Wavelet Approximation of the Solutions of Some Stochastic Differential Equations

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Abstract

The aim of this paper is to approximate a stochastic integral with respect to a fractional Brownian motion using wavelet approximation and fractional integration. The approximation of the stochastic integral is illustrated through some examples.

Keywords: Stochastic differential equations, wavelets, approximation, fractional Brownian motion, fractional integral.

1 Introduction

The aim of this paper is to approximate a stochastic integral of the type $\int_0^t S(u)dB(u)$, where $(B(t))_{t\in[0,1]}$ is a fractional Brownian motion and S is a stochastic process. For this we use an optimal wavelet approximation of the fractional Brownian motion $(B(t))_{t\in[0,1]}$ following the ideas from [8], [2], [1]. According to the papers [9] and [10] of M. Zähle we can represent the stochastic integral with respect to fractional Brownian motion by using fractional integrals. The new achievements of this paper are the results contained in

Theorem 4.1 and in the approximation results stated in Corollary 5.1, Corollary 5.2 and Corollary 5.3. In order to illustrate our approach we give as an example the Ornstein-Uhlenbeck process. Another example is the solution of a linear stochastic equation, which is known to have explicit solution. The approximation method can be applied also for stochastic differential equations driven by fractional noise, which do not have explicit solution (see Corollary 5.3). The authors use optimal wavelet approximation in order to develop efficient computer simulations, because the method given in this paper can be applied to simulate the solutions of stochastic differential equations driven by fractional noise as follows: first one approximates the fractional noise, then one uses a numerical scheme (e.g. implicit or explicit Euler scheme, or Milstein approximation etc.) to get approximations of the solution and finally one proves the a.s. convergence of the approximations to the solution (as in [7], where a trigonometric series approximation for the fractional Brownian motion is used).

2 Wavelet Approximation for B

Let $(B(t))_{t\in[0,1]}$ be an one dimensional fractional Brownian motion with Hurst index $H \in (0,1)$, i.e. a Gaussian random process, which has zero mean, continuous sample paths and covariance function

$$E\Big(B(s)B(t)\Big) = \frac{1}{2}\Big(t^{2H} + s^{2H} - |s-t|^{2H}\Big).$$

Note that if $H = \frac{1}{2}$, then the fractional Brownian motion is the ordinary standard Brownian motion. The fractional Brownian motion B has on any finite interval $[0,T]$ Hölder continuous paths with exponent $\gamma \in (0,H)$ (see [3]). Note that, if $H \neq \frac{1}{2}$, then B is not a semimartingale, so the classical stochastic integration does not work. But the Hölder continuity of B will ensure the existence of the integrals $\int_0^T S(u)dB(u)$, defined in terms of fractional integration (see Section 3).

We use the following optimal wavelet approximation of the fractional Brownian motion $(B(t))_{t\in[0,1]}$ with Hurst index H investigated in [8] and [2]:

$$B(t) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} 2^{-jH}(\Psi(2^j t - k) - \Psi(-k))\epsilon_{j,k}, \tag{1}$$

where Ψ is the mother function of the wavelets approximation, and $\epsilon_{j,k}$ are independent identically distributed $N(0,1)$ random variables.

As in [8] and [2] we consider the following assumptions for Ψ: $\Psi \in C^1(\mathbb{R})$ and there exists a constant $c > 0$ such that

$$|\Psi(t)| \leq \frac{c}{(2+|t|)^2} \text{ and } |\Psi'(t)| \leq \frac{c}{(2+|t|)^3} \text{ for all } t \in \mathbb{R}. \tag{2}$$

We consider the following *high frequency component* of the wavelet representation in (1)

$$V_1(t) = \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} 2^{-jH}(\Psi(2^j t - k) - \Psi(-k))\epsilon_{j,k}$$

and the *low frequency component*

$$V_2(t) = \sum_{j=-\infty}^{-1} \sum_{k=-\infty}^{\infty} 2^{-jH}(\Psi(2^j t - k) - \Psi(-k))\epsilon_{j,k}.$$

