Selfsimilar random fractal measure using contraction method in probabilistic metric spaces

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Abstract

We use contraction method in probabilistic metric spaces to prove existence and uniqueness of selfsimilar random fractal measures.

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1 Introduction

Contraction methods for proving the existence and uniqueness of nonrandom selfsimilar fractal sets and measures were first applied by Hutchinson [7]. Further results and applications to image compression were obtained by Barnsley and Demko [2] and Barnsley [3]. At the same time Falconer [5], Graf [6], and Mauldin and Williams [13] randomized each step in the approximation process to obtain sefsimilar random fractal sets. Atbeiter [1] and Olsen [15] studied selfsimilar random fractal measures applying nonrandom metrics. More recently Hutchinson and Rüschendorf [8, 9, 10] introduced probability metrics defined by expectation for random measure and established existence, uniqueness and approximation properties of selfsimilar random fractal measures. In these works a finite first moment condition is essential.

In this paper it will shown that, using probabilistic metric spaces techniques, we can weak the first moment condition for existence and uniqueness of selfsimilar measures.

The theory of probabilistic metric spaces, introduced in 1942 by K. Menger [14], was developed by numerous authors, as it can be realized upon consulting the list of references in [4], as well as those in [18]. The study of contraction mappings for probabilistic metric spaces was initiated by V. M. Sehgal [19], and H. Sherwood [20].

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2 Selfsimilar random fractal measures

Recently Hutchinson and Rüschendorf [8, 9, 10] gave a simple proof for the existence and uniqueness of invariant random measures using the L^q-metric, $0 < q \le \infty$. The underlying probability space for the iteration procedure is generated by selecting independent and identically distributed scaling laws. Let (X, d) be a complete separable metric space. A *scaling law with weights* $\mathbf{S}$ is a 2N-tuple $(p_1, S_1, ..., p_n, S_N)$, $N \ge 1$, of positive real numbers p_i such that $\sum_{i=1}^N p_i = 1$ and of Lipschitz maps $S_i : X \to X$ with Lipschitz constant $r_i = LipS_i$, i∈{1,...,N}.

Denote $M = M(X)$ the set of finite mass Borel regular measures on X with the weak topology. If $\mu \in M$, then the measure $\mathbf{S}\mu$ is defined by

$$\mathbf{S}\mu = \sum_{i=1}^N p_i S_i \mu,$$

where $S_i\mu$ is the usual push forward measure, i.e.

$$S_i\mu(A) = \mu(S_i^{-1}(A)), \text{ for } A \subseteq X.$$

We say μ *satisfies the scaling law* $\mathbf{S}$ or *is a selfsimilar fractal measure* if $\mathbf{S}\mu = \mu$.

Let M_q denote the set of unit mass Borel regular measures μ on X with finite q-th moment. That is,

$$M_q = \{\mu \in M \,|\, \mu(X) = 1, \int d^q(x, a) d\mu(x) < \infty\}$$

for some (and hence any) $a \in X$. Note that, if $p \ge q$ then $M_p \subset M_q$.

The *minimal metric* l_q on M_q is defined by

$$l_q(\mu, \nu) = \inf\{(\int d^q(x, y) d\gamma(x, y))^{\frac{1}{q} \wedge 1} \,|\, \pi_1\gamma = \mu,\, \pi_2\gamma = \nu\}$$

where $\wedge$ denotes the minimum of the relevant numbers and $\pi_i\gamma$ denotes the i-th marginal of γ, i.e. projection of the measure γ on $X \times X$ onto the i-th component.

We have the following properties of l_q (see [16]):

a) Suppose α is a positive real, $S : X \to X$ is Lipschitz, and $\vee$ denotes the maximum of the relevant numbers. Then for $q > 0$ and for measures μ, ν we have the following properties:

$$l_q^{q \vee 1}(\alpha\mu, \alpha\nu) = \alpha l_q^{q \vee 1}(\mu, \nu),$$

$$l_q^{q \vee 1}(\mu_1 + \mu_2, \nu_1 + \nu_2) \le l_q^{q \vee 1}(\mu_1, \nu_1) + l_q^{q \vee 1}(\mu_2, \nu_2),$$

$$l_q(S\mu, S\nu) \le (LipS)^{q \wedge 1} l_q(\mu, \nu)$$

b) (M_q, l_q) is a complete separable metric space and $l_q(\mu_n, \mu) \to 0$ if and only if
(i) $\mu_n \to \mu$ (weak convergence) and
(ii) $\int d^q(x, a) d\mu_n(x) \to \int d^q(x, a) d\mu(x)$ (convergence of q-th moments).

c) If δ_a is the Dirac measure at $a \in X$, then

$$l_q(\mu, \mu(X)\delta_a) = (\int d^q(x,a)d\mu(x))^{\frac{1}{q}\wedge 1},$$

$$l_q(\delta_a, \delta_b) = d^{1\wedge q}(a,b).$$

Let $\mathbf{M}$ denote the set of all random measures μ with value in M, i.e. random variables $\mu : \Omega \to M$. Let $\mathbf{M_q}$ denote the space of random measures $\mu : \Omega \to M_q$ with finite expected q-th moment i.e.

