, $g_2g_1^{-1} \in \langle G_{d,i}; G_{d,j}\rangle$ and $g_2g_1^{-1}(\overline{\sigma}) = \tilde{\sigma}$, i.e., the group $\langle G_{d,i}; G_{d,j}\rangle$ acts transitively on S^d. This completes the proof. □

Let $d \ge 5$ and fix $G_{d,i}$ for some $i \in \{1, ..., s_d\}$. We define the function $\tau_i : S^d \to S^d$ associated to $G_{d,i}$ by

$$\tau_i(\sigma) = \begin{cases} (\sigma_3, \sigma_2, \sigma_1), \text{if } i \neq \frac{d-1}{2}, \text{ and } \sigma = (\sigma_1, \sigma_2, \sigma_3) \text{ with } \sigma_1, \sigma_3 \in \mathbb{R}^{i+1}, \ \sigma_2 \in \mathbb{R}^{d-2i-1}; \\ (\sigma_3, \sigma_1), \text{if } i = \frac{d-1}{2}, \text{ and } \sigma = (\sigma_1, \sigma_3) \text{ with } \sigma_1, \sigma_3 \in \mathbb{R}^{\frac{d+1}{2}}. \end{cases}$$

It is clear by construction that $\tau_i \notin G_{d,i}$, $\tau_i G_{d,i} \tau_i^{-1} = G_{d,i}$ and $\tau_i^2 = id_{\mathbb{R}^{d+1}}$.

Inspired by [2], [3], we introduce the action of the group $G_{d,i}^{\tau_i} = \langle G_{d,i}, \tau_i \rangle \subset O(d+1)$ on the space $H_1^2(S^d)$. Due to the above properties of τ_i, only two types of elements in $G_{d,i}^{\tau_i}$ can be distinguished; namely, $\tilde{g} = g \in G_{d,i}$, and $\tilde{g} = \tau_i g \in G_{d,i}^{\tau_i} \setminus G_{d,i}$ (with $g \in G_{d,i}$), respectively. Therefore, the action $G_{d,i}^{\tau_i} \times H_1^2(S^d) \to H_1^2(S^d)$ given by

$$gu(\sigma) = u(g^{-1}\sigma), \quad (\tau_i g)u(\sigma) = -u(g^{-1}\tau_i^{-1}\sigma), \tag{5}$$

for $g \in G_{d,i}$, $u \in H_1^2(S^d)$ and $\sigma \in S^d$, is well-defined, continuous and linear. We define the subspace of $H_1^2(S^d)$ containing all symmetric points with respect to the compact group $G_{d,i}^{\tau_i}$, i.e.,

$$H_{G_{d,i}^{\tau_i}}(S^d) = \{u \in H_1^2(S^d) : \tilde{g}u = u \text{ for every } \tilde{g} \in G_{d,i}^{\tau_i}\}.$$

For further use, we also introduce

$$H_{G_{d,i}}(S^d) = \{u \in H_1^2(S^d) : gu = u \text{ for every } g \in G_{d,i}\},$$

where the action of the group $G_{d,i}$ on $H_1^2(S^d)$ is defined by the first relation of (5).

Remark 1. Every nonzero element of the space $H_{G_{d,i}^{\tau_i}}(S^d)$ changes sign. To see this, let $u \in H_{G_{d,i}^{\tau_i}}(S^d) \setminus \{0\}$. Due to the $G_{d,i}^{\tau_i}$-invariance of u and (5) we have $u(\sigma) = -u(\tau_i^{-1}\sigma)$ for every $\sigma \in S^d$. Since $u \neq 0$, it should change the sign.

The next result shows us how can we construct mutually distinct subspaces of $H_1^2(S^d)$ which cannot be compared by symmetrical point of view.

Theorem 3.1. *For every $i, j \in \{1, ..., s_d\}$ with $i \neq j$, one has*
a) $H_{G_{d,i}}(S^d) \cap H_{G_{d,j}}(S^d) = \{$*constant functions on* $S^d\}$;
b) $H_{G_{d,i}^{\tau_i}}(S^d) \cap H_{G_{d,j}^{\tau_j}}(S^d) = \{0\}$.

Proof. a) Let $u \in H_{G_{d,i}}(S^d) \cap H_{G_{d,j}}(S^d)$. In particular, u is both $G_{d,i}$- and $G_{d,j}$-invariant, i.e. $g_i u = g_j u = u$ for every $g_i \in G_{d,i}$ and $g_j \in G_{d,j}$, respectively. Consequently, u is also $\langle G_{d,i}, G_{d,j} \rangle$-invariant; thus, $u(\sigma) = u(g_{ij}\sigma)$ for every $g_{ij} \in \langle G_{d,i}, G_{d,j} \rangle$ and $\sigma \in S^d$. Due to Proposition 3.1, for every fixed $\sigma \in S^d$, the orbit of $g_{ij}\sigma$ is the whole sphere S^d whenever g_{ij} runs through $\langle G_{d,i}, G_{d,j} \rangle$. Therefore, u should be constant.

b) Let $u \in H_{G_{d,i}^{\tau_i}}(S^d) \cap H_{G_{d,j}^{\tau_j}}(S^d)$. The second relation of (5) shows that $u(\sigma) = -u(\tau_i^{-1}\sigma) = -u(\tau_j^{-1}\sigma)$, $\sigma \in S^d$. But, due to a), u is constant. Thus, u should be 0. ☐

To conclude this section, we construct explicit functions belonging to $H_{G_{d,i}^{\tau_i}}(S^d)$ which will be essential in the proof of Theorems 2.1 and 2.2, but it is of interest in its own right as well. Before we give the class of functions we are speaking about, we say that a set $D \subset S^d$ is $G_{d,i}^{\tau_i}$-invariant, if $\tilde{g}D \subseteq D$ for every $\tilde{g} \in G_{d,i}^{\tau_i}$.

Proposition 3.2. *Let $i \in \{1, ..., s_d\}$ and $s > 0$ be fixed. Then there exist a number $C_i > 0$ and a $G_{d,i}^{\tau_i}$-invariant set $D_i \subset S^d$ with $\mathrm{Vol}_h(D_i) > 0$, both independent on the number s, and a function $w \in H_{G_{d,i}^{\tau_i}}(S^d)$ such that*

i) $\|w\|_\infty \le s$;
ii) $|\nabla w(\sigma)| \le C_i s$ *for a.e.* $\sigma \in S^d$;
iii) $|w(\sigma)| = s$ *for every* $\sigma \in D_i$.