Obviously,

$$B(t) = V_1(t) + V_2(t) \text{ for each } t \in [0,1].$$

Let $N \in \mathbb{N}$. In the following we use two approximation components, corresponding to the components V_1, respectively V_2, namely

$$B_1^N(t) = \sum_{j=0}^{N} \sum_{|k| \leq \frac{2^{N+4}}{(N-j+1)^2}} 2^{-jH}(\Psi(2^j t - k) - \Psi(-k))\epsilon_{j,k}$$

and

$$B_2^N(t) = \sum_{j=-2^{[N/2]}}^{-1} \sum_{|k| \leq 2^{[N/2]}} 2^{-jH}(\Psi(2^j t - k) - \Psi(-k))\epsilon_{j,k}.$$

We denote

$$B^N(t) = B_1^N(t) + B_2^N(t) \text{ for each } t \in [0,1].$$

Using Theorem 2 and Theorem 3 from [2] we have the following result:

Theorem 2.1 *The sequence $(B^N)_{N\in\mathbb{N}}$ converges to B almost surely in $\omega \in \Omega$ and uniformly in $t \in [0,1]$, i.e.*

$$\mathbb{P}\Big(\lim_{N\to\infty} \sup_{t\in[0,1]} |B^N(t) - B(t)| = 0\Big) = 1.$$

In the sequel we need the following result:

Theorem 2.2 *For all $N \in \mathbb{N}$ the approximating processes $(B^N(t))_{t\in[0,1]}$ are with probability* 1 *Lipschitz continuous.*

Proof: We write

$$\begin{aligned} |B^N(s) - B^N(t)| &\leq |B_1^N(s) - B_1^N(t)| + |B_2^N(s) - B_2^N(t)| \\ &\leq \sum_{j=0}^{N} \sum_{|k|\leq \frac{2^{N+4}}{(N-j+1)^2}} 2^{-jH} |\Psi(2^j s - k) - \Psi(2^j t - k))||\epsilon_{j,k}| \\ &\quad + \sum_{j=-2^{[N/2]}}^{-1} \sum_{|k|\leq 2^{[N/2]}} 2^{-jH} |\Psi(2^j s - k) - \Psi(2^j t - k)||\epsilon_{j,k}|. \end{aligned}$$

Using the assumption (2) for Ψ and using that the set of indices of j and k is bounded, it follows that there exists a $c_N > 0$ (depending on ω) such that

$$|B^N(s) - B^N(t)| \leq c_N |s-t| \text{ for all } s,t \in [0,1] \text{ and all } n \in \mathbb{N}.$$

■

3 Fractional Integrals and Derivatives

Let $a, b \in \mathbb{R}$, $a < b$ and $f, g : \mathbb{R} \to \mathbb{R}$. We use notions and results about fractional calculus, from [9] and [4]:

$$f_{a+}(x) = \mathbb{I}_{(a,b)}(f(x) - f(a+)), \quad g_{b-}(x) = \mathbb{I}_{(a,b)}(g(x) - g(b-)).$$

For $f \in L_1(a,b)$ and $\alpha > 0$ the *left-* and the *right-sided fractional Riemann-Liouville integral of f of order α* on (a,b) is given for a.e. $x \in (a,b)$ by

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y) dy$$

and

$$I_{b-}^{\alpha} f(x) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y) dy.$$

For $p > 1$ let $I^\alpha_{a+}(L_p(a,b))$, be the class of functions f which have the representation $f = I^\alpha_{a+}\Phi$, where $\Phi \in L_p(a,b)$, and let $I^\alpha_{b-}(L_p(a,b))$ be the class of functions g which have the representation $g = I^\alpha_{b-}\varphi$, where $\varphi \in L_p(a,b)$. If $0 < \alpha < 1$, then the function Φ, respectively φ, in the representations above agree a.s. with the *left-sided* and respectively *right-sided fractional derivative of f of order α* (in the Weyl representation)

$$\Phi(x) = D^\alpha_{a+}f(x) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(x)}{(x-a)^\alpha} + \alpha\int_a^x \frac{f(x)-f(y)}{(x-y)^{\alpha+1}}dy\right)\mathbb{I}_{(a,b)}(x)$$

and

$$\varphi(x) = D^\alpha_{b-}g(x) = \frac{(-1)^\alpha}{\Gamma(1-\alpha)}\left(\frac{g(x)}{(b-x)^\alpha} + \alpha\int_x^b \frac{g(x)-g(y)}{(y-x)^{\alpha+1}}dy\right)\mathbb{I}_{(a,b)}(x).$$