$$\mathbf{M_q} := \{\mu \in \mathbf{M} | \, \mu^\omega(X) = 1 a.s., \, E_\omega \int_X d^q(x,a)d\mu^\omega(x) < \infty\} \tag{1}$$

The notation E_ω indicate that the expectation is with respect to the variable ω. It follows from (1) that $\mu^\omega \in M_q$ a.s. Note that $\mathbf{M_p} \subset \mathbf{M_q}$ if $q \le p$.Moreover, since $E^{\frac{1}{q}}|f|^q \to exp(E\log|f|)$ as $q \to 0$,

$$\mathbf{M_0} := \cup_{q>0}\mathbf{M_q} = \{\mu \in \mathbf{M}|\mu_\omega(X) = 1\, a.s.\,, E_\omega \int_X \log d(x,a)d\mu^\omega(x) < \infty\}.$$

For random measures $\mu, \nu \in \mathbf{M_q}$, define

$$l_q^*(\mu,\nu) := \begin{cases} E_\omega^{\frac{1}{q}} l_q^q(\mu^\omega, \nu^\omega), & q \ge 1 \\ E_\omega l_q(\mu^\omega, \nu^\omega), & 0 < q < 1. \end{cases}$$

One can check as in [16], that $(\mathbf{M_q}, \mathbf{l_q^*})$ is a complete separable metric space. Note that $l_q^*(\mu,\nu) = l_q(\mu,\nu)$ if μ and ν are constant random measures.

Let $\mathcal{M}$ denote the class of probability distributions on $\mathbf{M}$. i.e.

$$\mathcal{M} = \{\mathcal{D} = dist\mu \,|\, \mu \in \mathbf{M}\}.$$

Let $\mathcal{M}_q$ be the set of probability distributions of random measures $\mu \in \mathbf{M}_q$. If $q \le p$ then $\mathcal{M}_p \subset \mathcal{M}_q$. Let

$$\mathcal{M}_0 := \cup_{q>0}\mathcal{M}_q.$$

The minimal metric on $\mathcal{M}_q$ is defined by

$$l_q^{**}(\mathcal{D}_1, \mathcal{D}_2) = \inf\{l_q^*(\mu,\nu)|\, \mu \overset{d}{=} D_1, \, \nu \overset{d}{=} D_2\}.$$

It follows that $(\mathcal{M}_q, l_q^{**})$ is a complete separable metric space with the next properties:

$$a)\, l_q^{**}(\alpha\mathcal{D}_1, \alpha\mathcal{D}_2) = \alpha l_q^{**}(\mathcal{D}_1, \mathcal{D}_2),$$

$$b)\, l_q^{**}(\mathcal{D}_1 + \mathcal{D}_2, \mathcal{D}_3 + \mathcal{D}_4) \le l_q^{**q}(\mathcal{D}_1, \mathcal{D}_3) + l_q^{**q}(\mathcal{D}_2, \mathcal{D}_4)$$

for $\mathcal{D}_i \in \mathcal{M}_q$, $i = 1,2,3,4$.

A random scaling law $\mathbf{S} = (p_1, S_1, p_2, S_2, ..., p_n, S_N)$ is a random variable whose values are scaling laws, with $\sum_{i=1}^N p_i = 1$ a.s. We write $\mathcal{S} = dist\mathbf{S}$ for the probability distribution determined by $\mathbf{S}$ and $\overset{d}{=}$ for the equality in distribution.

If μ is a random measure, then the random measure $\mathbf{S}\mu$ is defined (up to probability distribution) by

$$\mathbf{S}\mu := \sum_{i=1}^{N} p_i S_i \mu^{(i)},$$

where $\mathbf{S}, \mu^{(1)}, ..., \mu^{(N)}$ are independent of one another and $\mu^{(i)} \overset{d}{=} \mu$. If $\mathcal{D} = dist\mu$ we define $\mathcal{SD} = dist\mathbf{S}\mu$.

We say μ *satisfies the scaling law* $\mathbf{S}$, or is a *selfsimilar random fractal measure,* if

$$\mathbf{S}\mu \overset{d}{=} \mu, \text{ or equivalently } \mathcal{SD} = \mathcal{D}$$

and $\mathcal{D}$ is called a selfsimilar random fractal distribution.

To generate random selfsimilaar fractal measure we use the next **iterative procedure** (see [8]):

Fix $q > 0$.

