definition remark Example equationsection

Approximation of SDE Driven by Fractional Brownian Motion]Approximation of Stochastic Differential Equations Driven by Fractional Brownian Motion H. Lisei]Hannelore Lisei Babeş-Bolyai University
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Abstract

The aim of this paper is to approximate the solution of a stochastic differential equation driven by fractional Brownian motion using a series expansion for the noise. We prove that the solution of the approximating equations converge in probability to the solution of the given equation. We illustrate the approximation through an example from mathematical finance.

Primary 60H10; Secondary: 60H40, 60H05 Stochastic differential equations, approximation, fractional Brownian motion.

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1 Introduction

We consider the following stochastic differential equation driven by fractional Brownian motion

$$X(t) = X_0 + \int_0^t F(X(s), s)ds + \int_0^t G(X(s), s)dB(s),\ t \in [t_0]. \tag{1}$$

We assume that with probability 1 we have $F \in C(R^n \times [0,T], R^n), G \in C^1(R^n \times [0,T], R^n)$ and for each $t \in [0,T]$ the functions $F(\cdot, t), \dfrac{\partial G(\cdot,t)}{\partial x}, \dfrac{\partial G(\cdot,t)}{\partial t}$ are locally Lipschitz.

The fractional Brownian motion $\Big(B(t)\Big)_{t\in[0,1]}$ with Hurst index $H \in (0,1)$ we approximate using the series expansion given in [5]. Let J_ν be the Bessel function of first type of order ν and let $x_1 < x_2 < \dots$ be the positive, real zeros of J_{-H}, while $y_1 < y_2 < \dots$ are the positive, real zeros of J_{1-H}. We consider $(X_n)_{n\in N}$ and $(Y_n)_{n\in N}$ to be two independent sequences of centered Gaussian random variables such that for each $n \in N$ we have

$$\mathrm{Var} X_n = \frac{2c_H^2}{x_n^{2H} J_{1-H}^2(x_n)}, \quad \mathrm{Var} Y_n = \frac{2c_H^2}{y_n^{2H} J_{-H}^2(y_n)},$$

where

$$c_H^2 = \frac{\sin(\pi H)}{\pi}\Gamma(1+2H).$$

In [5] it is proved that a fractional Brownian motion $\Big(B(t)\Big)_{t\in[0,1]}$ with Hurst index $H \in (0,1)$ can be written as

$$B(t) = \sum_{n=1}^{\infty} \frac{\sin(x_n t)}{x_n} X_n + \sum_{n=1}^{\infty} \frac{1-\cos(y_n t)}{y_n} Y_n, \quad t \in [0,1].$$

The equation (1) we approximate for each $N \in N$ through

$$X_N(t) = X_0 + \int_0^t F(X_N(s), s)ds + \int_0^t G(X_N(s), s)dB_N(s), \tag{2}$$

$$\tag{3}$$

where

$$B_N(t) = \sum_{n=1}^{N} \frac{\sin(x_n t)}{x_n} X_n + \sum_{n=1}^{N} \frac{1-\cos(y_n t)}{y_n} Y_n, \quad t \in [0,1], N \in N.$$

We will show that the equation (2) has a local solution, which converges in probability to the solution of (1) in the interval, where the solutions exist. We illustrate the approximation through the model for the price of risky assets from mathematical finance. The figures are generated by a Matlab program.

Investigations concerning stochastic differential equations driven by a fractional Brownian motion or more general fractional process have been done by L. Coutin and L. Decreusefond [2], L. Coutin and Z. Qian [3], M.L. Kleptsyna, P.E. Kloeden and V.V. Anh [7], F. Klingenhöfer and M. Zähle [8], M. Zähle [15], [16], M. Errami and F. Russo [6] and many others. These studies were motivated by the problems occurring in mathematical finance, telecommunication networks, biology, hydrology etc. The main difficulty raised by the fractional Brownian motion and the processes related to it, is that they are not Markovian, even more, they are not semimartingales. Hence a new approach to stochastic fractional calculus was developed. There exist several ways to define the stochastic integral, pathwise and related techniques, Dirichlet forms, anticipating techniques using Malliavin calculus and Skorohod integration (e.g. [1], [14], [10],[4]). In this paper we use the approach of M. Zähle [14], based on the ideas of Lebesgue-Stieltjes integrals and fractional calculus [12].

2 Series Expansion for Fractional Brownian Motion B

A Gaussian random process $\Big(B(t)\Big)_{t\geq 0}$ is called **fractional Brownian motion with Hurst index** $H \in (0,1)$, if it has zero mean, continuous sample paths and covariance function

$$E\Big(B(s)B(t)\Big) = \frac{1}{2}\Big(t^{2H} + s^{2H} - |s-t|^{2H}\Big).$$

Note that if $H = \frac{1}{2}$, then the fractional Brownian motion is the ordinary standard Brownian motion.