An explicit function $w : S^d \to \mathbb{R}$ fulfilling all the requirements of Proposition 3.2 is given by

$$w(\sigma) = \frac{8s}{(R-r)}\mathrm{sgn}(|\sigma_1| - |\sigma_3|)\max\left(0, \min\left(\frac{R-r}{8}, \frac{R-r}{4} - \right.\right.$$
$$\left.\left. - \max\left(\left||\sigma_1| + |\sigma_3| - \frac{R+3r}{4}\right|, \left|\left||\sigma_1| - |\sigma_3|\right| - \frac{R+3r}{4}\right|\right)\right)\right) \qquad (6)$$

where $R > r$, and $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in S^d$ with $\sigma_1, \sigma_3 \in \mathbb{R}^{i+1}$, $\sigma_2 \in \mathbb{R}^{d-2i-1}$ whenever $i \ne \frac{d-1}{2}$, and $\sigma = (\sigma_1, \sigma_3) \in S^d$ with $\sigma_1, \sigma_3 \in \mathbb{R}^{\frac{d+1}{2}}$ whenever $i = \frac{d-1}{2}$. The $G_{d,i}^{\tau_i}$-invariant set $D_i \subset S^d$ can be defined as

$$D_i = \left\{\sigma \in S^d : \left||\sigma_1| + |\sigma_3| - \frac{R+3r}{4}\right| \le \frac{R-r}{8}, \left|\left||\sigma_1| - |\sigma_3|\right| - \frac{R+3r}{4}\right| \le \frac{R-r}{8}\right\}.$$

The geometrical image of the function w from (6) is shown by Fig. 1.

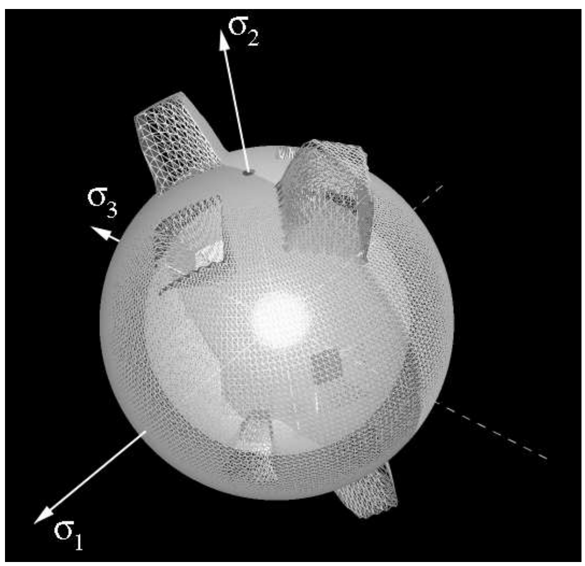

FIGURE 1. The image of the function $w : S^d \to \mathbb{R}$ from (6) with parameters $r = 0.2, R = 1.5, s = 0.4$; the value $w(\sigma)$ is represented (radially) on the line determined by $0 \in \mathbb{R}^{d+1}$ and $\sigma \in S^d$, the 'zero altitude' being $c\sigma$, i.e., the sphere cS^d, with $c = 1.3$. The union of those 8 disconnected holes on the sphere S^d where the function w takes values s and $(-s)$ corresponds to the $G_{d,i}^{\tau_i}$–invariant set D_i. (Note that the figure describes the case $i \ne \frac{d-1}{2}$. When $i = \frac{d-1}{2}$ the coordinate σ_2 vanishes and the figure becomes simpler.)

4. **Proof of Theorem 2.1.** Throughout this section we assume the hypotheses of Theorem 2.1 are fulfilled. Let $\tilde{s} > 0$ be so small that f is odd on $[-\tilde{s}, \tilde{s}]$, and let $\tilde{f}(s) = \text{sgn}(s) f(\min(|s|, \tilde{s}))$. Clearly, $\tilde{f}$ is continuous and odd on $\mathbb{R}$. Define also $\tilde{F}(s) = \int_0^s \tilde{f}(t)dt$, $s \in \mathbb{R}$.

On account of (f_2^0), one may fix $c_0 > 0$ such that

$$\liminf_{s \to 0^+} \frac{f(s)}{s} < -c_0 < 0. \tag{7}$$

In particular, there is a sequence $\{\overline{s}_k\}_k \subset (0, \tilde{s})$ converging (decreasingly) to 0, such that

$$\tilde{f}(\overline{s}_k) = f(\overline{s}_k) < -c_0 s_k. \tag{8}$$

Let us define the functions

$$\psi(s) = \tilde{f}(s) + c_0 s \quad \text{and} \quad \Psi(s) = \int_0^s \psi(t)dt = \tilde{F}(s) + \frac{c_0}{2}s^2, \quad s \in \mathbb{R}. \tag{9}$$

Due to (8), $\psi(\overline{s}_k) < 0$; so, there are two sequences $\{a_k\}_k$, $\{b_k\}_k \subset (0, \tilde{s})$, both converging to 0, such that $b_{k+1} < a_k < \overline{s}_k < b_k$ for every $k \in \mathbb{N}$ and

$$\psi(s) \leq 0 \text{ for every } s \in [a_k, b_k]. \tag{10}$$

Since $c_0 > 0$, see (7), the norm

$$\|u\|_{c_0} = \left(\int_{S^d} |\nabla u|^2 d\sigma_h + c_0 \int_{S^d} u^2 d\sigma_h \right)^{1/2} \tag{11}$$

is equivalent to the standard norm $\|\cdot\|_{H_1^2}$. Now, we define $\mathcal{E} : H_1^2(S^d) \to \mathbb{R}$ by

$$\mathcal{E}(u) = \frac{1}{2}\|u\|_{c_0}^2 - \int_{S^d} \Psi(u(\sigma))d\sigma_h,$$

which is well-defined since ψ has a subcritical growth, and $H_1^2(S^d)$ is compactly embedded into $L^p(S^d)$, $p \in [1, 2^*)$, see Hebey [9, Theorem 2.9, p. 37]. Moreover, $\mathcal{E}$ belongs to $C^1(H_1^2(S^d))$, it is even, and it coincides with the energy functional associated to (P) on the set $B^\infty(\tilde{s}) = \{u \in L^\infty(S^d) : \|u\|_\infty \leq \tilde{s}\}$ because the functions f and $\tilde{f}$ coincide on $[-\tilde{s}, \tilde{s}]$.

From now on, we fix $i \in \{1, ..., s_d\}$ and the corresponding subspace $H_{G_{d,i}^{\tau_i}}(S^d)$ of $H_1^2(S^d)$ introduced in the previous section. Let us denote by $\mathcal{E}_i$ the restriction of the functional $\mathcal{E}$ to $H_{G_{d,i}^{\tau_i}}(S^d)$ and for every $k \in \mathbb{N}$, consider the set

$$T_k^i = \{u \in H_{G_{d,i}^{\tau_i}}(S^d) : \ \|u\|_\infty \leq b_k\}, \tag{12}$$

where b_k is from (10).