The convergence at the singularity $y = x$ holds in the L_p-sense. Recall that

$$I^\alpha_{a+}(D^\alpha_{a+}f) = f \text{ for } f \in I^\alpha_{a+}(L_p(a,b)), \quad I^\alpha_{b-}(D^\alpha_{b-}g) = g \text{ for } g \in I^\alpha_{b-}(L_p(a,b)) \tag{3}$$

and

$$D^\alpha_{a+}(I^\alpha_{a+}f) = f, \quad D^\alpha_{b-}(I^\alpha_{b-}g) = g \text{ for } f,g \in L_1(a,b).$$

For completeness we denote

$$D^0_{a+}f(x) = f(x), D^0_{b-}g(x) = g(x), D^1_{a+}f(x) = f'(x), D^1_{b-}g(x) = g'(x).$$

Let $0 \le \alpha \le 1$. The *fractional integral* of f with respect to g is defined as

$$\begin{aligned}\int_a^b f(x)dg(x) &= (-1)^\alpha \int_a^b D^\alpha_{a+}f_{a+}(x)D^{1-\alpha}_{b-}g_{b-}(x)dx \\ &\quad + f(a+)(g(b-)-g(a+))\end{aligned} \tag{4}$$

if $f_{a+} \in I^\alpha_{a+}(L_p(a,b)), g_{b-} \in I^{1-\alpha}_{b-}(L_q(a,b))$ for $\frac{1}{p} + \frac{1}{q} \le 1$ (see [9]).

4 Approximation of the Stochastic Integral

Without loss of generality we consider fractional integrals over $[0,T]$, where $0 < T \le 1$, because for arbitrary $T > 0$ we can rescale the time variable such that we obtain a fractional Brownian motion on $[0,1]$ (we use the H-self similar property).

Since B has continuous trajectories and $B(0)=0$, it follows by (4) that

$$\int_0^T S(u)dB(u) = (-1)^\alpha \int_0^T D_{0+}^\alpha S_{0+}(u) D_{T-}^{1-\alpha} B_{T-}(u)du + S(0+)B(T) \tag{5}$$

for $S_{0+} \in I_{0+}^\alpha(L_2(0,T))$ and $B_{T-} \in I_{T-}^{1-\alpha}(L_2(0,T))$.

The condition $S_{0+} \in I_{0+}^\alpha(L_2(0,T))$ (with probability 1) means that $S_{0+} \in L_2(0,T)$ and

$$\mathcal{I}_\varepsilon(x) = \int_0^{x-\varepsilon} \frac{S(x)-S(y)}{(x-y)^{\alpha+1}} dy \text{ for } x \in (0,T)$$

converges in $L^2(0,T)$ as $\varepsilon \searrow 0$.

The condition $B_{T-} \in I_{T-}^{1-\alpha}(L_2(0,T))$ means $B_{T-} \in L_2(0,T)$ and

$$\mathcal{J}_\varepsilon(x) = \int_{x+\varepsilon}^{T} \frac{B(x)-B(y)}{(y-x)^{2-\alpha}} dy \text{ for } x \in (0,T)$$

converges in $L_2(0,T)$ as $\varepsilon \searrow 0$. This condition for B is fulfilled for $\alpha > 1-H$, since the Brownian motion B is a.s. Hölder continuous with exponent $\gamma \in (0,H)$.