The fractional Brownian motion B has on any finite interval $[0,T]$ Hölder continuous paths with exponent $\gamma \in (0,H)$ (see [4]). Moreover, the quadratic variation on $[a,b] \subseteq [0,T]$ is

$$\lim_{|\Delta_n|\to 0} \sum_{i=1}^{n} \Big(B(t_i^n) - B(t_{i-1}^n)\Big)^2 \quad = \quad \begin{cases} \infty & \text{if } \ H < \frac{1}{2}, \\ b-a & \text{if } \ H = \frac{1}{2}, \\ 0 & \text{if } \ H > \frac{1}{2}, \end{cases} \tag{4}$$

where $\Delta_n = (a = t_0^n < \ldots < t_n^n = b)$ is a partition of $[a,b]$ with $|\Delta_n| = \max\limits_{1\le i\le n}(t_i^n - t_{i-1}^n)$.

If $H \neq \frac{1}{2}$, then the convergence in (4) holds with probability 1 uniformly in the set of all partitions of $[a,b]$, while for $H = \frac{1}{2}$ the convergence in (4) holds in mean square uniformly in the set of all partitions of $[a,b]$. Note that, if $H \neq \frac{1}{2}$, then B is not a semimartingale, so the classical stochastic integration does not work. But the Hölder continuity of B will ensure the existence of integrals

$$\int_0^T G(u)dB(u),$$

defined in terms of fractional integration (see Section 4) as investigated in [14] and [16].

For $\nu \neq -1, -2, \ldots$ **the Bessel function J_ν of the first type of order ν** is defined on the region $\{z \in C : |\arg z| < \pi\}$ as the absolutely convergent sum

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{\nu+2k}.$$

It is known that for $\nu > -1$ the function J_ν has a countable number of real, positive simple zeros (see [13], Chapter 15). Let $x_1 < x_2 < \ldots$ be the positive, real zeros of J_{-H} and let $y_1 < y_2 < \ldots$ be the positive, real zeros of J_{1-H}.

Let $(X_n)_{n\in N}$ and $(Y_n)_{n\in N}$ be two independent sequences of independent Gaussian random variables such that for each $n \in N$ we have

$$E(X_n) = E(Y_n) = 0$$

and

$$\mathrm{Var} X_n = \frac{2c_H^2}{x_n^{2H} J_{1-H}^2(x_n)}, \quad \mathrm{Var} Y_n = \frac{2c_H^2}{y_n^{2H} J_{-H}^2(y_n)},$$

where

$$c_H^2 = \frac{\sin(\pi H)}{\pi}\Gamma(1+2H).$$

In [5] Theorem 4.5 it is proved that the random process $\Big(B(t)\Big)_{t\in[0,1]}$ given by

$$B(t) = \sum_{n=1}^{\infty} \frac{\sin(x_n t)}{x_n} X_n + \sum_{n=1}^{\infty} \frac{1-\cos(y_n t)}{y_n} Y_n, \quad t \in [0,1]$$

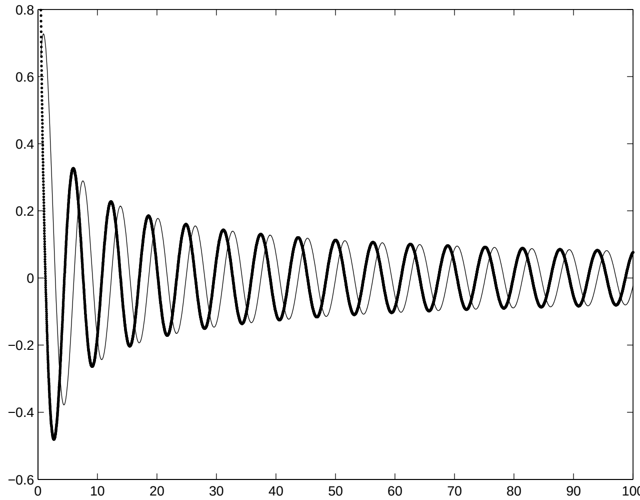
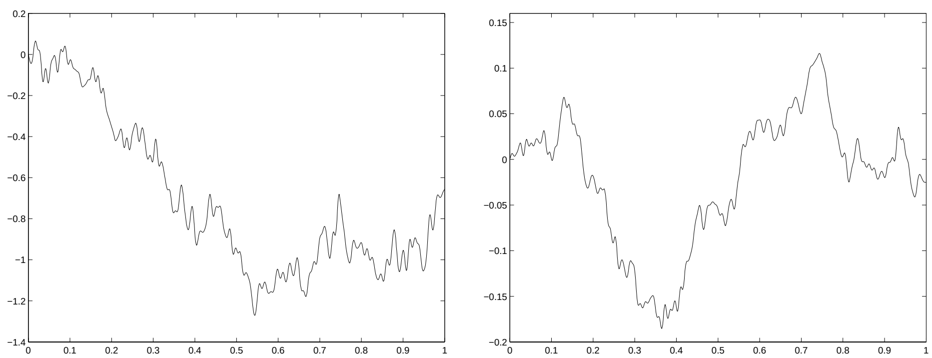


Figure 1: Bessel functions: J_{-H} (with '$\cdot$'), J_{1-H} (with '-'), $H = 0.65$



a) $H = 0.65$ b) $H = 0.90$

Figure 2: Approximation B_N of fractional Brownian motion

is well defined and both series converge absolutely and uniformly in $t \in [0, 1]$. The process B is a fractional Brownian motion with Hurst index H.

