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BROWNIAN MOTION USING CONTRACTION METHOD IN PROBABILISTIC METRIC SPACES

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Abstract. Paper's abstract
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Recently Hutchinson and Rüschendorf gave a simple proof for the existence and uniqueness of invariant fractal sets and fractal functions using probability metrics defined by expectation. In these works a finite first moment condition is essential.

In this paper, using probabilistic metric spaces techniques, we can weak the first moment condition for existence and uniqueness of selfsimilar fractal functions.

The theory of probabilistic metric spaces, introduced in 1942 by K. Menger, was developed by numerous authors. The study of contraction mappings for probabilistic metric spaces was initiated by V. M. Sehgal, and H. Sherwood.

1 Selfsimilar fractal functions

Denote (X,d) a complete separable metric space Let $g : I \to X$, where $I \subset \mathbb{R}$ is a closed bounded interval, $N \in \mathbb{N}$ and let $I = I_1 \cup I_2 \cup \cdots \cup I_N$ be a partition of I into disjoint subintervals. Let $\Phi_i : I \to I_i$ be increasing Lipschitz maps with $p_i = Lip\Phi_i$. We have $\sum_{i=1}^{N} p_i \geq 1$ and if the Φ_i are affine then $\sum_{i=1}^{N} p_i = 1$. If $g_i : I_i \to \mathbf{X}$, for $i \in \{1, ..., N\}$ define $\sqcup_i g_i : I \to \mathbf{X}$ by

$$(\sqcup_i g_i)(x) = g_j(x) \, \text{for} \, x \in I_j.$$

A *scaling law* $\mathbf{S}$ is an N-tuple $(S_1, ..., S_N)$, $N \geq 2$, of Lipschitz maps $S_i : \mathbf{X} \to \mathbf{X}$. Denote $r_i = LipS_i$. *A random scaling law* $\mathbf{S} = (S_1, S_2, ..., S_N)$ is a random variable whose values are scaling laws. We write $\mathcal{S} = dist\mathbf{S}$ for the probability distribution determined by $\mathbf{S}$ and $\stackrel{d}{=}$ for the equality in distribution.

Let $\mathbf{S} = (S_1, ..., S_N)$ be a scaling law. For the function $g : I \to X$ define the function $\mathbf{S}g : I \to X$ by

$$\mathbf{S}g = \sqcup_i S_i \circ g \circ \Phi_i^{-1}.$$

We say g *satisfies the scaling law* $\mathbf{S}$, or is a *selfsimilar fractal function*, if

$$\mathbf{S}g \stackrel{d}{=} g.$$

Fix $0 < p \leq \infty$. Let

$$L_\infty = \{g : I \to X \mid \ esssup_{x\in X} d(\, g(x), a) < \infty\},$$

$$L_p = \{g : I \to X \mid \int d(g(x), a)^p < \infty\}, \text{ if } 0 < p < \infty,$$

for some a$\in \mathbb{R}$.

The metric $d_p$on L_p is the complete metric defined by

$$\begin{aligned} d_\infty(f, g) &= es\sup_x d(f(x), g(x)), \\ d_p(f, g) &= \left(\int d(f(x), g(x))\right)^{\frac{1}{p}\wedge 1} \text{ if } 0 < p < \infty. \end{aligned}$$

Let $\lambda_\infty = \max_i r_i$ and $\lambda_p = \sum_i p_i r_i^p$,for $0 < p < \infty$.

In Hutchinson and Rüschendorf prove the following:

Theorem 1.1 *([8]) If* $\mathbf{S} = (S_1, S_2, ..., S_N)$ *is a scaling law with* $\lambda_p < 1$ *for some* $0 < p \leq \infty$ *then there is a unique* $g^* \in L^p$ *such that* g^* *satisfies* $\mathbf{S}$.

Moreover, for any $g_0 \in L^p$,

$$esssup d_\infty(\mathbf{S}^k g_0, g^*) \leq \frac{\lambda_\infty^k}{1 - \lambda_\infty} esssup d_\infty(g_0, \mathbf{S}g_0) \to 0,$$

$$d_p(\mathbf{S}^k g_0, g^*) \leq \frac{\lambda_p^{k(\frac{1}{p}\wedge 1)}}{1 - \lambda_p^{\frac{1}{p}\wedge 1}} d_p(g_0, \mathbf{S}g_0) \to 0, \; 0 < p < 1$$

as $k \to \infty$.