It is easy to verify that

$$D_{T-}^{1-\alpha} B_{T-}(u) = D_{T-}^{1-\alpha} B(u) - \frac{(-1)^{1-\alpha}}{\Gamma(\alpha)} \cdot \frac{B(T)}{(T-u)^{1-\alpha}} \mathbb{I}_{(0,T)}(u).$$

Then using (3) and (5) it follows that

$$\int_0^T S(u)dB(u) = (-1)^\alpha \int_0^T D_{0+}^\alpha S_{0+}(u) D_{T-}^{1-\alpha} B(u)du + S(T-)B(T). \tag{6}$$

We will use (6) for the integrals with respect the approximating processes. Observe that $B^N \in I_{T-}^{1-\alpha}(L_2(0,T))$, which follows from the Lipschitz continuity property in Theorem 2.2. Then we write

$$\int_0^T S(u)dB^N(u) = (-1)^\alpha \int_0^T D_{0+}^\alpha S_{0+}(u) D_{T-}^{1-\alpha} B^N(u)du + S(T-)B^N(T). \tag{7}$$

We write the difference of the stochastic integrals from (6) and (7)

$$\int_0^T S(u)dB(u) - \int_0^T S(u)dB^N(u) = \tag{8}$$

$$= (-1)^\alpha \int_0^T D_{0+}^\alpha S_{0+}(u)\Big(D_{T-}^{1-\alpha}B(u) - D_{T-}^{1-\alpha}B^N(u)\Big)du$$

$$+S(T-)(B(T) - B^N(T)).$$

From the uniform convergence property in Theorem 2.1 we have for a.e. $\omega \in \Omega$

$$\lim_{N\to\infty} S(T-)(B(T) - B^N(T)) = 0.$$

Using the definition of fractional derivatives we write

$$\int_0^T D_{0+}^\alpha S_{0+}(u)\Big(D_{T-}^{1-\alpha}B(u) - D_{T-}^{1-\alpha}B^N(u)\Big)du =$$

$$= \int_0^T D_{0+}^\alpha S_{0+}(u)\frac{(-1)^{1-\alpha}}{\Gamma(\alpha)}\Bigg(\frac{B(u) - B^N(u)}{(T-u)^{1-\alpha}}$$

$$+(1-\alpha)\int_u^T \frac{B(u) - B^N(u) - (B(y) - B^N(y))}{(y-u)^{2-\alpha}}dy\Bigg)du.$$

Denote

$$F(u) = D_{0+}^\alpha S_{0+}(u)\frac{(-1)^{1-\alpha}}{\Gamma(\alpha)}.$$

We prove now that for a.e. $\omega \in \Omega$

$$\lim_{N\to\infty}\int_0^T F(u)\Bigg(\frac{B(u) - B^N(u)}{(T-u)^{1-\alpha}}$$

$$+(1-\alpha)\int_u^T \frac{B(u) - B^N(u) - (B(y) - B^N(y))}{(y-u)^{2-\alpha}}dy\Bigg)du = 0.$$

From the uniform convergence property in Theorem 2.1 we have

$$\left| F(u) \frac{B(u) - B^N(u)}{(T-u)^{1-\alpha}} \right| \leq \frac{|F(u)|}{(T-u)^{1-\alpha}} \varepsilon \tag{9}$$

for all $u \in [0, T)$ and N sufficiently large and $\varepsilon > 0$ arbitrary fixed. We assume that F (i.e. S) satisfies the condition

$$\int_0^T \frac{|F(u)|}{(T-u)^{1-\alpha}} du < \infty.$$

For example, if $\sup_{u \in [0,T]} |F(u)| < \infty$, then the condition above holds. Therefore by using Theorem 2.1 and the dominated convergence theorem we get

$$\lim_{N \to \infty} \int_0^T F(u) \frac{B(u) - B^N(u)}{(T-u)^{1-\alpha}} du = 0 \text{ for a.e. } \omega \in \Omega.$$

It is known that B is a.s. Hölder continuous with exponent $\gamma \in (0, H)$ (see [3]). By construction, the approximations B^N have the same property of Hölder continuity with exponent $\gamma \in (0, H)$ (moreover they are Lipschitz continuous, see Theorem 2.2). Therefore, the difference $B - B^N$ has the same property.