Proposition 4.1. *The functional $\mathcal{E}_i$ is bounded from below on T_k^i and its infimum m_k^i on T_k^i is attained at $u_k^i \in T_k^i$. Moreover, $m_k^i = \mathcal{E}_i(u_k^i) < 0$ for every $k \in \mathbb{N}$.*

Proof. For every $u \in T_k^i$ we have

$$\mathcal{E}_i(u) = \frac{1}{2}\|u\|_{c_0}^2 - \int_{S^d} \Psi(u(\sigma))d\sigma_h \geq -\max_{[-b_k, b_k]} \Psi \cdot \text{Vol}_h(S^d) > -\infty.$$

It is clear that T_k^i is convex and closed, thus weakly closed in $H_{G_{d,i}^{\tau_i}}(S^d)$. Let $m_k^i = \inf_{T_k^i} \mathcal{E}_i$, and $\{u_n\}_n \subset T_k^i$ be a minimizing sequence of $\mathcal{E}_i$ for m_k^i. Then, for large $n \in \mathbb{N}$, we have

$$\frac{1}{2}\|u_n\|_{c_0}^2 \le m_k^i + 1 + \max_{[-b_k,b_k]} \Psi \cdot \mathrm{Vol}_h(S^d),$$

thus $\{u_n\}_n$ is bounded in $H_{G_{d,i}^{\tau_i}}(S^d)$. Up to a subsequence, $\{u_n\}_n$ weakly converges in $H_{G_{d,i}^{\tau_i}}(S^d)$ to some $u_k^i \in T_k^i$. Since ψ has a subcritical growth, by using the compactness of the embedding $H_{G_{d,i}^{\tau_i}}(S^d) \subset H_1^2(S^d) \hookrightarrow L^p(S^d)$, $1 \le p < 2^*$, one can conclude the sequentially weak continuity of the function $u \mapsto \int_{S^d} \Psi(u(\sigma))d\sigma_h$, $u \in H_{G_{d,i}^{\tau_i}}(S^d)$. Consequently, $\mathcal{E}_i$ is sequentially weak lower semicontinuous. Combining this fact with the weak closedness of the set T_k^i, we obtain $\mathcal{E}_i(u_k^i) = m_k^i = \inf_{T_k^i} \mathcal{E}_i$.

The next task is to prove that $m_k^i < 0$ for every $k \in \mathbb{N}$. First, due to (9) and (f_1^0), we have

$$-\infty < \liminf_{s\to 0^+} \frac{\Psi(s)}{s^2} \le \limsup_{s\to 0^+} \frac{\Psi(s)}{s^2} = +\infty. \tag{13}$$

Therefore, the left-hand side of (13) and the evenness of Ψ implies the existence of $\underline{l} > 0$ and $\varrho \in (0, \tilde{s})$ such that

$$\Psi(s) \ge -\underline{l}s^2 \text{ for every } s \in (-\varrho, \varrho). \tag{14}$$

Let $D_i \subset S^d$ and $C_i > 0$ be from Proposition 3.2 (which depend only on $G_{d,i}$ and τ_i), and fix a number $\overline{l} > 0$ large enough such that

$$\overline{l}\mathrm{Vol}_h(D_i) > \left(\underline{l} + \frac{c_0}{2}\right) \mathrm{Vol}_h(S^d) + \frac{C_i^2}{2}, \tag{15}$$

$c_0 > 0$ being from (7). Taking into account the right-hand side of (13), there is a sequence $\{s_k\}_k \subset (0, \varrho)$ such that $s_k \le b_k$ and $\Psi(s_k) = \Psi(-s_k) > \overline{l}s_k^2$ for every $k \in \mathbb{N}$.

Let $w_k := w_{s_k} \in H_{G_{d,i}^{\tau_i}}(S^d)$ be the function from Proposition 3.2 corresponding to the value $s_k > 0$. Then $w_k \in T_k^i$ and one has

$$\begin{aligned}\mathcal{E}_i(w_k) &= \frac{1}{2}\|w_k\|_{c_0}^2 - \int_{S^d} \Psi(w_k(\sigma))d\sigma_h \\ &\le \frac{1}{2}\left(C_i^2 + c_0\mathrm{Vol}_h(S^d)\right) s_k^2 - \int_{D_i} \Psi(w_k(\sigma))d\sigma_h - \int_{S^d\setminus D_i} \Psi(w_k(\sigma))d\sigma_h.\end{aligned}$$

On account of Proposition 3.2 iii), we have

$$\int_{D_i} \Psi(w_k(\sigma))d\sigma_h = \Psi(s_k)\mathrm{Vol}_h(D_i) > \overline{l}\mathrm{Vol}_h(D_i)s_k^2.$$

On the other hand, due to (14) and Proposition 3.2 i), we have

$$\int_{S^d\setminus D_i} \Psi(w_k(\sigma))d\sigma_h \ge -\underline{l}\int_{S^d\setminus D_i} w_k^2(\sigma)d\sigma_h > -\underline{l}\mathrm{Vol}_h(S^d)s_k^2.$$

Combining (15) with the above estimations, we obtain that $m_k^i = \inf_{T_k^i} \mathcal{E}_i \le \mathcal{E}_i(w_k) < 0$, which proves our claim. □

Proposition 4.2. *Let $u_k^i \in T_k^i$ from Proposition 4.1. Then, $\|u_k^i\|_\infty \le a_k$. (The number a_k is from (10).)*

Proof. Let $A = \{\sigma \in S^d : u_k^i(\sigma) \notin [-a_k, a_k]\}$ and suppose that meas$(A) > 0$. Define the function $\gamma(s) = \operatorname{sgn}(s)\min(|s|, a_k)$ and set $w_k = \gamma \circ u_k^i$. Since γ is Lipschitz continuous, then $w_k \in H_1^2(S^d)$, see Hebey [9, Proposition 2.5, p. 24].