For the random version we start with the random scaling law. Let $\mathbf{S} = (S_1, ..., S_N)$ be a random scaling law and let $G = (G_t)_{t\in I}$ be a stochastic process or a random function with state space $(X, \mathcal{X})$,where $\mathcal{X}$.is the Borel σ-algebra on X. The trajectory of the process G is the function $g : I \to$ X.The trajectory of the random function $\mathbf{S}g$ is defined up to probability distribution by

$$\mathbf{S}g = \sqcup_i S_i \circ g^{(i)} \circ \Phi_i^{-1},$$

where $\mathbf{S}, g^{(1)}, ..., g^{(N)}$ are independent of one another and $g^{(i)} \stackrel{d}{=} g$, for $i \in \{1, ..., N\}$. If $\mathcal{G} = distg$ we define

$$\mathcal{S}\mathcal{G} = dist\mathbf{S}g.$$

We say g or $\mathcal{G}$ *satisfies the scaling law* $\mathbf{S}$, or is a *selfsimilar random fractal function*, if

$$\mathbf{S}g \overset{d}{=} g, \quad \text{or equivalently} \quad \mathcal{S}\mathcal{G} = \mathcal{G}.$$

Beginning from any $g_0 \in L_p$ Hutchinson and Rüschendorf define [8] a sequence of random functions

$$\mathbf{S}g_0 = \sqcup_i S_i \circ g_0 \circ \Phi_i^{-1},$$

$$\mathbf{S}^2 g_0 = \sqcup_{i,j} S_i \circ S_j^i \circ g_0 \circ \Phi_j^{-1} \circ \Phi_i^{-1},$$

$$\mathbf{S}^3 g_0 = \sqcup_{i,j,k} S_i \circ S_j^i \circ S_k^{ij} \circ g_0 \circ \Phi_k^{-1} \circ \Phi_j^{-1} \circ \Phi_i^{-1},$$

etc.; where $\mathbf{S^i} = (S_1^i, S_2^i, ..., S_N^i)$, for $i \in \{1, ..., N\}$, are independent of each other and of $\mathbf{S}$, the $\mathbf{S}^{ij} = (S_1^{ij}, S_2^{ij}, ..., S_N^{ij})$, for $i, j \in \{1, ..., N\}$ are independent of each other and of $\mathbf{S}$ and $\mathbf{S^i}$, etc.

Theorem 1.2 *(Hutchinson and Ruschendorf ([8])) If there exists a random function h such that*

$$esssup_\omega d_\infty(h^\omega, \delta_a^\omega) < \infty \qquad or \tag{1}$$

$$E_\omega^{\frac{1}{p}} d_p^p(h^\omega, \delta_h^\omega) < \infty \quad for 1 \leq p < \infty \quad or \tag{2}$$

$$E_\omega d_p(h^\omega, \delta_h^\omega) < \infty \quad for\ 0 < p < 1, \tag{3}$$

and if $\mathbf{S} = (S_1, ..., S_N)$ *is a random scaling law which satisfies either*

$$\lambda_p := E\sum_{i=1}^N p_i r_i^p < 1 \quad and \quad E\sum_{i=1}^N p_i d^p(S_i(a), a) < \infty,\ or \tag{4}$$

$$\lambda_\infty := esssup_\omega \max_i r_i < 1 \quad and \quad esssup_\omega \max_i d^p(S_i(a), a) < \infty, \tag{5}$$

then there exists a unique g^* *such that* $\mathbf{S}g^* \overset{d}{=} g^*$ *and for any* $g_0 \in L_p$*,*

$$esssup d_\infty(\mathbf{S}^k g_0, g^*) \leq \frac{\lambda_\infty^k}{1-\lambda_\infty} esssup d_\infty(g_0, \mathbf{S}g_0) \to 0,$$

$$E^{\frac{1}{p}} d_p^p(\mathbf{S}^k g_0, g^*) \leq \frac{\lambda_p^{\frac{k}{p}}}{1-\lambda_p^{\frac{1}{p}}} E^{\frac{1}{p}} d_p^p(g_0, \mathbf{S}g_0) \to 0,\ 1 \leq p < \infty$$

$$E d_p(\mathbf{S}^k g_0, g^*) \leq \frac{\lambda_p^k}{1-\lambda_p} E_p^d(g_0, \mathbf{S}g_0) \to 0,\ 0 < p < 1$$

as $k \to \infty$*, where* g^* *does not depend on* g_0*. In particular,* $\mathbf{S}^k g_0 \to g^*$ *a.s. Moreover, up to probability distribution,* g^* *is the unique function such that* $E\int log|g^*| < \infty$ *and which satisfies* $\mathbf{S}$*.*

However, using contraction method in probabilistic metric spaces,instead of (1) we can give weaker conditions for the existence and uniqueness of invariant random function.