For $\alpha \in (1 - H, 1)$ and for $\gamma \in (0, H)$ chosen such that $1 - \alpha < \gamma$, there exists $\eta \in (0, 1)$ such that $\eta\gamma - 1 + \alpha > 0$. For a.e. $\omega \in \Omega$ we have

$$\frac{|B(u) - B^N(u) - (B(y) - B^N(y))|}{|y-u|^\gamma} \leq \delta \tag{10}$$

for every $u, y \in [0, T]$, where δ is a random variable with finite moments. We have by the uniform convergence property in Theorem 2.1 and by (10)

$$\begin{aligned} &\left| \frac{B(u) - B^N(u) - (B(y) - B^N(y))}{(y-u)^{2-\alpha}} \right| \leq \\ &\leq |B(u) - B^N(u) - (B(y) - B^N(y))|^{1-\eta} \\ &\quad \cdot \left| \frac{|B(u) - B^N(u) - (B(y) - B^N(y))|}{(y-u)^\gamma} \right|^\eta (y-u)^{\eta\gamma - 2 + \alpha} \leq \\ &\leq \varepsilon^{1-\eta} \delta^\eta (y-u)^{\eta\gamma - 2 + \alpha} \end{aligned}$$

for all $u, y \in [0,T]$ and N sufficiently large. The integral

$$\int_u^T (y-u)^{\eta\gamma-2+\alpha} dy = \frac{(T-u)^{\eta\gamma-1+\alpha}}{\eta\gamma-1+\alpha}$$

is finite, because $\eta\gamma - 1 + \alpha > 0$. Another condition imposed on F (i.e. on S) is

$$\int_0^T \frac{|F(u)|}{(T-u)^{1-\eta\gamma-\alpha}} du < \infty.$$

This condition is satisfied if

$$\int_0^T |F(u)| du < \infty \text{ and } \int_0^T \frac{|F(u)|}{(T-u)^{1-\alpha}} du < \infty.$$

Then for a.e. $\omega \in \Omega$ we have

$$\lim_{N\to\infty} \int_0^T F(u) \left(\int_u^T \frac{B(u) - B^N(u) - (B(y) - B^N(y))}{(y-u)^{2-\alpha}} dy \right) du = 0.$$

These results prove the main result of our paper:

Theorem 4.1 *Let $\alpha \in (1-H, 1)$. We assume that the stochastic process S satisfies a.s. the following conditions:* **(1)** $S_{0+} \in I^{\alpha}_{0+}(L_2(0,T))$*;* **(2)** $\int_0^T |D^{\alpha}_{0+} S_{0+}(u)| du < \infty$*;* **(3)** $\int_0^T \frac{|D^{\alpha}_{0+} S_{0+}(u)|}{(T-u)^{1-\alpha}} du < \infty$.
Then the following approximation holds with probability 1

$$\lim_{N\to\infty} \int_0^t S(u) dB^N(u) = \int_0^t S(u) dB(u) \quad \textit{for all} \quad t \in [0,T].$$

5 Applications

Let $(B(t))_{t\in[0,1]}$ be a fractional Brownian motion with Hurst index H and let $(B^N(t))_{t\in[0,1]}$ be the approximations defined in Section 2.

A first application we give for the Ornstein-Uhlenbeck process

$$X(t) = X_0 + a \int_0^t X(s)ds + B(t), \ t \in [0,1], \tag{11}$$

where $a \in \mathbb{R}$. The solution of (11) has the representation

$$X(t) = X_0 e^{at} + \int_0^t e^{a(t-s)} dB(s), \ t \in [0,1].$$

Then the process X can be approximated by

$$X^N(t) = X_0 e^{at} + \int_0^t e^{a(t-s)} dB^N(s), \ t \in [0,1], \ N \in \mathbb{N}$$

and Theorem 4.1 implies that the following result holds:

Corollary 5.1 *The sequence $(X^N)_{N \in \mathbb{N}}$ converges to X almost surely in ω and uniformly in t, i.e.*

$$\mathbb{P}\Big(\lim_{N \to \infty} \sup_{t \in [0,1]} |X(t) - X^N(t)| = 0 \Big) = 1.$$

Such kind of Ornstein-Uhlenbeck processes were used in [5] for the approximation of Volterra type stochastic integrals, used in statistical parameter estimation. The wavelet estimation used in the mentioned paper is based also on the results in [1] and [8].