We first claim that $w_k \in H_{G_{d,i}^{\tau_i}}(S^d)$. To see this, it suffices to prove that $\tilde{g}w_k = w_k$ for every $\tilde{g} \in G_{d,i}^{\tau_i}$. First, let $\tilde{g} = g \in G_{d,i}$. Since $gu_k^i = u_k^i$, we have

$$gw_k(\sigma) = w_k(g^{-1}\sigma) = (\gamma \circ u_k^i)(g^{-1}\sigma) = \gamma(u_k^i(g^{-1}\sigma)) = \gamma(u_k^i(\sigma)) = w_k(\sigma)$$

for every $\sigma \in S^d$. Now, let $\tilde{g} = \tau_i g \in G_{d,i}^{\tau_i} \setminus G_{d,i}$ (with $g \in G_{d,i}$). Since γ is an odd function and $(\tau_i g)u_k^i = u_k^i$, on account of (5) we have

$$\begin{aligned}(\tau_i g)w_k(\sigma) &= -w_k(g^{-1}\tau_i^{-1}\sigma) = -(\gamma \circ u_k^i)(g^{-1}\tau_i^{-1}\sigma)\\ &= \gamma(-u_k^i(g^{-1}\tau_i^{-1}\sigma)) = \gamma((\tau_i g)u_k^i(\sigma)) = \gamma(u_k^i(\sigma))\\ &= w_k(\sigma)\end{aligned}$$

for every $\sigma \in S^d$. In conclusion, the claim is true, and $w_k \in H_{G_{d,i}^{\tau_i}}(S^d)$. Moreover, $\|w_k\|_\infty \le a_k$. Consequently, $w_k \in T_k^i$.

We introduce the sets

$$A_1 = \{\sigma \in A : \ u_k^i(\sigma) < -a_k\} \qquad \text{and} \qquad A_2 = \{\sigma \in A : \ u_k^i(\sigma) > a_k\}.$$

Thus, $A = A_1 \cup A_2$, and we have that $w_k(\sigma) = u_k^i(\sigma)$ for all $\sigma \in S^d \setminus A$, $w_k(\sigma) = -a_k$ for all $\sigma \in A_1$, and $w_k(\sigma) = a_k$ for all $\sigma \in A_2$. Moreover,

$$\begin{aligned}&\mathcal{E}_i(w_k) - \mathcal{E}_i(u_k^i) =\\ &\quad = -\frac{1}{2}\int_A |\nabla u_k^i|^2 d\sigma_h + \frac{c_0}{2}\int_A [w_k^2 - (u_k^i)^2]d\sigma_h - \int_A [\Psi(w_k) - \Psi(u_k^i)]d\sigma_h\\ &\quad = -\frac{1}{2}\int_A |\nabla u_k^i(\sigma)|^2 d\sigma_h + \frac{c_0}{2}\int_A [a_k^2 - (u_k^i(\sigma))^2]d\sigma_h\\ &\qquad - \int_{A_1}[\Psi(-a_k) - \Psi(u_k^i(\sigma))]d\sigma_h - \int_{A_2}[\Psi(a_k) - \Psi(u_k^i(\sigma))]d\sigma_h.\end{aligned}$$

Note that $\int_A [w_k^2 - (u_k^i)^2]d\sigma_h \le 0$. Next, by the mean value theorem, for a.e. $\sigma \in A_2$, there exists $\theta_k(\sigma) \in [a_k, b_k]$ such that $\Psi(a_k) - \Psi(u_k^i(\sigma)) = \psi(\theta_k(\sigma))(a_k - u_k^i(\sigma))$. Thus, on account of (10), one has

$$\int_{A_2}[\Psi(a_k) - \Psi(u_k^i(\sigma))]d\sigma_h \ge 0.$$

In the same way, using the oddness of ψ, we conclude that

$$\int_{A_1}[\Psi(-a_k) - \Psi(u_k^i(\sigma))]d\sigma_h \ge 0.$$

In conclusion, every term of the expression $\mathcal{E}_i(w_k) - \mathcal{E}_i(u_k^i)$ is nonpositive. On the other hand, since $w_k \in T_k^i$, then $\mathcal{E}_i(w_k) \ge \mathcal{E}_i(u_k^i) = \inf_{T_k^i} \mathcal{E}_i$. So, every term in $\mathcal{E}_i(w_k) - \mathcal{E}_i(u_k^i)$ should be zero. In particular,

$$\int_A |\nabla u_k^i(\sigma)|^2 d\sigma_h = \int_A [a_k^2 - (u_k^i(\sigma))^2]d\sigma_h = 0.$$

These equalities imply that meas(A) should be 0, contradicting our initial assumption. $\square$

Proposition 4.3. $\lim_{k\to\infty} m_k^i = \lim_{k\to\infty} \|u_k^i\|_\infty = \lim_{k\to\infty} \|u_k^i\|_{H_1^2} = 0.$

Proof. Using Proposition 4.2, we have that $\|u_k^i\|_\infty \leq a_k < \tilde{s}$ for a.e. $\sigma \in S^d$. Therefore, we readily have that $\lim_{k\to\infty} \|u_k^i\|_\infty = 0$.

Moreover, the mean value theorem shows that

$$\begin{aligned} m_k^i &= \mathcal{E}_i(u_k^i) \geq -\int_{S^d} \Psi(u_k^i(\sigma))d\sigma_h \geq -\max_{[-\tilde{s},\tilde{s}]} |\psi| \int_{S^d} |u_k^i(\sigma)|d\sigma_h \\ &\geq -\max_{[-\tilde{s},\tilde{s}]} |\psi| \mathrm{Vol}_h(S^d)a_k. \end{aligned}$$

Since $\lim_{k\to\infty} a_k = 0$, we have $\lim_{k\to\infty} m_k \geq 0$. On the other hand, $m_k < 0$ for every $k \in \mathbb{N}$, see Proposition 4.1, which implies $\lim_{k\to\infty} m_k^i = 0$.

Note that

$$\frac{\|u_k^i\|_{c_0}^2}{2} = m_k^i + \int_{S^d} \Psi(u_k^i(\sigma))d\sigma_h \leq m_k^i + \max_{[-\tilde{s},\tilde{s}]} |\psi| \mathrm{Vol}_h(S^d)a_k,$$

thus $\lim_{k\to\infty} \|u_k^i\|_{c_0} = 0$. But $\|\cdot\|_{c_0}$ and $\|\cdot\|_{H_1^2}$ are equivalent norms. □

Now, we prove the key result of this section where the non-smooth principle of symmetric criticality for Szulkin-type functions plays a crucial role.