Theorem 1.3 *Let $\mathcal{E}_p$ be the set of random functions $(G_t)_{t\in I}$ with state space X and let $\mathbf{S}$ be a random scaling law. Suppose there exists $h \in \mathcal{E}_p$ and a positive number γ such that either*

$$P(\{\omega\in\Omega \,|\, esssup_x d(h^\omega(x), \mathbf{S}h^\omega(x)) \geq t\}) \leq \frac{\gamma}{t} \textit{ for all } \quad t>0$$

and $\lambda_\infty := esssup_\omega \max_i r_i^\omega < 1$ or

$$P(\{\omega\in\Omega \,|\, \int_I d(h^\omega(x), \mathbf{S}h^\omega(x))^{\frac{1}{p}\wedge 1} \geq t\}) \leq \frac{\gamma}{t} \textit{ for all } \quad t>0$$

and $\lambda_p := E\sum_{i=1}^N p_i r_i^{\omega p} < 1$. Then there exists $G^* \in \mathcal{E}_p$ such that $\mathbf{S}g^* = g^*$. Moreover, up to a probability distribution g^* is the unique function in $\mathcal{E}_0 = \cup_{p>0}\mathcal{E}_p$.

The Brownian motion can be characterized as the fixed point of a scaling operator.

For each $\alpha > 0$, let $B^\alpha : [0,1] \to \mathbb{R}$ denote the constrained Brownian motion given by

$$B^\alpha(t+h) - B^\alpha(t) \overset{d}{=} N(0, \alpha h), \quad for \quad t>0\, and\, h>0,$$
$$B^\alpha(0) = 0 \; a.s., \quad \text{and} \quad B^\alpha(1) = 1 \; a.s.,$$

where $N(0,\alpha h)$ denotes the normal distribution with mean 0 and variance αh. For fix $p\in\mathbb{R}$ consider the Brownian motion $B^\alpha|_{B^\alpha(\frac{1}{2})=p}$ constrained by $B^\alpha(\frac{1}{2}) = p$.

Let $S_1, S_2 : \mathbb{R}\to\mathbb{R}$ the affine transformation characterized by

$$S_1(0) = 0,\; S_1(1) = S_2(0) = p,\; S_2(1) = 1.$$

If

$$r_1 = LipS_1 = |p|, \quad r_2 = LipS_2 = |1-p|,$$

then

$$B^\alpha|_{B^\alpha(\frac{1}{2})=p}(t) \overset{d}{=} S_1 \circ B^{\frac{\alpha}{2r_1^2}}(2t), \quad t\in[0,\frac{1}{2}].$$

Similarly

$$B^\alpha|_{B^\alpha(\frac{1}{2})=p}(t) \overset{d}{=} S_2 \circ B^{\frac{\alpha}{2r_1^2}}(2t-1), \quad t\in[\frac{1}{2},1].$$

Now define

$$\Phi_1 : [0,1]\to[0,\frac{1}{2}], \quad \Phi_1(s) = \frac{s}{2},$$
$$\Phi_2 : [0,1]\to[\frac{1}{2},1], \quad \Phi_1(s) = \frac{s+1}{2}.$$

It follows that

$$B^\alpha|_{B^\alpha(\frac{1}{2})}(t) \overset{d}{=} \sqcup_{i=1}^2 S_i \circ B^{\frac{\alpha}{2r_i^2}} \circ \Phi_i^{-1}(t), \quad t\in[0,1].$$

Now let p^α be random point with distribution $N(0,\frac{\alpha}{2})$ and let $\mathbb{S}^\alpha = (S_1^\alpha, S_2^\alpha)$ be the random scaling law obtained by defining (S_1^α, S_2^α) from the random point p^α in the same manner as (S_1, S_2) was previously defined from p. Let $r_i^\alpha = Lip_i^\alpha$ for $i = 1, 2$ and let $r^\alpha = \max\{r_1^\alpha, r_2^\alpha\}$. Denote $\mathbb{S} = \{\mathbb{S}^\alpha | \alpha > 0.\}$

It follows for each $\alpha > 0$ that

$$B^\alpha \stackrel{d}{=} \sqcup_{i=1}^2 S_i^\alpha \circ B^{\frac{\alpha}{2r_i^2}(i)} \circ \Phi_i^{-1},$$

where $B^{\frac{\alpha}{2r_1^2}(1)} \stackrel{d}{=} B^{\frac{\alpha}{2r_1^2}}$ and $B^{\frac{\alpha}{2r_2^2}(2)} \stackrel{d}{=} B^{\frac{\alpha}{2r_2^2}}$ are chosen independently of one another.