Beginning with a nonrandom measure $\mu_0 \in M_q$ one iteratively applies iid scaling laws with distribution $\mathcal{S}$ to obtain a sequence μ_n of random measures in $\mathbf{M_q}$ and a corresponding sequence $\mathcal{D}_n$ of distributions in $\mathcal{M}_q$, as follows:

(i) Select a scaling law $\mathbf{S}$ via the distribution $\mathbf{S}$ and define.

$$\mu_1 = \mathbf{S}\mu_0 = \sum_{i=1}^{n} p_i S_i \mu_0, i.e.\ \mu_1(\omega) = \mathbf{S}\mu_0 = \sum_{i=1}^{n} p_i(\omega) S_i(\omega)\mu_0,\ \mathcal{D}_1 \overset{d}{=} \mu_1,$$

(ii) Select $\mathbf{S_1}, ..., \mathbf{S_n}$ via $\mathbf{S}$ with $\mathbf{S^i} = (p_1^i, S_1^i, ..., p_N^i, S_N^i), i \in \{1, 2, ..., N\}$ independent of each other and of $\mathbf{S}$ and define

$$\mu_2 := \mathbf{S}^2\mu_0 = \sum_{i,j} p_i p_j^i S_i \circ S_j^i \mu_0,\ \mathcal{D}_2 \overset{d}{=} \mu_0$$

(iii) Select $\mathbf{S}^{ij} = (p_1^i, S_1^{ij}, ..., p_N^i, S_N^{i,j})$ via $\mathcal{S}$, independent of one another and of $\mathbf{S^1}, ..., \mathbf{S^N}, \mathbf{S}$ and define

$$\mu_3 = \mathbf{S}^3\mu_0 = \sum_{i,j,k} p_i p_j^i p_k^{ij} S_i \circ S_j^i \circ S_k^{ij} \mu_0,\ \mathcal{D}_3 \overset{d}{=} \mu_3,$$

etc.

Thus $\mu_{n+1} = \sum_{i=1}^N p_i S_i \mu_n^{(i)}$ where $\mu_n^{(i)} \overset{d}{=} \mu_n \overset{d}{=} \mathcal{D}_n$, $\mathbf{S} =\overset{\mathbf{d}}{=} \mathcal{S}$, and the $\mu_n^{(i)}$ and $\mathbf{S}$ are independent. It follows that $\mathcal{D}_n = \mathcal{SD}_{n-1} = \mathcal{S}^n\mathcal{D}_\prime$, where $\mathcal{D}_0$ is the distribution of μ_0. If $\mu_0 \in M_q$, then $\mathcal{D}_0$ is constant.

The underlying probability space for a.s. convergence is defined above (see [10]).

A *construction tree* (or a construction process) is a map $\omega : \{1,...,N\}^* \to \Gamma$, where Γ is the set of (nonrandom) scaling laws. A construction tree specifies at each node of the scaling law used to define constructively a recursive sequence of random measures. Denote the scaling law of ω at the node σ by the 2N-tuple

$$\mathbf{S}^\sigma(\omega) = \omega(\sigma) = (p_1^\sigma(\omega), S_1^\sigma(\omega), ..., p_N^\sigma(\omega), S_N^\sigma(\omega))$$

where p_i^σ are weights and S_i^σ Lipschitz maps. The sample space of all construction trees is denoted by $\tilde{\Omega}$. The underlying probability space $(\tilde{\Omega}, \tilde{\mathcal{K}}, \tilde{P})$ for the iteration procedure is generated by selecting identical distributed and independent scaling laws $\omega(\sigma) \stackrel{d}{=} \mathbf{S}$ for each $\sigma \in \{1,...,N\}^*$.

In [9] it is proved the following theorem:

Theorem 2.1 *Let* $\mathbf{S} = (p_1, S_1, p_2 S_2, ..., p_n, S_N)$ *be a random scaling law, with* $\sum_{i=1}^N p_i = 1$ *a.s. Assume* $\lambda_q := E_\omega(\sum_{i=1}^N p_i r_i^q) < 1$ *and*

$$E_\omega(\sum_{i=1}^N p_i d^q(S_i a, a) < \infty \, for\ some\, q > 0, \ and\ for\, a \in X. \tag{2}$$

Then

a) the operator $\mathbf{S} : \mathbf{M_q} \to \mathbf{M_q}$ *is a contraction map with respect to* l_q^*.

b) If μ^* *is the unique fixed point of* $\mathbf{S}$ *and* $\mu_0 \in M_p$ *(or more generally* $\mathbf{M}_q$*), then*

$$E_\omega^{\frac{1}{q}} l_q^q(\mu_n, \mu^*) \leq \frac{\lambda_q^{\frac{k}{q}}}{1 - \lambda_q^{\frac{1}{q}}} E_\omega^{\frac{1}{q}} l_q^q(\mu_0, \mathbf{S}\mu_0) \to 0, \, q \geq 1$$

$$E_\omega l_q(\mu_n, \mu^*) \leq \frac{\lambda_q^k}{1 - \lambda_q} E_\omega l_q(\mu_0, \mathbf{S}\mu_0) \to 0, \, 0 < q < 1$$

as $k \to \infty$. *In particular* $\mu_n \to \mu^*$ *a.s. in the sense of weak convergence of measures.*

Moreover, up to probability distribution, μ^* *is the unique unit mass random measure with* $E_\omega \int \ln d(x,a) d_\mu^\omega < \infty$ *which satisfies* $\mathbf{S}$.