For each $N \in N$ we define the process

$$B_N(t) = \sum_{n=1}^{N} \frac{\sin(x_n t)}{x_n} X_n + \sum_{n=1}^{N} \frac{1 - \cos(y_n t)}{y_n} Y_n, \quad t \in [0, 1], \tag{5}$$

then using the above mentioned result from [5] we have

$$P(\lim_{N\to\infty} \sup_{t\in[0,1]} |B(t) - B_N(t)| = 0) = 1. \tag{6}$$

In the sequel we need the following result:

Theorem 2.1 *For all $N \in N$ the approximating processes $(B_N(t))_{t\in[0,1]}$ are with probability* 1 *Lipschitz continuous.*

Let $N \in N$ be fixed. We write

$$|B_N(t) - B_N(s)| \leq \sum_{n=1}^{N} \left| \frac{\sin(x_n t) - \sin(x_n s)}{x_n} X_n \right| + \sum_{n=1}^{N} \left| \frac{\cos(x_n s) - \cos(x_n t)}{x_n} Y_n \right|.$$

But the functions sin and cos are Lipschitz continuous, therefore

$$|B_N(t) - B_N(s)| \leq |t-s| \sum_{n=1}^{N} \Big(|X_n| + |Y_n| \Big) = C_N |t-s| forall s, t \in [0,1],$$

where $C_N = \sum_{n=1}^{N} \Big(|X_n| + |Y_n| \Big) < \infty$ is a random variable.

3 Fractional Integrals and Derivatives

Let $a, b \in R$, $a < b$ and $f, g : R \to R$. We use notions and results about fractional calculus, from [12] and [14]:

$$f(a+) := \lim_{\delta \searrow 0} f(a+\delta), \quad f(b-) := \lim_{\delta \searrow 0} f(b-\delta),$$

$$f_{a+}(x) = I_{(a,b)}(f(x) - f(a+)), \quad g_{b-}(x) = I_{(a,b)}(g(x) - g(b-)).$$

Note that for $\alpha > 0$ we have $(-1)^\alpha = e^{i\pi\alpha}$.

For $f \in L_1(a,b)$ and $\alpha > 0$ the **left-** and **right-sided fractional Rieman-Liouville integral of f of order** α on (a,b) is given for a.e. x by

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y) dy$$

and

$$I_{b-}^{\alpha} f(x) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y) dy.$$

For $p > 1$ let $I_{a+}^{\alpha}(L_p(a,b))$, be the class of functions f which have the representation $f = I_{a+}^{\alpha}\Phi$, where $\Phi \in L_p(a,b)$, and let $I_{b-}^{\alpha}(L_p(a,b))$ be the class of functions g which have the representation $g = I_{b-}^{\alpha}\varphi$, where $\varphi \in L_p(a,b)$. If $0 < \alpha < 1$, then the function Φ, respectively φ, in the representations above agree a.s. with the **left-sided** and respectively **right-sided fractional derivative of f of order** α (in the Weyl representation)

$$\Phi(x) = D_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^\alpha} + \alpha \int_a^x \frac{f(x) - f(y)}{(x-y)^{\alpha+1}} dy \right) I_{(a,b)}(x)$$

and

$$\varphi(x) = D_{b-}^{\alpha} g(x) = \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)} \left(\frac{g(x)}{(b-x)^{\alpha}} + \alpha \int_x^b \frac{g(x) - g(y)}{(y-x)^{\alpha+1}} dy \right) I_{(a,b)}(x).$$

The convergence at the singularity $y = x$ holds in the L_p-sense. Recall that

$$I_{a+}^{\alpha}(D_{a+}^{\alpha} f) = f \text{ for } f \in I_{a+}^{\alpha}(L_p(a,b)), \quad I_{b-}^{\alpha}(D_{b-}^{\alpha} g) = g \text{ for } g \in I_{b-}^{\alpha}(L_p(a,b))$$

and

$$D_{a+}^{\alpha}(I_{a+}^{\alpha} f) = f, \quad D_{b-}^{\alpha}(I_{b-}^{\alpha} g) = g \text{ for } f, g \in L_1(a,b).$$