Proposition 4.4. *u_k^i is a weak solution of* (P) *for every $k \in \mathbb{N}$.*

Proof. We divide the proof into two parts. First, let

$$T_k = \{u \in H_1^2(S^d) : \|u\|_\infty \leq b_k\}.$$

Step 1. $\mathcal{E}'(u_k^i)(w - u_k^i) \geq 0$ for every $w \in T_k$.
The set T_k is closed and convex in $H_1^2(S^d)$. Let ζ_{T_k} be the indicator function of the set T_k (i.e., $\zeta_{T_k}(u) = 0$ if $u \in T_k$, and $\zeta_{T_k}(u) = +\infty$, otherwise). We define the Szulkin-type functional $\mathcal{I}_k : H_1^2(S^d) \to \mathbb{R} \cup \{+\infty\}$ by $\mathcal{I}_k = \mathcal{E} + \zeta_{T_k}$, see [18], i.e., $\mathcal{E}$ is of class $C^1(H_1^2(S^d))$, and ζ_{T_k} is convex, lower semicontinuous and proper. On account of (12), we have that $T_k^i = T_k \cap H_{G_{d,i}^{\tau_i}}(S^d)$; therefore, the restriction of ζ_{T_k} to $H_{G_{d,i}^{\tau_i}}(S^d)$ is precisely the indicator function $\zeta_{T_k^i}$ of the set T_k^i. Since u_k^i is a local minimum point of $\mathcal{E}_i$ relative to T_k^i (see Proposition 4.1), then u_k^i is a critical point of the functional $\mathcal{I}_k^i := \mathcal{E}_i + \zeta_{T_k^i}$ in the sense of Szulkin [18, p. 78], i.e.,

$$0 \in \mathcal{E}_i'(u_k^i) + \partial\zeta_{T_k^i}(u_k^i) \quad \text{in} \quad (H_{G_{d,i}^{\tau_i}}(S^d))^*, \tag{16}$$

where $\partial\zeta_{T_k^i}$ stands for the subdifferential of the convex function $\zeta_{T_k^i}$.

Since $\mathcal{E}$ is even, by means of (5) one can easily check that it is $G_{d,i}^{\tau_i}$-invariant. The function ζ_{T_k} is also $G_{d,i}^{\tau_i}$-invariant since $\tilde{g}T_k \subseteq T_k$ for every $\tilde{g} \in G_{d,i}^{\tau_i}$ (we use again (5)). Finally, since $G_{d,i}^{\tau_i} \subset O(d+1)$ is compact, and $\mathcal{E}_i$ and $\zeta_{T_k^i}$ are the restrictions of $\mathcal{E}$ and ζ_{T_k} to $H_{G_{d,i}^{\tau_i}}(S^d)$, respectively, we may apply – via relation (16) – the principle of symmetric criticality proved by Kobayaski-Ôtani [11, Theorem 3.16, p. 443]. Thus, we obtain

$$0 \in \mathcal{E}'(u_k^i) + \partial\zeta_{T_k}(u_k^i) \quad \text{in} \quad (H_1^2(S^d))^*.$$

Consequently, for every $w \in H_1^2(S^d)$, we have

$$\mathcal{E}'(u_k^i)(w - u_k^i) + \zeta_{T_k}(w) - \zeta_{T_k}(u_k^i) \geq 0,$$

which implies our claim.

Step 2. (Proof concluded) u_k^i *is a weak solution of* (P).
By Step 1, we have

$$\int_{S^d} \langle \nabla u_k^i, \nabla (w - u_k^i) \rangle d\sigma_h + c_0 \int_{S^d} u_k^i (w - u_k^i) d\sigma_h$$

$$- \int_{S^d} \psi(u_k^i)(w - u_k^i) d\sigma_h \geq 0, \quad \forall\, w \in T_k.$$

Recall from (9) that $\psi(s) = \tilde{f}(s) + c_0 s$, $s \in \mathbb{R}$. Moreover, f and $\tilde{f}$ coincide on $[-\tilde{s}, \tilde{s}]$ and $u_k^i(\sigma) \in [-a_k, a_k] \subset (-\tilde{s}, \tilde{s})$ for a.e. $\sigma \in S^d$ (see Proposition 4.2). Consequently, the above inequality reduces to

$$\int_{S^d} \langle \nabla u_k^i, \nabla (w - u_k^i) \rangle d\sigma_h - \int_{S^d} f(u_k^i)(w - u_k^i) d\sigma_h \geq 0, \quad \forall\, w \in T_k. \tag{17}$$

Let us define the function $\gamma(s) = \operatorname{sgn}(s) \min(|s|, b_k)$, and fix $\varepsilon > 0$ and $v \in H_1^2(S^d)$ arbitrarily. Since γ is Lipschitz continuous, $w_k = \gamma \circ (u_k^i + \varepsilon v)$ belongs to $H_1^2(S^d)$, see Hebey [9, Proposition 2.5, p. 24]. The explicit expression of w_k is

$$w_k(\sigma) = \begin{cases} -b_k, & \text{if} \quad \sigma \in \{u_k^i + \varepsilon v < -b_k\} \\ u_k^i(\sigma) + \varepsilon v(\sigma), & \text{if} \quad \sigma \in \{-b_k \leq u_k^i + \varepsilon v < b_k\} \\ b_k, & \text{if} \quad \sigma \in \{b_k \leq u_k^i + \varepsilon v\}. \end{cases}$$

Therefore, $w_k \in T_k$. Taking $w = w_k$ as a test function in (17), we obtain

$$\begin{aligned} 0 \quad \leq \quad & -\int_{\{u_k^i + \varepsilon v < -b_k\}} |\nabla u_k^i|^2 + \int_{\{u_k^i + \varepsilon v < -b_k\}} f(u_k^i)(b_k + u_k^i) \\ & + \varepsilon \int_{\{-b_k \leq u_k^i + \varepsilon v < b_k\}} \langle \nabla u_k^i, \nabla v \rangle - \varepsilon \int_{\{-b_k \leq u_k^i + \varepsilon v < b_k\}} f(u_k^i) v \\ & - \int_{\{b_k \leq u_k^i + \varepsilon v\}} |\nabla u_k^i|^2 - \int_{\{b_k \leq u_k^i + \varepsilon v\}} f(u_k^i)(b_k - u_k^i). \end{aligned}$$

After a suitable rearrangement of the terms in this inequality, we obtain that

$$\begin{aligned} 0 \quad \leq \quad & \varepsilon \int_{S^d} \langle \nabla u_k^i, \nabla v \rangle - \varepsilon \int_{S^d} f(u_k^i) v \\ & - \int_{\{u_k^i + \varepsilon v < -b_k\}} |\nabla u_k^i|^2 - \int_{\{b_k \leq u_k^i + \varepsilon v\}} |\nabla u_k^i|^2 \\ & + \int_{\{u_k^i + \varepsilon v < -b_k\}} f(u_k^i)(b_k + u_k^i + \varepsilon v) + \int_{\{b_k \leq u_k^i + \varepsilon v\}} f(u_k^i)(-b_k + u_k^i + \varepsilon v) \\ & - \varepsilon \int_{\{u_k^i + \varepsilon v < -b_k\}} \langle \nabla u_k^i, \nabla v \rangle - \varepsilon \int_{\{b_k \leq u_k^i + \varepsilon v\}} \langle \nabla u_k^i, \nabla v \rangle. \end{aligned}$$