Using contraction method in probabilistic metric spaces, instead of condition (2) we can give weaker condition for the existence and uniqueness of invariant measure. More precisely, in Section 4 we will prove the following

Theorem 2.2 *Let* $\mathbf{S} = (p_1, S_1, p_2 S_2, ..., p_n, S_N)$ *be a random scaling law, which satisfies* $\sum_{i=1}^N p_i = 1$ *a.s. and suppose* $\lambda_q := esssup(\sum_{i=1}^N p_i r_i^q) < 1$ *for some* $q > 0$. *If there exist* $\alpha \in M_q$ *and a positive number* γ *such that*

$$P(\{\omega \in \Omega | \, l_q(\alpha(\omega), \mathbf{S}\alpha(\omega)) \geq t\}) \leq \frac{\gamma}{t}, \, for\ all\, t > 0, \tag{3}$$

then there exists μ^* *such that* $\mathbf{S}\mu^* = \mu^* a.s.$ *and exponentially fast.*

Moreover up to probability distribution μ^* *is the unique unit mass random measure which satisfies* $\mathbf{S}$.

Remark: If condition (2) is satisfied then condition (3) hold also. To see this, let $a \in X$ and $\alpha(\omega) := \delta_a$ for all $\omega \in \Omega$. We have:

$$P(\{\omega \in \Omega |\, l_q(\delta_a(\omega), \mathbf{S}\delta_a(\omega)) \geq t\}) =$$

$$= P(\{\omega \in \Omega |\, l_q(\sum_{i=1}^{N} p_i\delta_a(\omega), \sum_{i=1}^{N} p_i S_i\delta_a(\omega)) \geq t\}) \leq$$

$$\leq P(\{\omega \in \Omega |\, \sum_{i=1}^{N} p_i l_q(\delta_a(\omega), S_i\delta_a(\omega)) \geq t\}) =$$

$$= P(\{\omega \in \Omega |\, \sum_{i=1}^{N} p_i d^q(S_i a, a)) \geq t\}) \leq \frac{1}{t} E_\omega(\sum_{i=1}^{N} p_i d^q(S_i a, a)) = \frac{\gamma}{t}$$

However, condition (3) can be satisfied also if

$$E_\omega(\sum_{i=1}^{N} p_i d^q(S_i a, a)) = \infty \text{ for all}\, q > 0.$$

Let $\Omega =]0, 1]$ with the Lebesque measure, let X be the interval $[0, \infty[$ and $N = 1$. Define $S : X \to X$ by $S^\omega(x) = \frac{x}{2} + e^{\frac{1}{\omega}}$. This map is a contraction with ratio $\frac{1}{2}$. For $q > 0$, the expectation $E_\omega d^q(S0, 0) = \infty$, however

$$P(\{\omega \in \Omega | l_q(S0, 0) \geq t\}) = \frac{1}{t}$$

for all $t > 0$.

3 Invariant sets in E-spaces

3.1 Menger spaces

Let $\mathbf{R}$ denote the set of real numbers and $\mathbf{R}_+ := \{x \in \mathbf{R} : x \geq 0\}$. A mapping $F : \mathbf{R} \to [0, 1]$ is called a *distribution function* if it is non-decreasing, left continuous with $\inf_{t\in\mathbf{R}} F(t) = 0$ and $\sup_{t\in\mathbf{R}} F(t) = 1$ (see [4]). By Δ we shall denote the set of all distribution functions F. Let Δ be ordered by the relation "$\leq$", i.e. $F \leq G$ if and only if $F(t) \leq G(t)$ for all real t. Also $F < G$ if and only if $F \leq G$ but $F \neq G$. We set $\Delta^+ := \{F \in \Delta : F(0) = 0\}$.

Throughout this paper H will denote the Heviside distribution function defined by

$$H(x) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0. \end{cases}$$

Let X be a nonempty set. For a mapping $\mathcal{F} : X \times X \to \Delta^+$ and $x, y \in X$ we shall denote $\mathcal{F}(x, y)$ by $F_{x,y}$, and the value of $F_{x,y}$ at $t \in \mathbf{R}$ by $F_{x,y}(t)$, respectively. The pair $(X,\mathcal{F})$ is a *probabilistic metric space* (briefly *PM space*) if X is a nonempty set and $\mathcal{F}: X\times X\to\Delta^+$ is a mapping satisfying the following conditions:

1^0. $F_{x,y}(t)=F_{y,x}(t)$ for all $x,y\in X$ and $t\in\mathbf{R}$;
2^0. $F_{x,y}(t)=1$, for every $t>0$, if and only if $x=y$;
3^0. if $F_{x,y}(s)=1$ and $F_{y,z}(t)=1$ then $F_{x,z}(s+t)=1$.