For completeness we denote

$$D_{a+}^{0} f(x) = f(x), D_{b-}^{0} g(x) = g(x), D_{a+}^{1} f(x) = f'(x), D_{b-}^{1} g(x) = g'(x).$$

Let $0 \leq \alpha \leq 1$. The **fractional integral** of f with respect to g is defined as

$$\begin{aligned} \int_a^b f(x) dg(x) &= (-1)^{\alpha} \int_a^b D_{a+}^{\alpha} f_{a+}(x) D_{b-}^{1-\alpha} g_{b-}(x) dx \\ &\quad + f(a+)(g(b-) - g(a+)) \end{aligned} \tag{7}$$

if $f_{a+} \in I_{a+}^{\alpha}(L_p(a,b)), g_{b-} \in I_{b-}^{1-\alpha}(L_q(a,b))$ for $\frac{1}{p} + \frac{1}{q} \leq 1$.

In our investigations we will take $p = q = 2$. If $0 \leq \alpha < \frac{1}{2}$, then the integral in (7) can be written as

$$\int_a^b f(x) dg(x) = (-1)^{\alpha} \int_a^b D_{a+}^{\alpha} f(x) D_{b-}^{1-\alpha} g_{b-}(x) dx \tag{8}$$

if $f \in I_{a+}^{\alpha}(L_2(a,b))$, $f(a+)$ exists, $g_{b-} \in I_{b-}^{1-\alpha}(L_2(a,b))$ (see [14]).

4 The Stochastic Integral

Without loss of generality we consider $0 < T \leq 1$, because for arbitrary $T > 0$ we can rescale the time variable using the H-self similar property of the fractional Brownian motion meaning that $\Big(B(ct)\Big)_{t \geq 0}$ and $\Big(c^H B(t)\Big)_{t \geq 0}$ are equal in distribution for every $c > 0$.

We will define the $\int_0^T G(u) dB(u)$ Ito integral instead of $\int_0^t G(u) dB(u)$ and use

$$\int_0^t G(u) dB(u) = \int_0^T I_{[0,t]}(u) G(u) dB(u) \text{ for } t \in [0,T]$$

(by Theorem 2.5, p. 345, in [14]).

We consider $\alpha > 1 - H$. It follows by (8) that

$$\int_0^T G(u)dB(u) = (-1)^\alpha \int_0^T D_{0+}^\alpha G(u) D_{T-}^{1-\alpha} B_{T-}(u)du \tag{9}$$

for $G \in I_{0+}^\alpha(L_2(0,T))$, where $G(0+)$ exists and $B_{T-} \in I_{T-}^{1-\alpha}(L_2(0,T))$.

The condition $G \in I_{0+}^\alpha(L_2(0,T))$ (with probability 1) means that $G \in L_2(0,T)$ and

$$\mathcal{I}_\varepsilon(x) = \int_0^{x-\varepsilon} \frac{G(x) - G(y)}{(x-y)^{\alpha+1}} dy \text{ for } x \in (0,T)$$

converges in $L^2(0,T)$ as $\varepsilon \searrow 0$.

The condition $B_{T-} \in I_{T-}^{1-\alpha}(L_2(0,T))$ means $B_{T-} \in L_2(0,T)$ and

$$\mathcal{J}_\varepsilon(x) = \int_{x+\varepsilon}^T \frac{B(x) - B(y)}{(y-x)^{2-\alpha}} dy \text{ for } x \in (0,T)$$

converges in $L_2(0,T)$ as $\varepsilon \searrow 0$ This condition for B is fulfilled for $\alpha > 1 - H$, since the fractional Brownian motion B is a.s. Hölder continuous with exponent $\gamma \in (0, H)$ (see [4]).

We will use (8) for the integrals with respect the approximating processes $\Big(B_N(t)\Big)_{t\in[0,T]}$. Observe that $B_{N,T-} \in I_{T-}^{1-\alpha}(L_2(0,T))$, which follows from the Lipschitz continuity property in Theorem 2.1. We have

$$\int_0^T G(u)dB_N(u) = (-1)^\alpha \int_0^T D_{0+}^\alpha G(u) D_{T-}^{1-\alpha} B_{N,T-}(u)du \tag{10}$$

for $G \in I_{0+}^\alpha(L_2(0,T))$, where $G(0+)$ exists.

Let $\Big(Z(t)\Big)_{t\in[0,T]}$ be a cádlág process. Its **generalized quadratic variation process** $\Big([Z](t)\Big)_{t\in[0,T]}$ is defined as

$$[Z](t) = \lim_{\varepsilon \searrow 0} \varepsilon \int_0^1 \int_0^t \frac{1}{u}(Z_{t-}(s+u) - Z_{t-}(s))^2 ds du + (Z(t) - Z(t-))^2,$$

if the limit exists uniformly in probability (see [11], also in [16] Section 5).