Let $M_k = \max_{[-a_k, a_k]} |f|$. Since $u_k^i(\sigma) \in [-a_k, a_k] \subset [-b_k, b_k]$ for a.e. $\sigma \in S^d$, we have

$$\int_{\{u_k^i + \varepsilon v < -b_k\}} f(u_k^i)(b_k + u_k^i + \varepsilon v) \leq -\varepsilon M_k \int_{\{u_k^i + \varepsilon v < -b_k\}} v$$

and

$$\int_{\{b_k \leq u_k^i + \varepsilon v\}} f(u_k^i)(-b_k + u_k^i + \varepsilon v) \leq \varepsilon M_k \int_{\{b_k \leq u_k^i + \varepsilon v\}} v.$$

Using the above estimates and dividing by $\varepsilon > 0$, we obtain

$$\begin{aligned} 0 \quad \leq \quad & \int_{S^d} \langle \nabla u_k^i, \nabla v \rangle d\sigma_h - \int_{S^d} f(u_k^i) v d\sigma_h \\ & - M_k \int_{\{u_k^i + \varepsilon v < -b_k\}} v d\sigma_h + M_k \int_{\{b_k \leq u_k^i + \varepsilon v\}} v d\sigma_h \\ & - \int_{\{u_k^i + \varepsilon v < -b_k\}} \langle \nabla u_k^i, \nabla v \rangle d\sigma_h - \int_{\{b_k \leq u_k^i + \varepsilon v\}} \langle \nabla u_k^i, \nabla v \rangle d\sigma_h. \end{aligned}$$

Now, letting $\varepsilon \to 0^+$, and taking into account Proposition 4.2 (i.e., $-a_k \leq u_k^i(\sigma) \leq a_k$ for a.e. $\sigma \in S^d$), we have

$$\text{meas}(\{u_k^i + \varepsilon v < -b_k\}) \to 0 \quad \text{and} \quad \text{meas}(\{b_k \leq u_k^i + \varepsilon v\}) \to 0,$$

respectively. Consequently, the above inequality reduces to

$$0 \quad \leq \quad \int_{S^d} \langle \nabla u_k^i, \nabla v \rangle d\sigma_h - \int_{S^d} f(u_k^i) v d\sigma_h.$$

Putting $(-v)$ instead of v, we see that u_k^i is a weak solution of (P), which completes the proof. □

Proof of Theorem 2.1. Fix $i \in \{1, ..., s_d\}$. Combining Propositions 4.1 and 4.3, one can see that there are infinitely many distinct elements in the sequence $\{u_k^i\}_k$. These elements are weak solutions of (P) as Proposition 4.4 shows, and they change sign, see Remark 1. Moreover, due to Theorem 3.1 b), solutions in different spaces $H_{G_{d,i}^{\tau_i}}(S^d)$, $i \in \{1, ..., s_d\}$, cannot be compared from symmetrical point of view. The L^∞– and H_1^2–asymptotic behaviour of the sequences of solutions are described in Proposition 4.3. □

5. **Proof of Theorem 2.2.** Certain parts of the proof are similar to that of Theorem 2.1; so, we present only the differences. We assume throughout of this section that the hypotheses of Theorem 2.2 are fulfilled. Due to (f_2^∞), one can fix $c_\infty > 0$ such that

$$\liminf_{s \to \infty} \frac{f(s)}{s} < -c_\infty < 0.$$

Let $\{\overline{s}_k\} \subset (0, \infty)$ be a sequence converging (increasingly) to $+\infty$, such that $f(\overline{s}_k) < -c_\infty \overline{s}_k$. We define the functions

$$\psi(s) = f(s) + c_\infty s \quad \text{and} \quad \Psi(s) = \int_0^s \psi(t) dt = F(s) + \frac{c_\infty}{2} s^2, \quad s \in \mathbb{R}. \tag{18}$$

By construction, $\psi(\overline{s}_k) < 0$; consequently, there are two sequences $\{a_k\}_k$, $\{b_k\}_k \subset (0, \infty)$, both converging to ∞, such that $a_k < \overline{s}_k < b_k < a_{k+1}$ for every $k \in \mathbb{N}$ and

$$\psi(s) \leq 0 \text{ for every } s \in [a_k, b_k]. \tag{19}$$

Since $c_\infty > 0$, the norm $\|\cdot\|_{c_\infty}$ defined in the same way as (11) with c_∞ instead of c_0, is equivalent to the standard norm $\|\cdot\|_{H_1^2}$. Now, we define the energy functional $\mathcal{E} : H_1^2(S^d) \to \mathbb{R}$ associated with (P) by

$$\mathcal{E}(u) = \frac{1}{2} \|u\|_{c_\infty}^2 - \int_{S^d} \Psi(u(\sigma)) d\sigma_h.$$

Since $H_1^2(S^d)$ is continuously embedded into $L^p(S^d)$, $1 \leq p \leq 2^*$, see [9, Corollary 2.1, p. 33]), using hypothesis (f_3^∞), the functional $\mathcal{E}$ is well-defined, and it belongs to $C^1(H_1^2(S^d))$. Moreover, since f is odd on the whole $\mathbb{R}$, the functional $\mathcal{E}$ is even.

We fix $i \in \{1,...,s_d\}$ and the subspace $H_{G_{d,i}^{\tau_i}}(S^d)$ of $H_1^2(S^d)$. Let $\mathcal{E}_i$ be the restriction of the functional $\mathcal{E}$ to $H_{G_{d,i}^{\tau_i}}(S^d)$ and for every $k \in \mathbb{N}$, define the set

$$Z_k^i = \{u \in H_{G_{d,i}^{\tau_i}}(S^d) : \ \|u\|_\infty \leq b_k\},$$

where b_k is from (19).