Thus the family of constrained Brownian motion $\{B^\alpha | \alpha > 0\}$ satisfies the family of scaling laws $\mathbb{S}$.

2 Invariant sets in E-spaces

2.1 Menger spaces

Let $\mathbf{R}$ denote the set of real numbers and $\mathbf{R}_+ := \{x \in \mathbf{R} : x \geq 0\}$. A mapping $F : \mathbf{R} \to [0, 1]$ is called a *distribution function* if it is non-decreasing, left continuous with $\inf_{t \in \mathbf{R}} F(t) = 0$ and $\sup_{t \in \mathbf{R}} F(t) = 1$ (see [2]). By Δ we shall denote the set of all distribution functions F. Let Δ be ordered by the relation "$\leq$", i.e. $F \leq G$ if and only if $F(t) \leq G(t)$ for all real t. Also $F < G$ if and only if $F \leq G$ but $F \neq G$. We set $\Delta^+ := \{F \in \Delta : F(0) = 0\}$.

Throughout this paper H will denote the Heviside distribution function defined by

$$H(x) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0. \end{cases}$$

Let X be a nonempty set. For a mapping $\mathcal{F} : X \times X \to \Delta^+$ and $x, y \in X$ we shall denote $\mathcal{F}(x, y)$ by $F_{x,y}$, and the value of $F_{x,y}$ at $t \in \mathbf{R}$ by $F_{x,y}(t)$, respectively. The pair $(X, \mathcal{F})$ is a *probabilistic metric space* (briefly *PM space*) if X is a nonempty set and $\mathcal{F} : X \times X \to \Delta^+$ is a mapping satisfying the following conditions:

1^0. $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X$ and $t \in \mathbf{R}$;

2^0. $F_{x,y}(t) = 1$, for every $t > 0$, if and only if $x = y$;

3^0. if $F_{x,y}(s) = 1$ and $F_{y,z}(t) = 1$ then $F_{x,z}(s + t) = 1$.

A mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a *t-norm* if the following conditions are satisfied:

4^0. $T(a, 1) = a$ for every $a \in [0, 1]$;

5^0. $T(a, b) = T(b, a)$ for every $a, b \in [0, 1]$

6^0. if $a \geq c$ and $b \geq d$ then $T(a, b) \geq T(c, d)$;

7^0. $T(a, T(b, c)) = T(T(a, b), c)$ for every $a, b, c \in [0, 1]$.

A *Menger space* is a triplet $(X, \mathcal{F}, T)$, where $(X, \mathcal{F})$ is a probabilistic metric space, where T is a t-norm, and instead of 3^0 we have the stronger condition

8^0. $F_{x,y}(s + t) \geq T(F_{x,z}(s), F_{z,y}(t))$ for all $x, y, z \in X$ and $s, t \in \mathbf{R}_+$.

The (t,ϵ)-topology in a Menger space was introduced in 1960 by B. Schweizer and A. Sklar [?]. The base for the neighbourhoods of an element $x \in X$ is given by

$$\{U_x(t,\epsilon) \subseteq X : t > 0, \epsilon \in]0,1[\},$$

where

$$U_x(t,\epsilon) := \{y \in X : F_{x,y}(t) > 1-\epsilon\}.$$

In 1966, V.M. Sehgal [13] introduced the notion of a contraction mapping in PM spaces. The mapping $f : X \to X$ is said to be a *contraction* if there exists $r \in]0,1[$ such that

$$F_{f(x),f(y)}(rt) \geq F_{x,y}(t)$$

for every $x, y \in X$ and $t \in \mathbf{R}_+$.

A sequence $(x_n)_{n\in\mathbf{N}}$ from X is said to be *fundamental* if

$$\lim_{n,m\to\infty} F_{x_m,x_n}(t) = 1$$

for all $t > 0$. The element $x \in X$ is called *limit* of the sequence $(x_n)_{n\in\mathbf{N}}$, and we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$, if $\lim_{n\to\infty} F_{x,x_n}(t) = 1$ for all $t > 0$. A probabilistic metric (Menger) space is said to be *complete* if every fundamental sequence in that space is convergent.