A mapping $T:[0,1]\times[0,1]\to[0,1]$ is called a *t-norm* if the following conditions are satisfied:

4^0. $T(a,1)=a$ for every $a\in[0,1]$;
5^0. $T(a,b)=T(b,a)$ for every $a,b\in[0,1]$
6^0. if $a\geq c$ and $b\geq d$ then $T(a,b)\geq T(c,d)$;
7^0. $T(a,T(b,c))=T(T(a,b),c)$ for every $a,b,c\in[0,1]$.

A *Menger space* is a triplet $(X,\mathcal{F},T)$, where $(X,\mathcal{F})$ is a probabilistic metric space, where T is a t-norm, and instead of 3^0 we have the stronger condition

8^0. $F_{x,y}(s+t)\geq T(F_{x,z}(s),F_{z,y}(t))$ for all $x,y,z\in X$ and $s,t\in\mathbf{R}_+$.

The (t,ϵ)-topology in a Menger space was introduced in 1960 by B. Schweizer and A. Sklar [17]. The base for the neighbourhoods of an element $x\in X$ is given by

$$\{U_x(t,\epsilon)\subseteq X: t>0, \epsilon\in]0,1[\},$$

where

$$U_x(t,\epsilon):=\{y\in X: F_{x,y}(t)>1-\epsilon\}.$$

In 1966, V.M. Sehgal [19] introduced the notion of a contraction mapping in PM spaces. The mapping $f:X\to X$ is said to be a *contraction* if there exists $r\in]0,1[$ such that

$$F_{f(x),f(y)}(rt)\geq F_{x,y}(t)$$

for every $x,y\in X$ and $t\in\mathbf{R}_+$.

A sequence $(x_n)_{n\in\mathbf{N}}$ from X is said to be *fundamental* if

$$\lim_{n,m\to\infty}F_{x_m,x_n}(t)=1$$

for all $t>0$. The element $x\in X$ is called *limit* of the sequence $(x_n)_{n\in\mathbf{N}}$, and we write $\lim_{n\to\infty}x_n=x$ or $x_n\to x$, if $\lim_{n\to\infty}F_{x,x_n}(t)=1$ for all $t>0$. A probabilistic metric (Menger) space is said to be *complete* if every fundamental sequence in that space is convergent.

Let A and B nonempty subsets of X. The *probabilistic Hausdorff-Pompeiu distance* between A and B is the function $F_{A,B}:\mathbf{R}\to[\mathbf{0},\mathbf{1}]$ defined by

$$F_{A,B}(t):=\sup_{s<t}T(\inf_{x\in A}\sup_{y\in B}F_{x,y}(s),\inf_{y\in B}\sup_{x\in A}F_{x,y}(s)).$$

In the following we remember some properties proved in [11, 12]:

Proposition 3.1 *If $\mathcal{C}$ is a nonempty collection of nonempty closed bounded sets in a Menger space $(X, \mathcal{F}, T)$ with T continuous, then $(\mathcal{C}, \mathcal{F}_{\mathcal{C}}, T)$ is also Menger space, where $\mathcal{F}_{\mathcal{C}}$ is defined by $\mathcal{F}_{\mathcal{C}}(A, B) := F_{A,B}$ for all $A, B \in \mathcal{C}$.*

Proof. See [11, 19]. □

Proposition 3.2 *Let $T_m(a, b) := \max\{a+b-1, 0\}$. If $(X, \mathcal{F}, T_m)$ is a complete Menger space and $\mathcal{C}$ is the collection of all nonempty closed bounded subsets of X in $(t, \epsilon)-$ topology, then $(\mathcal{C}, \mathcal{F}_{\mathcal{C}}, T_m)$ is also a complete Menger space.*

Proof. See [12]. □

3.2 E-spaces

The notion of E-space was introduced by Sherwood [20] in 1969. Next we recall this definition. Let $(\Omega, \mathcal{K}, P)$ be a probability space and let (Y, ρ) be a metric space. The ordered pair $(\mathcal{E}, \mathcal{F})$ is an *E-space over the metric space* (Y, ρ) (briefly, an E-space) if the elements of $\mathcal{E}$ are random variables from Ω into Y and $\mathcal{F}$ is the mapping from $\mathcal{E} \times \mathcal{E}$ into Δ^+ defined via $\mathcal{F}(x, y) = F_{x,y}$, where

$$F_{x,y}(t) = P(\{\omega \in \Omega |\ d(x(\omega), y(\omega)) < t\})$$

for every $t \in \mathbf{R}$. Usually $(\Omega, \mathcal{K}, P)$ is called the base and (Y, ρ) the target space of the E-space. If $\mathcal{F}$ satisfies the condition

$$\mathcal{F}(x, y) \neq H, \quad for \quad x \neq y,$$

with H defined in paragraf 3.1., then $(\mathcal{E}, \mathcal{F})$ is said to be a *canonical E-space.* Sherwood [20] proved that every canonical E-space is a Menger space under $T = T_m$, where $T_m(a, b) = \max\{a + b - 1, 0\}$. In the following we suppose that $\mathcal{E}$ is a canonical E-space.