Proposition 5.1. *The functional $\mathcal{E}_i$ is bounded from below on Z_k^i and its infimum $\tilde{m}_k^i$ on Z_k^i is attained at $\tilde{u}_k^i \in Z_k^i$. Moreover, $\lim_{k\to\infty} \tilde{m}_k^i = -\infty$.*

Proof. It is easy to check that $\mathcal{E}_i$ is bounded from below on Z_k^i. In order to see that it attains its infimum on Z_k^i we show that the function $u \mapsto \int_{S^d} \Psi(u(\sigma))d\sigma_h$, $u \in H_{G_{d,i}^{\tau_i}}(S^d)$ is sequentially weak continuous; in such case, $\mathcal{E}_i$ is sequentially weak lower semicontinuous and we may proceed in the standard way. On one hand, due to (f_3^∞), (18) and the oddness of ψ, one can find $c_1 > 0$ such that

$$|\psi(s)| \leq c_1(1 + |s|^{2^*-1}), \quad s \in \mathbb{R}. \tag{20}$$

On the other hand, the definition of $G_{d,i}$ shows that the $G_{d,i}$-orbit of every point $\sigma \in S^d$ has at least dimension 1, i.e., $\dim(G_{d,i}\sigma) \geq 1$ for every $\sigma \in S^d$. Thus

$$d_G = \min\{\dim(G_{d,i}\sigma) : \sigma \in S^d\} \geq 1.$$

Applying [2, Lemma 3.2], we conclude in particular that $H_{G_{d,i}}(S^d)$ is compactly embedded into $L^q(S^d)$, whenever $q \in \left[1, \frac{2d-2}{d-3}\right)$. Since $\frac{2d-2}{d-3} > 2^*$, the embedding $H_{G_{d,i}^{\tau_i}}(S^d) \subset H_{G_{d,i}}(S^d) \hookrightarrow L^{2^*}(S^d)$ is compact. Combining (20) with the above compactness property, we conclude the sequentially weak continuity of the function $u \mapsto \int_{S^d} \Psi(u(\sigma))d\sigma_h$, $u \in H_{G_{d,i}^{\tau_i}}(S^d)$. Consequently, we may assert that the infimum $\tilde{m}_k^i$ on Z_k^i is attained at the point $\tilde{u}_k^i \in Z_k^i$.

We will prove $\lim_{k\to\infty} \tilde{m}_k^i = -\infty$. First, due to (18) and (f_1^∞), we have

$$-\infty < \liminf_{s\to\infty} \frac{\Psi(s)}{s^2} \leq \limsup_{s\to\infty} \frac{\Psi(s)}{s^2} = +\infty. \tag{21}$$

The left inequality of (21) and the evenness of Ψ implies the existence of $\underline{l}, \varrho > 0$ such that

$$\Psi(s) \geq -\underline{l}s^2 \text{ for every } |s| > \varrho. \tag{22}$$

Let $D_i \subset S^d$ and $C_i > 0$ be from Proposition 3.2 (which depend only on $G_{d,i}$ and τ_i), and fix a number $\overline{l} > 0$ large enough such that

$$\overline{l}\mathrm{Vol}_h(D_i) > \left(\underline{l} + \frac{c_\infty}{2}\right)\mathrm{Vol}_h(S^d) + \frac{C_i^2}{2}. \tag{23}$$

Taking into account the right-hand side of (21), there is a sequence $\{\tilde{s}_k\}_k \subset (0,\infty)$ such that $\lim_{k\to\infty} \tilde{s}_k = \infty$ and $\Psi(\tilde{s}_k) = \Psi(-\tilde{s}_k) > \overline{l}\tilde{s}_k^2$ for every $k \in \mathbb{N}$.

Let $\{b_{n_k}\}_k$ be an increasing subsequence of $\{b_k\}_k$ such that $\tilde{s}_k \leq b_{n_k}$ for every $k \in \mathbb{N}$. Let $\tilde{w}_k := w_{\tilde{s}_k} \in H_{G_{d,i}^{\tau_i}}(S^d)$ be the function from Proposition 3.2 corresponding to the value $\tilde{s}_k > 0$. Then $\tilde{w}_k \in Z^i_{n_k}$ and one has

$$\begin{aligned}\mathcal{E}_i(\tilde{w}_k) &= \frac{1}{2}\|\tilde{w}_k\|^2_{c_\infty} - \int_{S^d} \Psi(\tilde{w}_k(\sigma))d\sigma_h \\ &\leq \frac{1}{2}\left(C_i^2 + c_\infty \mathrm{Vol}_h(S^d)\right)\tilde{s}_k^2 - \int_{D_i} \Psi(\tilde{w}_k(\sigma))d\sigma_h - \int_{S^d\setminus D_i} \Psi(\tilde{w}_k(\sigma))d\sigma_h.\end{aligned}$$

On account of Proposition 3.2 iii), we have

$$\int_{D_i} \Psi(\tilde{w}_k(\sigma))d\sigma_h = \Psi(\tilde{s}_k)\mathrm{Vol}_h(D_i) > \bar{l}\mathrm{Vol}_h(D_i)\tilde{s}_k^2.$$

Due to Proposition 3.2 i) and (22), we have

$$\begin{aligned}\int_{S^d\setminus D_i} \Psi(\tilde{w}_k(\sigma))d\sigma_h &= \int_{(S^d\setminus D_i)\cap\{|\tilde{w}_k|\leq\varrho\}} \Psi(\tilde{w}_k(\sigma))d\sigma_h \\ &\quad + \int_{(S^d\setminus D_i)\cap\{|\tilde{w}_k|>\varrho\}} \Psi(\tilde{w}_k(\sigma))d\sigma_h \\ &\geq -\left(\max_{[-\varrho,\varrho]}|\Psi| + \underline{l}\tilde{s}_k^2\right)\mathrm{Vol}_h(S^d).\end{aligned}$$

Combining these estimates, we obtain that

$$\mathcal{E}_i(\tilde{w}_k) \leq \tilde{s}_k^2\left(-\bar{l}\mathrm{Vol}_h(D_i) + \left(\underline{l} + \frac{c_\infty}{2}\right)\mathrm{Vol}_h(S^d) + \frac{C_i^2}{2}\right) + \max_{[-\varrho,\varrho]}|\Psi|\mathrm{Vol}_h(S^d).$$

Taking into account (23) and that $\lim_{k\to\infty}\tilde{s}_k = \infty$, we obtain $\lim_{k\to\infty}\mathcal{E}_i(\tilde{w}_k) = -\infty$. Since $\tilde{m}^i_{n_k} = \mathcal{E}_i(\tilde{u}^i_{n_k}) = \inf_{Z^i_{n_k}}\mathcal{E}_i \leq \mathcal{E}_i(\tilde{w}_k)$, then $\lim_{k\to\infty}\tilde{m}^i_{n_k} = -\infty$. Since the sequence $\{\tilde{m}^i_k\}_k$ is non-increasing, the claim follows. □

Proposition 5.2. $\lim_{k\to\infty}\|\tilde{u}^i_k\|_\infty = \lim_{k\to\infty}\|\tilde{u}^i_k\|_{H^2_1} = \infty.$

Proof. Assume first by contradiction that there exists a subsequence $\{\tilde{u}^i_{n_k}\}_k$ of $\{\tilde{u}^i_k\}_k$ such that $\|\tilde{u}^i_{n_k}\|_\infty \leq M$ for some $M > 0$. In particular, $\{\tilde{u}^i_{n_k}\} \subset Z^i_l$ for some $l \in \mathbb{N}$. Therefore, for every $n_k \geq l$, we have

$$\tilde{m}^i_l \geq \tilde{m}^i_{n_k} = \inf_{Z^i_{n_k}}\mathcal{E}_i = \mathcal{E}_i(\tilde{u}^i_{n_k}) \geq \inf_{Z^i_l}\mathcal{E}_i = \tilde{m}^i_l.$$

Consequently, $\tilde{m}_{n_k} = \tilde{m}_l$ for every $n_k \geq l$, and since the sequence $\{\tilde{m}^i_k\}_k$ is non-increasing, there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ we have $\tilde{m}^i_k = \tilde{m}^i_l$, contradicting Proposition 5.1.