In particular, if B is a fractional Brownian motion with Hurst index $H \in (\frac{1}{2}, 1)$ and B_N is an approximation of B as given in (5), it is easy to verify that

$$[B](t) = 0 \quad \text{and} \quad [B_N](t) = 0 \quad \text{for each } \ t \in [0,T], \tag{11}$$

because B is locally Hölder continuous and B^N is Lipschitz continuous. The **Ito formula** for change of variable for fractional integrals is given in the next theorem.

Theorem 4.1 ([16], Theorem 5.8, p. 170) *Let $\Big(Z(t)\Big)_{t\in[0,T]}$ be a continuous process with generalized quadratic variation $[Z]$. Let $Q : R \times [0,T] \to R$ be a random function such that a.s. we have $Q \in \mathcal{C}^1(R \times [0,T])$ and $\frac{\partial^2 Q}{\partial x^2} \in \mathcal{C}(R \times [0,T])$. Then, for $t_0, t \in [0,T]$ we have*

$$\begin{aligned} Q(Z(t),t) - Q(Z(t_0),t_0) &= \int_{t_0}^{t} \frac{\partial Q}{\partial x}(Z(s),s)dZ(s) + \int_{t_0}^{t} \frac{\partial Q}{\partial t}(Z(s),s)ds \\ &\quad + \int_{t_0}^{t} \frac{\partial^2 Q}{\partial^2 x}(Z(s),s)d[Z]s. \end{aligned}$$

Let $1 - H < \alpha < \frac{1}{2}$ and let $G \in I^{\alpha}_{0+}(L_2(0,T))$ such that $G(0+)$ exists. We define the processes

$$Z(t) = \int_0^t G(s)dB(s) \text{ and } Z_N(t) = \int_0^t G(s)dB_N(s), \quad t \in (0,T].$$

Then by Theorem 5.6, p. 167 in [16] it follows that

$$[Z](t) = 0 \text{ and } [Z_N](t) = 0.$$

Using Theorem 4.1, it follows that, if $Q : R \times [0,T] \to R$ is a random function such that a.s. we have $Q \in \mathcal{C}^1(R \times [0,T])$ and $\frac{\partial^2 Q}{\partial x^2} \in \mathcal{C}(R \times [0,T])$, then for $t_0, t \in [0,T]$ we have

$$\begin{aligned} Q(Z(t),t) - Q(Z(t_0),t_0) &= \int_{t_0}^{t} \frac{\partial Q}{\partial x}(Z(s),s)G(s)dB(s) \qquad (12) \\ &\quad + \int_{t_0}^{t} \frac{\partial Q}{\partial t}(Z(s),s)ds \end{aligned}$$

and

$$\begin{aligned} Q(Z_N(t),t) - Q(Z_N(t_0),t_0) &= \int_{t_0}^{t} \frac{\partial Q}{\partial x}(Z_N(s),s)G(s)dB_N(s) \qquad (13) \\ &\quad + \int_{t_0}^{t} \frac{\partial Q}{\partial t}(Z_N(s),s)ds. \end{aligned}$$

5 Stochastic Differential Equations Driven by Fractional Brownian Motion

Let $\Big(B(t)\Big)_{t\geq 0}$ be a fractional Brownian motion with Hurst parameter H such that $H > \frac{1}{2}$. We investigate stochastic differential equations of the form

$$\begin{aligned} dX(t) &= F(X(t),t)dt + G(X(t),t)dB(t), \\ X(t_0) &= X_0, \end{aligned} \tag{14}$$

where $t_0 \in (0,T]$, X_0 is a random vector in R^n and the random functions F and G satisfy with probability 1 the following conditions:

(C1) $F \in C(R^n \times [0,T], R^n), G \in C^1(R^n \times [0,T], R^n)$;

(C2) for each $t \in [0,T]$ the functions $F(\cdot,t), \dfrac{\partial G(\cdot,t)}{\partial x^i}, \dfrac{\partial G(\cdot,t)}{\partial t}$ are locally Lipschitz for each $i \in \{1,\ldots,n\}$.

We consider the pathwise auxiliary partial differential equation on $R^n \times R \times [0,T]$

$$\begin{aligned} \frac{\partial K}{\partial z}(y,z,t) &= G(K(y,z,t),t), \\ K(Y_0,Z_0,t_0) &= X_0, \end{aligned} \tag{15}$$

where Y_0 is an arbitrary random vector in R^n and Z_0 an arbitrary random variable in R. From the theory of differential equations it follows that with probability 1 there exists a local solution $K \in C^1(R^n \times [0,T], R^n)$ in a neighbourhood V of (Y_0, Z_0, t_0) with partial derivatives being Lipschitz in the variable y and

$$\det\left(\frac{K^i}{\partial y^j}(y,z,t)\right)_{1\leq i,j\leq n} \neq 0.$$

We have for $(x,y,t) \in V$

$$\frac{\partial^2 K}{\partial z^2}(y,z,t) = \sum_{j=1}^{n} \frac{\partial G}{\partial x^j}(K(y,z,t),t)G^j(K(y,z,t),t).$$