The convergence in an E-space is exactly the probability convergence. The E-space $(\mathcal{E}, \mathcal{F})$ is said to be complete if the Menger space $(\mathcal{E}, \mathcal{F}, T_m)$ is complete.

Proposition 3.3 *If (Y, ρ) is a complete metric space then the E-space $(\mathcal{E}, \mathcal{F})$ is also complete.*

Proof. This property is well-known for $Y = R$ (see e.g. [21], Theorem VII.4.2.]). In the general case the proof is analogous.

Let $(x_n)_{n\in\mathbf{N}}$ be a Cauchy sequence of elements of $\mathcal{E}$, that is

$$\lim_{n,m\to\infty} F_{x_n,x_{n+m}}(t) = 1, \text{ for all } t > 0.$$

First we show that there exists a subsequence $(x_{n_k})_{k\in\mathbf{N}}$ of the given sequence which is convergent almost everywhere to a random variable x. Let as set positive numbers ϵ_i so that $\sum_{i=1}^{\infty}\epsilon_i < \infty$ and put $\delta_p = \sum_{i=p}^{\infty}\epsilon_i$, $p = 1, 2, ...$ For each i there is a natural number k_i, such that

$$P(\{\omega \in \Omega | \rho(x_k(\omega), x_l(\omega)) \geq \epsilon_i\}) < \epsilon_i \text{ for } k, l \geq k_i.$$

We can assume that $k_1 < k_2 < ... < k_i < ....$ Then

$$P(\{\omega \in \Omega | \rho(x_{k_{i+1}}(\omega), x_{k_i}(\omega)) \geq \epsilon_i\}) < \epsilon_i \text{ for } k, l \geq k_i.$$

Let us put

$$D_p = \cup_{i=p}^{\infty}\{\omega \in \Omega |\, \rho(x_{k_{i+1}}, x_{k_i}) \geq \epsilon_i\}.$$

Then $P(D_p) < \delta_p$. Lastly, for the intersection $D' = \cap_{p=1}^{\infty} D_p$ we obviously have $P(D') = 0$ since $\delta_p \to 0$. We shall show that the sequence $(x_{k_i}(\omega))$ has a finite limit $x(\omega)$ at every point $\omega \in \{\omega \in \Omega |\, \rho(x_k(\omega), x_m(\omega)) > t\} \setminus D'$. For some p we have $x \notin D_p$. Consequently, $\rho(x_{k_{i+1}}(\omega), x_{k_i}(\omega)) < \epsilon_i$, for all $i \geq p$. It follows that for any two indices i and j such that $j > i \geq p$ we have

$$\rho(x_{k_j}(\omega), x_{k_i}(\omega)) \leq \sum_{m=i}^{j-1} \rho(x_{k_{m+1}}(\omega), x_{k_m}(\omega)) <$$

$$< \sum_{m=i}^{j-1} \epsilon_m < \sum_{m=i}^{\infty} \epsilon_m = \delta_i.$$

Thus $\lim_{i,j\to\infty} \rho(x_{k_j}(\omega), x_{k_i}(\omega))) = 0$. This means that $(x_k(\omega))_{k\in\mathbf{N}}$ is a Chauchy sequence for every ω which implies the pointwise convergence of $(x_{k_i})_{i\in\mathbf{N}}$ to a finite limit function. Now it only remains to put

$$x(\omega) = \begin{cases} \lim x_{k_i}(\omega) & for \quad \omega \notin D' \\ 0 & for \quad \omega \in D' \end{cases}$$

to obtain the desired limit random variable. By Lebeque theorem (see e.g. [21] theorem VI.5.2) $x_{k_i} \to x$ with respect to d. Thus, every Cauchy sequence in $\mathcal{E}$ has a limit, which means that the space $\mathcal{E}$ is complete. □

The next result was proved in [12]:

Theorem 3.1 *Let $(\mathcal{E}, \mathcal{F})$ be a complete E- space, $N \in \mathbf{N}^*$, and let $f_1, ..., f_N : \mathcal{E} \to \mathcal{E}$ be contractions with ratio $r_1, ...r_N$, respectively. Suppose that there exists an element $z \in \mathcal{E}$ and a real number γ such that*

$$P(\{\omega \in \Omega | \rho(z(\omega), f_i(z(\omega)) \geq t\}) \leq \frac{\gamma}{t}, \tag{4}$$

for all $i \in \{1, .., N\}$ and for all $t > 0$. Then there exists a unique nonempty closed bounded and compact subset K of $\mathcal{E}$ such that

$$f_1(K) \cup ... \cup f_N(K) = K.$$

Corollary 3.1 *Let $(\mathcal{E}, \mathcal{F})$ be a complete E- space, and let $f : \mathcal{E} \to \mathcal{E}$ be a contraction with ratio r. Suppose there exists $z \in \mathcal{E}$ and a real number γ such that*

$$P(\{\omega \in \Omega |\ \rho(z(\omega), f(z)(\omega)) \geq t\}) \leq \frac{\gamma}{t} \textit{ for all } t > 0.$$

Then there exists a unique $x_0 \in \mathcal{E}$ such that $f(x_0) = x_0$.