It remains to prove that $\lim_{k\to\infty}\|\tilde{u}^i_k\|_{H^2_1} = \infty$. Note that (20) and the continuity of the embedding $H^2_1(S^d)$ into $L^{2^*}(S^d)$ implies that for come $C > 0$ we have

$$\left|\int_{S^d}\Psi(u(\sigma))d\sigma_h\right| \leq C(\|u\|_{H^2_1} + \|u\|^{2^*}_{H^2_1}), \quad \forall u \in H^2_1(S^d).$$

Similarly as above, we assume that there exists a subsequence $\{\tilde{u}^i_{n_k}\}_k$ of $\{\tilde{u}^i_k\}_k$ such that for some $M > 0$, we have $\|\tilde{u}^i_{n_k}\|_{H^2_1} \leq M$. Since $\|\cdot\|_{c_\infty}$ is equivalent with $\|\cdot\|_{H^2_1}$, due to the above inequality, the sequence $\{\mathcal{E}_i(\tilde{u}^i_{n_k})\}_k$ is bounded. But $\tilde{m}^i_{n_k} = \mathcal{E}_i(\tilde{u}^i_{n_k})$, thus, the sequence $\{\tilde{m}^i_{n_k}\}$ is also bounded. This fact contradicts Proposition 5.1. □

Proof of Theorem 2.2. Due to Proposition 5.1, we can find infinitely many distinct elements $\tilde{u}^i_k$; similar reasoning as in Propositions 4.2 and 4.4 show that $\tilde{u}^i_k$ are weak solutions of (P) for every $k \in \mathbb{N}$. The L^∞– and H_1^2–asymptotic behaviour of the sequences of solutions are described in Proposition 5.2. The rest is similar as in Theorem 2.1. □

6. **Final remarks.** **A.** (*Asymptotically critical problems on* $\mathbb{R}^{d+1}$) Theorems 2.1 and 2.2 can be successfully applied to treat equations of the form

$$-\Delta v = |x|^{\alpha-2} f(|x|^{-\alpha} v), \quad x \in \mathbb{R}^{d+1} \setminus \{0\} \ \ (\alpha < 0) \tag{24}$$

whenever $f : \mathbb{R} \to \mathbb{R}$ is enough smooth and oscillates either at zero or at infinity having an asymptotically critical growth. Finding solutions of (24) in the form $v(x) = v(r, \sigma) = r^\alpha u(\sigma)$, $(r, \sigma) = (|x|, x/|x|) \in (0, \infty) \times S^d$ being the spherical coordinates, we obtain

$$-\Delta_h u + \alpha(1 - d - \alpha)u = f(u) \ \text{ on } \ S^d. \tag{25}$$

Assuming (f_1^L) and $\liminf_{s\to L} \frac{f(s)}{s} < \alpha(1 - d - \alpha)$ with $L \in \{0^+, +\infty\}$, and (f_3^∞) whenever $L = +\infty$, we may formulate multiplicity results for (25), so for (24). Note that the obtained solutions of (24) are sign-changing and *non*-radial.

B. The minimal number of those sequences of solutions of (P) which contain mutually symmetrically distinct elements is $s_d = [d/2] + (-1)^{d+1} - 1$. Note that $s_d \sim d/2$ as $d \to \infty$. However, in lower dimensions, our results are not spectacular. For instance, $s_4 = 0$; therefore, on S^4 we have no analogous results as Theorems 2.1 and 2.2. Note that $s_3 = 1$; in fact, for $G_{3,1} = O(2) \times O(2)$ we may apply our arguments. Hence, on S^3 one can find a sequence of solutions of (P) with the described properties in our theorems.

We may compare our results with that of Bartsch-Willem [3]; they studied the lower bound of those sequences of solutions for a Schrödinger equation on $\mathbb{R}^{d+1}$ which contain elements in different $O(d + 1)$-orbits. Due to [3, Proposition 4.1, p. 457], we deduce that their lower bound is $s'_d = \left[\log_2 \frac{d+3}{3}\right]$ whenever $d \geq 3$ and $d \neq 4$.

C. Let $\alpha, \beta \in L^\infty(S^d)$ be two $G_{d,i}$-invariant functions such that $\text{essinf}_{S^d}\beta > 0$ and consider the problem

$$-\Delta_h u + \alpha(\sigma)u = \beta(\sigma)f(u) \ \text{ on } \ S^d. \tag{26}$$

If $f : \mathbb{R} \to \mathbb{R}$ has an asymptotically critical growth fulfilling (f_1^L) and

$$\liminf_{s\to L} \frac{f(s)}{s} < \text{essinf}_{S^d} \frac{\alpha}{\beta},$$

problem (26) admits a sequence of $G_{d,i}$-invariant (perhaps not sign-changing) weak solutions in both cases, i.e. $L \in \{0^+, \infty\}$. The proofs can be carried out following Theorems 2.1 and 2.2, respectively, considering instead of $H_{G_{d,i}^{\tau_i}}(S^d)$ the space $H_{G_{d,i}}(S^d)$. Note that $\alpha : S^d \to \mathbb{R}$ may change its sign. In particular, this type of result complements the paper of Cotsiolis-Iliopoulos [5].

D. The symmetry and compactness of the sphere S^d have been deeply exploited in our arguments. We intend to study a challenging problem related to (P) which is formulated on *non-compact Riemannian symmetric spaces* (for instance, on the hyperbolic space $H^d = SO_0(d, 1)/SO(d)$ which is the dual companion of $S^d = SO(d + 1)/SO(d)$). In order to handle this kind of problem, the action of the isometry group of the symmetric space seems to be essential, as shown by Hebey [9, Chapter 9], Hebey-Vaugon [10]. This problem will be treated in a forthcoming paper.

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