STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **XLIX**, Number 4, December 2004

FRACTIONAL BROWNIAN MOTION USING CONTRACTION METHOD IN PROBABILISTIC METRIC SPACES

A. SOÓS

Abstract. In this paper we show how the random scaling law can be generalized such that the fractional Brownian motion satisfies it. Using the contraction method in probabilistic metric spaces, we give existence and uniqueness conditions for fractional Brownian motion.

The fractional Brownian motion (fBm) has been introduced in 1968 by Mandelbrot and Van Ness. For any H in $[0,1]$ we denote by $\{B_t^H : t \in [0,1]\}$ the fractional Brownian motion of index H (Hurst parameter), and it is the centered Gaussian process whose covariance kernel is given by

$$R_H(s,t) = E(B_s^H B_t^H) := \frac{V_H}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right),$$

where

$$V_H := \frac{\Gamma(2-2H)cos(\pi H)}{\pi H(1-2H)}.$$

The theoretical study of the fractional Brownian motion was originally motivated by new problems in mathematical finance and telecommunication networks. In engineering applications of stochastic processes it is often used to model the input of system. These real inputs exhibit long-range dependence: the behavior of a real process after a given time t does not only depend on the situation at time t but also on the whole history of the process up to time t.

Another property of the fBm encountered in applications is the self similarity: the behavior of fBm is stochastically the same, up to a space-scaling, i.e. the process

Received by the editors: 15.01.2005.

2000 *Mathematics Subject Classification.* 60G57, 28A80, 60G18.

Key words and phrases. Fractal, invariant set, scaling law, probabilistic metric space, fractional Brownian motion.

$\{X_{\alpha t}, t \in [0,1]\}$ has the same law as the process $\{\alpha^H X_t, t \in [0,1]\}$, where $H \in]0,1[$ and $\alpha > 0$.

Since R_H is a positive definite operator, the Bochner-Milos theorem ensures that, for any value of $H \in [0,1]$, there exists a unique probability measure on $C_0([0,1];\mathbb{R})$ such that the canonical process is a fBm.

Using fractal theory methods Hutchinson and Rüschendorf [2] have obtained the classical Brownian motion ($H = \frac{1}{2}$) as the invariant set for an iterated function system.

A first theory of selfsimilar fractal sets and measures was developed in Hutchinson [1]. Falconer, Graf, Mouldin and Williams, and Arbeiter randomized each step in the approximation process to obtain self-similar random fractal sets and measures. Recently Hutchinson and Rüschendorf [3] gave a simple proof for the existence and uniqueness of random fractal sets, measures and fractal functions using probability metrics defined by expectation.

In this paper we use probabilistic metric spaces techniques in order to prove that the fBm can be characterized as the fixed point of a scaling law.

1. **Invariant sets in E-spaces**

Let X be a nonempty set. We denote by Δ^+ denote the set of all distribution functions F with $F(0) = 0$. A *Menger space* is a triplet $(X, \mathcal{F}, T)$, where $\mathcal{F} : X \times X \to \Delta^+$ is a mapping with the following properties:

1^0. $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X$ and $t \in \mathbb{R}$;

2^0. $F_{x,y}(t) = 1$, for every $t > 0$, if and only if $x = y$;

3^0. $F_{x,y}(s+t) \geq T(F_{x,z}(s), F_{z,y}(t))$ for all $x, y, z \in X$ and $s, t \in \mathbb{R}_+$,

and T is a t-norm.

A mapping $T : [0,1] \times [0,1] \to [0,1]$ is called a *t-norm* if the following conditions are satisfied:

4^0. $T(a, 1) = a$ for every $a \in [0,1]$;

5^0. $T(a,b) = T(b,a)$ for every $a, b \in [0,1]$

6^0. if $a \geq c$ and $b \geq d$ then $T(a,b) \geq T(c,d)$;

7^0. $T(a, T(b, c)) = T(T(a, b), c)$ for every $a, b, c \in [0, 1]$.

The mapping $f : X \to X$ is said to be a *contraction* if there exists $r \in]0, 1[$ such that

$$F_{f(x),f(y)}(rt) \geq F_{x,y}(t)$$

for every $x, y \in X$ and $t \in \mathbb{R}_+$.

A sequence $(x_n)_{n\in\mathbb{N}}$ from X is said to be *Cauchy* if

$$\lim_{n,m\to\infty} F_{x_m,x_n}(t) = 1 \,\text{for all}\, t > 0.$$

The element $x \in X$ is called *limit* of the sequence $(x_n)_{n\in\mathbf{N}}$ if $\lim_{n\to\infty} F_{x,x_n}(t) = 1$ for all $t > 0$. A probabilistic metric (Menger) space is said to be *complete* if every Cauchy sequence in this space is convergent.