4 Proof of Theorem 2.2

First we give two lemmas. Let $\mathcal{E}_q$ be the set of random variables with values in M_q and let $\mathcal{E}_q(\alpha)$ be the set

$$\mathcal{E}_q(\alpha) := \{\beta \in \mathcal{E}_q |\ \exists \gamma > 0\, P(\{\omega \in \Omega | l_q(\alpha(\omega), \beta(\omega)) \geq t\}) \leq \frac{\gamma}{t}, \text{for all } t > 0\}.$$

Lemma 4.1 $\mathbf{M}_q \subset \mathcal{E}_q(\alpha)$ *for all $\alpha \in M_q$.*

Proof: For $\beta \in \mathbf{M}_q$ we have

$$P(\{\omega \in \Omega | l_q(\alpha(\omega), \beta(\omega)) \geq t\}) = \int_{l_q(\alpha(\omega),\beta(\omega)) \geq t} dP \leq$$

$$\leq \frac{1}{t} \int_\Omega l_q(\alpha(\omega), \beta(\omega)) dP = \frac{1}{t} E_\omega l_q(\alpha(\omega), \beta(\omega)).$$

Hence $\beta \in \mathcal{M}_q$ we have $\gamma = E_\omega l_q(\alpha(\omega), \beta(\omega)) < \infty$ for all $t > 0$. □

Lemma 4.2 *$(\mathcal{E}_q, \mathcal{F})$ is a complete E-space.*

Proof: Choose $Y := \mathcal{E}_q$ and $\mathcal{F}_{\mu,\nu}(t) := P(\{\omega \in \Omega | l_q(\mu(\omega), \nu(\omega)) < t\})$ in the Proposition 3.3. □

Proof of Theorem 2.2: Let $\mathcal{S}$ be a random scaling law. Define $f : \mathcal{E}_q \to \mathcal{E}_q$ by $f(\mu) = \mathbf{S}\mu$, i.e.

$$\mathbf{S}\mu(\omega) = \sum_i p_i^\omega S_i^\omega \mu(\omega^{(i)}).$$

We first claim that if $\mu \in \mathcal{E}_q$ then $\mathbf{S}\mu \in \mathcal{E}_q$. For this, choose iid $\mu(\omega^{(i)}) \stackrel{d}{=} \mu(\omega)$ and $(p_1^\omega, S_1^\omega, ..., p_N^\omega, S_N^\omega) \stackrel{d}{=} \mathbf{S}$ independent of $\mu(\omega)$. For $q \geq 1$ and $b = S_i^\omega(a)$ we compute

$$\int d^q(x, a) d(\mathbf{S}\mu^(\omega)(x)) = l_q^q(\sum_{i=1}^N p_i^\omega S_i^\omega \mu(\omega^{(i)}), \delta_a) =$$

$$= l_q^q(\sum_{i=1}^N p_i^\omega S_i^\omega \mu(\omega^{(i)}), \sum_{i=1}^N p_i^\omega S_i^\omega \delta_b) \leq$$

$$\leq \sum_{i=1}^{N} p_i^{\omega} r_i^q l_q^q(\mu(\omega^{(i)}), \delta_b).$$

Since $\mu(\omega^{(i)}) \in M_q$ we have

$$\int d^q(x,a) d(\mathbf{S}\mu(x) < \infty.$$

The case $0 < q < 1$ is dealt similarly, replacing l_q^q by l_q:

$$\int d^q(x,a) d(\mathbf{S}\mu^(\omega)(x)) = l_q(\sum_{i=1}^{N} p_i^{\omega} S_i^{\omega} \mu(\omega^{(i)}), \delta_a) =$$

$$= l_q(\sum_{i=1}^{N} p_i^{\omega} S_i^{\omega} \mu(\omega^{(i)}), \sum_{i=1}^{N} p_i^{\omega} S_i^{\omega} \delta_b) \leq$$

$$\leq \sum_{i=1}^{N} p_i^{\omega} r_i^q l_q(\mu(\omega^{(i)}), \delta_b) < \infty.$$