We also consider the pathwise differential equation (in matrix representation) on $[0,T]$

$$\begin{aligned} dY(t) &= \left(\frac{K}{\partial y}(Y(t),B(t),t)\right)^{-1}\Big[F(K(Y(t),B(t),t),t) - \frac{\partial K}{\partial t}(Y(t),B(t),t)\Big]dt \\ Y(t_0) &= Y_0, \end{aligned}$$

which has a unique local solution on a maximal interval $(t_0^1, t_0^2) \subseteq [0,T]$ with $t_0 \in (t_0^1, t_0^2)$ (see Theorem 7.1 from Appendix).

Applying the Ito formula, see Theorem 4.1 and relation (12), to the random function $Q(z,t) = K(Y(t),z,t)$ (in fact, successively for $K^1,\ldots,K^n$) and the fractional Brownian motion B we obtain

$$\begin{aligned}
&K(Y(t),B(t),t) - K(Y(t_0),B(t_0),t_0)\\
&= \sum_{j=1}^{n} \int_{t_0}^{t} \frac{\partial K}{\partial y^j}(Y(s),B(s),s)dY^j(s) + \int_{t_0}^{t} \frac{\partial K}{\partial z}(Y(s),B(s),s)dB(s)\\
&\quad + \int_{t_0}^{t} \frac{\partial K}{\partial t}(Y(s),B(s),s)ds\\
&= \sum_{j=1}^{n} \int_{t_0}^{t} \frac{\partial K}{\partial y^j}(Y(s),B(s),s)dY^j(s) + \int_{t_0}^{t} G(K(Y(s),B(s),s),s)dB(s)\\
&\quad + \int_{t_0}^{t} \frac{\partial K}{\partial t}(Y(s),B(s),s)ds\\
&= \int_{t_0}^{t} F(K(Y(s),B(s),s),s)ds + \int_{t_0}^{t} G(K(Y(s),B(s),s),s)dB(s).
\end{aligned}$$

Therefore,

$$X(t) := K(Y(t),B(t),t)$$

satisfies

$$X(t) = X_0 + \int_{t_0}^{t} F(X(s),s)ds + \int_{t_0}^{t} G(X(s),s)dB(s).$$

Instead of the process $\Big(B(t)\Big)_{t\in[0,1]}$ we consider its approximations $\Big(B_N(t)\Big)_{t\in[0,1]}$ given in (5). For each $N \in N$ we consider the pathwise differential equation (in matrix representation)

$$\begin{aligned}
dY_N(t) &= \left(\frac{\partial K}{\partial y}(Y_N(t),B_N(t),t)\right)^{-1}\Big[F(K(Y_N(t),B_N(t),t),t)\\
&\quad -\frac{\partial K}{\partial t}(Y_N(t),B_N(t),t)\Big]dt\\
Y_N(t_0) &= Y_0,
\end{aligned}$$

which has a unique local solution Y_N on a maximal interval $(t^1,t^2) \subset (t_0^1,t_0^2)$ of existence which contains t_0 (see Theorem 7.2 from Appendix). Applying the Ito formula, see Theorem 4.1 and (13), to the random function $Q(z,t) = K(Y_N(t),z,t)$ (in fact, successively for $K^1,\ldots,K^n$) and the process B_N we obtain

$$
\begin{aligned}
&K(Y_N(t), B_N(t), t) - K(Y_N(t_0), B_N(t_0), t_0) \\
&= \sum_{j=1}^{n} \int_{t_0}^{t} \frac{\partial K}{\partial y^j}(Y_N(s), B_N(s), s) dY_N^j(s) + \int_{t_0}^{t} \frac{\partial K}{\partial z}(Y_N(s), B_N(s), s) dB_N(s) \\
&\quad + \int_{t_0}^{t} \frac{\partial K}{\partial t}(Y_N(s), B_N(s), s) ds
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \int_{t_0}^{t} \frac{\partial K}{\partial y^j}(Y_N(s), B_N(s), s) dY_N^j(s) + \int_{t_0}^{t} G(K(Y_N(s), B_N(s), s), s) dB_N(s) \\
&\quad + \int_{t_0}^{t} \frac{\partial K}{\partial t}(Y_N(s), B_N(s), s) ds \\
&= \int_{t_0}^{t} F(K(Y_N(s), B_N(s), s), s) ds + \int_{t_0}^{t} G(K(Y_N(s), B_N(s), s), s) dB_N(s).
\end{aligned}
$$