Let A and B nonempty subsets of X. The *probabilistic Hausdorff-Pompeiu distance* between A and B is the function $F_{A,B} : \mathbf{R} \to [\mathbf{0},\mathbf{1}]$ defined by

$$F_{A,B}(t) := \sup_{s<t} T(\inf_{x\in A} \sup_{y\in B} F_{x,y}(s), \inf_{y\in B} \sup_{x\in A} F_{x,y}(s)).$$

In the following we remember some properties proved in [11, ?]:

Proposition 2.1 *If $\mathcal{C}$ is a nonempty collection of nonempty closed bounded sets in a Menger space $(X,\mathcal{F},T)$ with T continuous, then $(\mathcal{C},\mathcal{F}_{\mathcal{C}},T)$ is also Menger space, where $\mathcal{F}_{\mathcal{C}}$ is defined by $\mathcal{F}_{\mathcal{C}}(A,B) := F_{A,B}$ for all $A, B \in \mathcal{C}$.*

Proof. See [11, 13]. □

Proposition 2.2 *Let $T_m(a,b) := \max\{a+b-1, 0\}$. If $(X,\mathcal{F},T_m)$ is a complete Menger space and $\mathcal{C}$ is the collection of all nonempty closed bounded subsets of X in $(t,\epsilon)-$ topology, then $(\mathcal{C},\mathcal{F}_{\mathcal{C}},T_m)$ is also a complete Menger space.*

Proof. See [?]. □

2.2 E-spaces

The notion of E-space was introduced by Sherwood [14] in 1969. Next we recall this definition. Let $(\Omega, \mathcal{K}, P)$ be a probability space and let (Y, ρ) be a metric space. The ordered pair $(\mathcal{E}, \mathcal{F})$ is an *E-space over the metric space* (Y, ρ) (briefly, an E-space) if the elements of $\mathcal{E}$ are random variables from Ω into Y and $\mathcal{F}$ is the mapping from $\mathcal{E} \times \mathcal{E}$ into Δ^+ defined via $\mathcal{F}(x, y) = F_{x,y}$, where

$$F_{x,y}(t) = P(\{\omega \in \Omega |\ d(x(\omega), y(\omega)) < t\})$$

for every $t \in \mathbf{R}$. Usually $(\Omega, \mathcal{K}, P)$ is called the base and (Y, ρ) the target space of the E-space. If $\mathcal{F}$ satisfies the condition

$$\mathcal{F}(x, y) \neq H, \quad for \ \ x \neq y,$$

with H defined in paragraf 3.1., then $(\mathcal{E}, \mathcal{F})$ is said to be a *canonical E-space.* Sherwood [14] proved that every canonical E-space is a Menger space under $T = T_m$, where $T_m(a, b) = \max\{a + b - 1, 0\}$. In the following we suppose that $\mathcal{E}$ is a canonical E-space.

The convergence in an E-space is exactly the probability convergence. The E-space $(\mathcal{E}, \mathcal{F})$ is said to be complete if the Menger space $(\mathcal{E}, \mathcal{F}, T_m)$ is complete.

The next result was proved in [**?**]:

Theorem 2.1 *Let $(\mathcal{E}, \mathcal{F})$ be a complete E- space, $N \in \mathbf{N}^*$, and let $f_1, ..., f_N : \mathcal{E} \to \mathcal{E}$ be contractions with ratio $r_1, ...r_N$, respectively. Suppose that there exists an element $z \in \mathcal{E}$ and a real number γ such that*

$$P(\{\omega \in \Omega | \rho(z(\omega), f_i(z(\omega)) \geq t\}) \leq \frac{\gamma}{t}, \tag{6}$$

for all $i \in \{1, .., N\}$ and for all $t > 0$. Then there exists a unique nonempty closed bounded and compact subset K of $\mathcal{E}$ such that

$$f_1(K) \cup ... \cup f_N(K) = K.$$

Corollary 2.1 *Let $(\mathcal{E}, \mathcal{F})$ be a complete E- space, and let $f : \mathcal{E} \to \mathcal{E}$ be a contraction with ratio r. Suppose there exists $z \in \mathcal{E}$ and a real number γ such that*

$$P(\{\omega \in \Omega |\ \rho(z(\omega), f(z)(\omega)) \geq t\}) \leq \frac{\gamma}{t} \, for \ all \, t > 0.$$

Then there exists a unique $x_0 \in \mathcal{E}$ such that $f(x_0) = x_0$.

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