To establish the contraction property let $\mu, \nu \in \mathcal{E}_q$, $\mu(\omega^{(i)}) \stackrel{d}{=} \mu(\omega), \nu(\omega^{(i)}) \stackrel{d}{=} \nu(\omega), i \in \{1, 2, ..., N\}$ and $q \geq 1$. We have

$$F_{f(\mu),f(\nu)}(t) = P(\{\omega \in \overline{\Omega} \,|\, l_q(f(\mu(\omega)), f(\nu(\omega)) < t\}) =$$

$$= P(\{\omega \in \overline{\Omega} \,|\, l_q(\sum_{i=1}^{N} p_i^{\omega} S_i^{\omega} \mu(\omega^{(i)}), \sum_{i=1}^{N} p_i^{\omega} S_i^{\omega} \nu(\omega^{(i)})) < t\}) \geq$$

$$\geq P(\{\omega \in \overline{\Omega} \,|\, [\sum_{i=1}^{N} p_i^{\omega} (r_i)^q l_q^q(\mu(\omega^{(i)}), \nu(\omega^{(i)}))]^{\frac{1}{q}} < t\}) \geq$$

$$\geq P(\{\omega \in \overline{\Omega} \,|\, [\lambda_q l_q^q(\mu(\omega), \nu(\omega))]^{\frac{1}{q}} < t\}) = F_{\mu,\nu}(\frac{t}{\lambda_q^{\frac{1}{q}}})$$

for all $t > 0$. In case $0 < q < 1$, one replaces l_q^q everywhere by l_q. Thus $\mathbf{S}$ is a contraction map with ratio $\lambda_q^{\frac{1}{q} \wedge 1}$. We can apply Corollary 3.1 for $r = \lambda_q^{\frac{1}{q} \wedge 1}$. If μ^* is the unique fixed point of $\mathbf{S}$ and $\mu_0 \in M_q$ then

$$F_{\mathbf{S}^n \mu_0, \mu^*}(t) = P(\{\omega \in \overline{\Omega} \,|\, l_q(\mathbf{S}^n \mu_0, \mu^*) < t\}) \geq$$

$$\geq P(\{\omega \in \overline{\Omega} \,|\, \frac{\lambda_q^{\frac{n}{q}}}{1 - \lambda_q^{\frac{1}{q}}} l_q(\mu_0, \mathbf{S}\mu_0) < t\}).$$

and

$$\lim_{n \to \infty} F_{\mathbf{S}^n \mu_0, \mu^*}(t) = 1 \,\text{for all}\; t > 0.$$

From $\mu_{n+1}(\omega) = \mathbf{S}\mu_n(\omega)$ it follows that $\mu_m \to \mu^*$exponentially fast. Moreover, for $q \geq 1$

$$\sum_{i=1}^{\infty} \overline{P}(l_q^q(\mathbf{S}^n \nu_0, \mu^*) \geq \epsilon) \leq \sum_{i=1}^{\infty} \frac{el_q^q(\mathbf{S}^n \mu_0, \mu^*)}{\epsilon} \leq c \sum_{i=1}^{\infty} \frac{\lambda_q^n}{\epsilon} < \infty.$$

This implies by Borel Catelli lemma that $l_q(\mu_n, \mu^*) \to 0$ a.s.

For the uniqueness let $\mathcal{D}$ the set of probability distribution of members of $\mathcal{E}_q$. We define on $\mathcal{D}$ the probability metric by

$$F_{\mathcal{A},\mathcal{B}}(t) = \sup_{s<t} \sup\{F_{\mu,\nu}(s) | \, \mu \stackrel{d}{=} \mathcal{A}, \nu \stackrel{d}{=} \mathcal{B}\}.$$

To establish the contraction property, let $\mathcal{A}, \mathcal{B} \in \mathcal{D}$. For $q \geq 1$, on has

$$F_{\mathcal{SA},\mathcal{SB}}(t) = \sup_{s<t} \sup\{F_{\mathbf{S}\mu,\mathbf{S}\nu}(s) | \, \mu \stackrel{d}{=} \mathcal{A}, \nu \stackrel{d}{=} \mathcal{B}\} \geq$$

$$\geq \sup_{s<t} \sup\{F_{\mu,\nu}(\frac{s}{\lambda_q}) | \, \mu \stackrel{d}{=} \mathcal{A}, \nu \stackrel{d}{=} \mathcal{B}\} = F_{\mathcal{A},\mathcal{B}}(\frac{t}{\lambda_q})$$

for all $t > 0$. In case $0 < q < 1$ on work similarly.

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ such that $\mathcal{SD}_1 = \mathcal{D}_1$ and $\mathcal{SD}_2 = \mathcal{D}_2$.

Since $\mathcal{D}_1 = \mathcal{S}^n(\mathcal{D}_1)$ and $\mathcal{D}_2 = \mathcal{S}^n(\mathcal{D}_2)$ we have

$$F_{\mathcal{D}_1,\mathcal{D}_2}(t) \geq F_{\mathcal{D}_1,\mathcal{D}_2}(\frac{t}{r^n})$$

for all $t > 0$. Using $\lim_{n\to\infty} r^n = 0$ it follows that

$$F_{\mathcal{D}_1,\mathcal{D}_2}(t) = 1,$$

for all $t > 0$. $\square$

Remark: Since $\lambda_q^{\frac{1}{q}} \to max_i r_i$ as $q \to \infty$, we can regard Theorem 3.1. from [12] as a limit case of Theorem 2.2. More precisely, if $max_i r_i < 1$ then $sprt\mu^*$ is the unique compact set satisfying $(S_1, ..., S_N)$.

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