Therefore,

$$X_N(t) := K(Y_N(t), B_N(t), t)$$

satisfies

$$X_N(t) = X_0 + \int_{t_0}^{t} F(X_N(s), s) ds + \int_{t_0}^{t} G(X_N(s), s) dB_N(s), \quad t \in (t_1, t_2).$$

By Theorem 7.2 it follows that we have the following pathwise property

$$\lim_{N \to \infty} \sup_{t \in (t_1, t_2)} \|Y_N(t) - Y(t)\| = 0.$$

Then the continuity properties of K and (6) imply that for a.e. $\omega \in \Omega$ it holds

$$\lim_{N \to \infty} \sup_{t \in (t_1, t_2)} \|X_N(t) - X(t)\| = 0.$$

By this we have proved the main result of our paper:

Theorem 5.1 *Let B be a fractional Brownian motion approximated through the processes B_N given in* (5) *and* (6)*. Let $F, G : R^n \times [0, T] \to R^n$ be random functions satisfying with probability* 1 *the conditions* (C1) *and* (C2)*. Let $t_0 \in (0, T]$ be fixed. Then, each of the stochastic equations*

$$X(t) = X_0 + \int_{t_0}^{t} F(X(s), s) ds + \int_{t_0}^{t} G(X(s), s) dB(s),$$

$$X_N(t) = X_0 + \int_{t_0}^{t} F(X_N(s), s)ds + \int_{t_0}^{t} G(X_N(s), s)dB_N(s), \quad N \in N$$

admits almost surely a unique local solution on a common interval (t_1, t_2) *(which is independent of* N *and contains* t_0*). Moreover, we have the following approximation result*

$$P(\lim_{N\to\infty} \sup_{t\in(t_1,t_2)} \|X_N(t) - X(t)\| = 0) = 1.$$

6 Application

We consider the one dimensional stochastic linear equation from finance mathematics, modeling the price S of a stock

$$S(t) = S_0 + \int_0^t \mu(s)S(s)ds + \int_0^t \sigma(s)S(s)dB(s),$$

where $(B(t))_{t\in[0,T]}$ is a fractional Brownian motion with Hurst index $H > \frac{1}{2}$, μ is the interest rate and σ the dispersion function.

It is known (see [8], p. 1022) that this equation has the following unique solution

$$S(t) = S_0 \exp\left\{ \int_0^t \mu(u)du + \int_0^t \sigma(u)dB(u) \right\} \text{ for all } t \in [0, T].$$

By the methods of the above section we approximate B through the processes B_N, via (5) and (6) and consider

$$S_N(t) = S_0 \exp\left\{ \int_0^t \mu(u)du + \int_0^t \sigma(u)dB_N(u) \right\} \text{ for all } t \in [0, T].$$

Using Theorem 5.1 it follows that

$$P(\lim_{N\to\infty} \sup_{t\in[0,T]} \|S_N(t) - S(t)\| = 0) = 1.$$

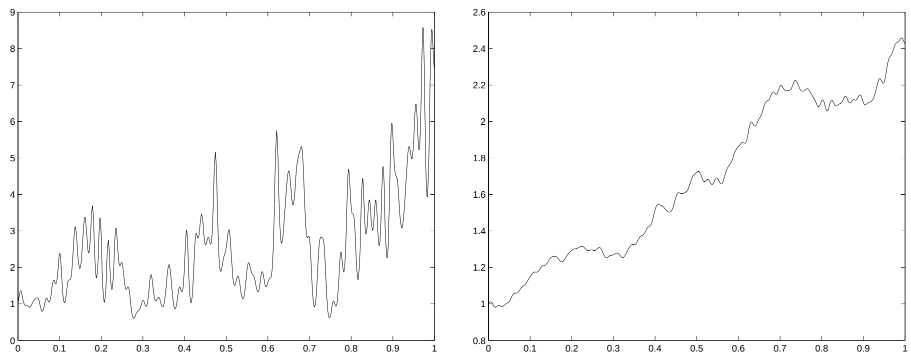
In the special case when μ and σ are constants, we have that the price of a stock is

$$S(t) = S_0 e^{\mu t + \sigma B(t)}$$

and we can simulate it by computer using

$$S_N(t) = S_0 e^{\mu t + \sigma B_N(t)}$$

as given in Figure 3.



a) $H = 0.26$ b) $H = 0.9$

Figure 3: Approximated solution S_N

7 Appendix

We prove the existence of the local solution of a deterministic equation with locally Lipschitz function (in the version we need in our paper). We adapt the ideas from the proof of Theorem 1.4 in [9]. We give the proof here in order to make the proof of Theorem 7.2 more understandable.

In what follows $\|\cdot\|$ denotes the norm in R^n.