A second application is the linear stochastic differential equation of the form

$$Y(t) = Y_0 + \int_0^t A(u)Y(u)du + \int_0^t S(u)Y(u)dB(u),$$

where A and S are stochastic processes, A is almost surely bounded and S satisfies the assumptions from Theorem 4.1. For example we can choose S such that with probability 1 it has Lipschitz continuous (or more general Hölder continuous with exponent less than H) trajectories, then condition (1) in Theorem 4.1 is satisfied. If

$$\mathbb{P}(\sup_{t \in [0,T]} |D^{\alpha}_{0+} S(t)| < \infty) = 1, \text{ with } \alpha > 1 - H,$$

then the conditions (2) and (3) in Theorem 4.1 are satisfied.

It is known that this equation has the following explicit solution (see [6])

$$Y(t) = Y_0 \exp\left\{ \int\limits_0^t \Big(A(u) - \frac{1}{2} S^2(u) \Big) du + \int\limits_0^t S(u) dB(u) \right\} \text{ for all } t \geq 0.$$

By the methods of the above section we approximate the stochastic integral by using fractional integration and consider

$$Y^N(t) = Y_0 \exp\left\{ \int\limits_0^t \Big(A(u) - \frac{1}{2} S^2(u) \Big) du + \int\limits_0^t S(u) dB^N(u) \right\} t \geq 0, N \in \mathbb{N}.$$

Theorem 4.1 implies the following result:

Corollary 5.2 *The sequence* $(Y^N)_{N\in\mathbb{N}}$ *converges to* Y *almost surely in* ω *and uniformly in* t*, i.e.*

$$\mathbb{P}\Big(\lim_{N\to\infty} \sup_{t\in[0,T]} |Y(t) - Y^N(t)| = 0 \Big) = 1.$$

This result can be used in numerical simulations of the solutions of stochastic differential equations from mathematical finance (e.g. long range dependencies in real stock market processes, [6]).

The approximation method can be applied also for stochastic differential equations driven by fractional noise, which do not have explicit solution. The method given in this paper is useful for simulations of the solutions of stochastic differential equations driven by fractional noise as follows: first one approximates the fractional noise, then one uses a numerical scheme to get approximations of the solution and finally one proves the a.s. convergence of the approximations to the solution (as in [7], where a trigonometric series approximation for the fractional Brownian motion is used).

We consider

$$Z(t) = Z_0 + \int\limits_0^t A(Z(s), s) dt + \int\limits_0^t S(Z(s), s) dB(s), t \geq 0, \tag{12}$$

where Z_0 is a random vector in $\mathbb{R}^n$ and the random functions A and S satisfy with probability 1 the following conditions:

(C1) $A \in C(\mathbb{R}^n \times [0,T], \mathbb{R}^n), S \in C^1(\mathbb{R}^n \times [0,T], \mathbb{R}^n)$;

(C2) for each $t \in [0,T]$ the functions $A(\cdot,t), \dfrac{\partial S(\cdot,t)}{\partial x^i}, \dfrac{\partial S(\cdot,t)}{\partial t}$ are locally Lipschitz for each $i \in \{1,\dots,n\}$.

Now we write the approximating equations

$$Z^N(t) = Z_0 + \int_0^t A(Z^N(s),s)ds + \int_0^t S(Z^N(s),s)dB^N(s), t \geq 0, \ N \in \mathbb{N}. \quad (13)$$

Following the ideas from Section 7 in [10] each of the equations (12) and (13) can be transformed (pathwise) into a random equation which has a unique local solution on a common (random) interval $(t_0, t_1) \subseteq [0,T]$ (this interval does not depend on N, see [7]). Estimating the difference $|Z(t) - Z^N(t)|$ and using the Gronwall lemma and Theorem 2.1 we obtain the following result:

Corollary 5.3 *The sequence $(Z^N)_{N\in\mathbb{N}}$ converges to Z almost surely in $\omega \in \Omega$ and uniformly in t, i.e.*

$$\mathbb{P}\Big(\lim_{N\to\infty} \sup_{t\in(t_0,t_1)} |Z(t) - Z^N(t)| = 0 \Big) = 1.$$

For details and proofs see the paper [7].

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