The notion of *E-space* was introduced by Sherwood [7] in 1969. Let $(\Omega, \mathcal{K}, P)$ be a probability space and let (Y, ρ) be a metric space. The ordered pair $(\mathcal{E}, \mathcal{F})$ is an *E-space over the metric space* (Y, ρ) if the elements of $\mathcal{E}$ are random variables from Ω into Y and $\mathcal{F} : \mathcal{E} \times \mathcal{E} \to \Delta^+$ defined by $\mathcal{F}(x, y) = F_{x,y}$, where

$$F_{x,y}(t) = P(\{\omega \in \Omega |\; d(x(\omega), y(\omega)) < t\})$$

for every $t \in \mathbb{R}$. The E-space $(\mathcal{E}, \mathcal{F})$ is said to be complete if the Menger space $(\mathcal{E}, \mathcal{F}, T_m)$ is complete, where $T_m(x, y) = \max\{x + y - 1, 0\}$.

In the sequel we will use the following result proved in [4]:

Theorem 1.1. *Let $(\mathcal{E}, \mathcal{F})$ be a complete E- space, $N \in \mathbb{N}^*$, and let $f_1, ..., f_N : \mathcal{E} \to \mathcal{E}$ be contractions with ratio $r_1, ...r_N$, respectively. Suppose that there exists an element $z \in \mathcal{E}$ and a real number γ such that*

$$P(\{\omega \in \Omega | \rho(z(\omega), f_i(z(\omega)) \geq t\}) \leq \frac{\gamma}{t}, \tag{1}$$

for all $i \in \{1, .., N\}$ and for all $t > 0$. Then there exists a unique nonempty closed bounded and compact subset K of $\mathcal{E}$ such that

$$f_1(K) \cup ... \cup f_N(K) = K.$$

Corollary 1.1. *Let $(\mathcal{E},\mathcal{F})$ be a complete E- space, and let $f:\mathcal{E}\to\mathcal{E}$ be a contraction with ratio r. Suppose that there exists $z\in\mathcal{E}$ and a real number γ such that*

$$P(\{\omega\in\Omega|\ \rho(z(\omega),f(z)(\omega))\geq t\})\leq\frac{\gamma}{t}\ \textit{for all}\ t>0.$$

Then there exists a unique $x_0\in\mathcal{E}$ such that $f(x_0)=x_0$.

2. **Scaling law and Brownian motion**

Denote by (X,d) a complete separable metric space. Let $g:I\to X$, where $I\subset\mathbb{R}$ is a closed bounded interval, $N\in\mathbb{N}$ and let $I=I_1\cup I_2\cup\cdots\cup I_N$ be a partition of I into disjoint subintervals. Let $\Phi_i:I\to I_i$ be increasing Lipschitz maps with $p_i=Lip\Phi_i$. If $g_i:I_i\to X$, for $i\in\{1,...,N\}$ define $\sqcup_i g_i:I\to X$ by

$$(\sqcup_i g_i)(x)=g_j(x)\quad\text{for}\quad x\in I_j.$$

A *scaling law* $\mathbb{S}$ is an N-tuple $(S_1,....,S_N)$, $N\geq 2$, of Lipschitz maps $S_i:X\to X$. Denote $r_i=LipS_i$.

A random scaling law $\mathbb{S}=(S_1,S_2,...,S_N)$ is a random variable whose values are scaling laws. We write $\mathcal{S}=dist\mathbb{S}$ for the probability distribution determined by $\mathbb{S}$ and $\overset{d}{=}$ for the equality in distribution.

Let $\mathbb{S}=(S_1,...,S_N)$ be a random scaling law and let $G=(G_t)_{t\in I}$ be a stochastic process or a random function with state space X. The trajectory of the process G is the function $g:I\to X$. The trajectory of the random function $\mathbb{S}g$ is defined up to probability distribution by

$$\mathbb{S}g\overset{d}{=}\sqcup_i S_i\circ g^{(i)}\circ\Phi_i^{-1},$$

where $\mathbb{S},g^{(1)},...,g^{(N)}$ are pairwise independent and $g^{(i)}\overset{d}{=}g$, for $i\in\{1,...,N\}$. We say that g or $\mathcal{G}$ *satisfies the scaling law* $\mathbb{S}$, or is a *random fractal function*, if

$$\mathbb{S}g\overset{d}{=}g,$$

The fBm can be characterized as the fixed point of a scaling law. Next we will contruct this scaling law.

Let $(\Omega, \mathcal{K}, P)$ be a probability space. The fBm with Hurst exponent H is a stochastic process $B^\alpha = (B^\alpha_t)_{t\in\mathbb{R}}$ characterised by $B^H_0(\omega) = 0$ a.s. and

$$B^H(t+h) - B^H(t) \overset{d}{=} N(0, h^H), \quad \text{for} \quad t > 0 \,\text{and}\, h > 0,$$

where $N(0, h^H)$ denotes the normal distribution with mean 0 and variance h^{2H}.

For each $H > 0$, let $B^H : [0,1] \to \mathbb{R}$ denote the *constrained fBm* given by

$$B^H(0) = 0 \; a.s. \quad \text{and} \quad B^H(1) = 1 \; a.s.$$

For a fixed $p \in \mathbb{R}$ consider the fBm $B^H \Big|_{B^H(\frac{1}{2})=p}$ constrained by $B^H(\frac{1}{2}) = p$.

Let $S_1, S_2 : \mathbb{R} \to \mathbb{R}$ be the unique affine transformations characterized by $S_1(0) = 0$, $S_1(1) = S_2(0) = p$, $S_2(1) = 1$. If $r_1 = LipS_1 = |p|, \quad r_2 = LipS_2 = |1-p|$, then

$$B^H|_{B^H(\frac{1}{2})=p}(t) \overset{d}{=} S_1 \circ B^{\frac{H}{2r_1^2}}(2t), \quad t \in [0, \frac{1}{2}].$$

Similarly

$$B^H|_{B^H(\frac{1}{2})=p}(t) \overset{d}{=} S_2 \circ B^{\frac{H}{2r_1^2}}(2t-1), \quad t \in [\frac{1}{2}, 1].$$

Let $I = [0,1]$, and define

$$\Phi_1 : I \to [0, \frac{1}{2}], \quad \Phi_1(s) = \frac{s}{2}, \, \text{and} \Phi_2 : I \to [\frac{1}{2}, 1], \quad \Phi_1(s) = \frac{s+1}{2}.$$

It follows that

$$B^H|_{B^H(\frac{1}{2})}(t) \overset{d}{=} \sqcup_i S_i \circ B^{\frac{H}{2r_i^2}} \circ \Phi_i^{-1}(t), \quad t \in [0,1].$$

Now let p^H be a random variable with distribution $N(0, \frac{H}{2})$. For each $H > 0$ let us define the random scaling law $\mathbb{S}^H = (S^H_1, S^H_2)$ in the same manner that (S_1, S_2) was previously defined from the point p.

Let $r^H_i = Lip^H_i$ for $i = 1, 2$ and let $r^\alpha = \max\{r^H_1, r^H_2\}$. It follows for each $H > 0$ that

$$B^H \overset{d}{=} \sqcup_i S^H_i \circ B^{\frac{H}{2r_i^2}(i)} \circ \Phi_i^{-1},$$

where $\mathbb{S}$ is first chosen as above, and then after conditioning on $\mathbb{S}$, $B^{\frac{H}{2r_1^2}(1)} \overset{d}{=} B^{\frac{H}{2r_1^2}}$ and $B^{\frac{H}{2r_2^2}(2)} \overset{d}{=} B^{\frac{H}{2r_2^2}}$ are chosen independently of one another.

Thus the family of constrained Brownian motion $\{B^H|H>0\}$ satisfies the family of scaling laws $\mathbb{S}=\{\mathbb{S}^H|H>0\}$.

3. **Generalized scaling law**

In this section we generelize the notion of random scaling law. Let p^H be a random variable in $\mathbb{R}$ with distribution $N(0,\frac{H}{2})$ and denote $I=[a,b]$. Let $S_1^H, S_2^H:\mathbb{R}\to\mathbb{R}$ be the unique affine transformations characterized by $S_1^H(a)=a$, $S_1^H(b)=S_2^H(a)=p^H$, $S_2H(b)=b$. Let $\Phi_i: I\to I_i,\quad i=1,2$ be increasing Lipschitz maps, such that $I_1\cup I_2=I$ and $\overset{\circ}{I}_1\cap\overset{\circ}{I}_2=\emptyset$.

The *generalized random scaling law* is a family of scaling laws

$$\mathbb{S}=\{\mathbb{S}^H|H>0\}.$$

If $f^{\omega,H}(t)=f^{\omega}(H,t)\ :]0,\infty[\times I\to\mathbb{R}$ is a stochastic process, then the stochastic process $(\mathbb{S}f)^H$ is defined up to probability distribution by

$$(\mathbb{S}f)^H\overset{d}{=}\sqcup_i S_i^H\circ f^{\frac{H}{2r_i^2}(i)}\circ\Phi_i^{-1},$$

where $\mathbb{S}$ is first chosen as before, and then after conditioning on $\mathbb{S}$, $f^{\frac{H}{2r_1^2}(1)}\overset{d}{=}f^{\frac{H}{2r_1^2}}$ and $f^{\frac{H}{2r_2^2}(2)}\overset{d}{=}f^{\frac{H}{2r_2^2}}$ are chosen independently of one another.

The family of stochastic processes or random functions f^H *satisfies the generalized scaling law* $\mathbb{S}$ or is a *fractal stochastic process* if

$$(\mathbb{S}f)^H\overset{d}{=}f^H.$$

Theorem 3.1. *Denote by $\mathcal{E}^H$ the set of random functions $g^H:\Omega\times I\to\mathbb{R}$ with the following property: there exist $h^H\in\mathcal{E}^H$ and a positive number γ such that*

$$P(\{\omega\in\Omega|\sup_H H^{-\frac{1}{2}}\int_I|h^H(x)|dx\geq t\})\leq\frac{\gamma}{t}$$

for all $t>0$.

Then there exists a family of stochastic processes $g*\in\mathcal{E}^H$ satisfying $\mathbb{S}$.

Proof. Let $f:\mathcal{E}^\alpha\to\mathcal{E}^\alpha$ defined by

$$f(g^H)=(\mathbb{S}g)^H=\sqcup_i S_i^H\circ g^{\frac{H}{2r_i^2}(i)}\circ\Phi_i^{-1},$$

where $\mathbb{S}$ is first chosen as in the previous section, and then after conditioning on $\mathbb{S}$, $g^{\frac{H}{2r_i^2}(i)} \overset{d}{=} g^{\frac{H}{2r_i^2}}$, $i = 1, 2$ are chosen independently of one another.

We first claim that, if $g^H \in \mathcal{E}^H$ then $f(g^H) \in \mathcal{E}^H$ as well. For this, choose $g^{\frac{H}{2r_i^2}(i)} \overset{d}{=} g^{\frac{H}{2r_i^2}}$, $i = 1, 2$, independently of one another and $\mathbb{S}^H = (S_1^H, S_2^H)$. Then, for $t > 0$,

$$P(\{\omega \in \Omega | \sup_H H^{-\frac{1}{2}} \int_I |(\mathbb{S}h)^H(x)| dx \geq t\}) \leq$$

$$\leq P(\{\omega \in \Omega | \frac{1}{2} \sup_H H^{-\frac{1}{2}} \sum_{i=1}^{2} r_i^H \int_{I_i} |h^{\frac{\alpha}{2(r_i^\alpha)^2}(i)}(x)| dx \geq t\}) \leq \frac{\gamma\sqrt{2}}{t}.$$

To establish the contraction property let us consider $g_1^H, g_2^H \in \mathcal{E}^H$. Since

$$F_{f(g_1^H), f(g_2^H)}(t) \geq F_{g_1^H, g_2^H}(\frac{t}{\sqrt{2}})$$

for all $t > 0$, f is a contraction. Then we can apply Corollary 1.1 and existence and uniqueness follows. □

References

[1] J. E. Hutchinson, *Fractals and Self Similarity*, Indiana University Mathematics Journal, 30(1981), no. 5, 713-747.

[2] J. E. Hutchinson, L. Rüschendorf, *Selfsimilar Fractals and Selfsimilar Random Fractals*, Progress in Probability, 46(2000), 109-124.

[3] J. E. Hutchinson, L. Rüschendorf, *Random fractals and probability metrics*, Advances of Appl. Probab. 32(2000), 925-947.

[4] J. Kolumbán, A. Soós, *Invariant sets of random variables in complete metric spaces*, Studia Univ. "Babes-Bolyai", Mathematica, 44, 4(2001), (accepted).

[5] B. Maldelbrot, J. Van Ness, *Fractional Brownian motions*, Fractional Noises and Applications, SIAM Review 10(1968), no. 4, 422-437.

[6] B. Schweizer, A. Sklar, *Probabilistic Mertic Spaces*, North Holland, New-York, Amsterdam, Oxford, 1983.

[7] H. Sherwood, *E-spaces and their relation to other classes of probabilistic metric spaces*, J. London Math. Soc., 44(1969), 441-448.

Babeş-Bolyai University, Faculty of Mathematics and Informatics
400084 Cluj-Napoca, str. M. Kogălniceanu, nr. 1
E-mail address: asoos@math.ubbcluj.ro