Theorem 7.1 *Let $A : R^n \times [0,\infty) \to R^n$ be such that for each $u \in R^n$ the function $A(u,\cdot)$ is continuous and for any $c, T > 0$ we have*

$$\|A(x,t) - A(y,t)\| \le L(c,T)\|x-y\|$$

for all $x, y \in R^n$ with $\|x\| \le c, \|y\| \le c$ and $t \in [0,T]$, where $L(c,T) > 0$ is the locally Lipschitz constant. We consider the equation

$$U(t) = U_0 + \int\limits_{t_0}^{t} A(U(s),s)ds, \tag{16}$$

where $U_0 \in R^n$ and $t_0 > 0$ fixed. Then equation (16) has a local solution, i.e. there exists a maximal interval $(t_1,t_2) \in [0,\infty)$ containing t_0 and a function $U : R^n \times (t_1,t_2) \to R^n$ such that (16) is satisfied for each $t \in (t_1,t_2)$.

For any $\tau > 0$ let $M(\tau) = \max\limits_{t\in[0,\tau+1]} \|A(0,t)\|$. We consider

$$\delta = \min\left\{1, \frac{\|U_0\|}{2\|U_0\|L(2\|U_0\|, t_0+1) + M(t_0)}, t_0\right\}.$$

We define the mapping $\mathcal{A} : C([t_0-\delta, t_0+\delta], R^n) \to C([t_0-\delta, t_0+\delta], R^n)$

$$(\mathcal{A}U)(t) := U_0 + \int\limits_{t_0}^{t} A(U(s),s)ds, \quad t \in [t_0-\delta, t_0+\delta].$$

We prove that $\mathcal{A}$ maps the ball $\mathcal{B}(0,R)$ of radius $R = 2\|U_0\|$ centered at 0 of the space $C([t_0-\delta, t_0+\delta], R^n)$ into itself. For $U \in \mathcal{B}(0,R)$ and for each $t \in [t_0-\delta, t_0+\delta]$ we have the following estimates

$$\begin{aligned} \|\mathcal{A}(U)(t)\| &\leq \|U_0\| + \left| \int_{t_0}^{t} \|A(U(s),s) - A(0,s)\| + \|A(0,s)\|\Big) ds \right| \\ &\leq \|U_0\| + (L(R, t_0+1)R + M(t_0))|t-t_0| \leq 2\|U_0\| = R. \end{aligned}$$

Therefore, $\mathcal{A}U \in \mathcal{B}(0,R)$. It is easy to verify that for each $U, V \in \mathcal{B}(0,R)$ and each $t \in [t_0-\delta, t_0+\delta]$ we have

$$\|\mathcal{A}(U)(t) - \mathcal{A}(V)(t)\| \leq L(R, t_0+1)|t-t_0| \sup_{t\in[t_0-\delta,t_0+\delta]} \|U(t) - V(t)\|.$$

For each $N \in N$ we denote

$$\mathcal{A}^N = \underbrace{\mathcal{A} \circ \ldots \circ \mathcal{A}}_{N \text{ times}}.$$

From the definition of $\mathcal{A}$ it then follows for each $N \in N$ and each $t \in [t_0-\delta, t_0+\delta]$ that

$$\|\mathcal{A}^N(U)(t) - \mathcal{A}^N(V)(t)\| \leq \frac{\Big(L(R,t_0+1)|t-t_0|\Big)^N}{N!} \sup_{t\in[t_0-\delta,t_0+\delta]} \|U(t) - V(t)\|.$$

Hence

$$\sup_{t\in[t_0-\delta,t_0+\delta]} \|\mathcal{A}^N(U)(t) - \mathcal{A}^N(V)(t)\| \leq \frac{\Big(L(R,t_0+1)\delta\Big)^N}{N!} \sup_{t\in[t_0-\delta,t_0+\delta]} \|U(t) - V(t)\|.$$

For N large enough we have $\dfrac{\Big(L(R,t_0+1)\delta\Big)^N}{N!} < 1$. By a well known extension of the contraction principle it follows that $\mathcal{A}$ has a unique fixed point in $\mathcal{B}(0,R)$.

We have proved that there exists a solution U defined on the interval $[t_0-\delta, t_0+\delta]$ satisfying (16). This solution can be extended to the interval $[t_0-\delta^*, t_0+\delta^*]$ $(\delta^* > \delta)$, where on $[t_0-\delta, t_0+\delta]$ we have the above solution U and for $t \geq t_0+\delta$ we use the above method to find a local solution for

$$U(t) = U(t_0+\delta) + \int_{t_0+\delta}^{t} A(U(s),s)ds,$$

and also for $t \leq t_0-\delta$ we use the above method to find a local solution for

$$U(t) = U(t_0-\delta) + \int_{t_0-\delta}^{t} A(U(s),s)ds.$$

Moreover, δ^* depends only on $\delta, \|U(t_0+\delta)\|, \|U(t_0-\delta)\|, M(t_0+\delta), M(t_0-\delta)$. Hence, there exists a maximal interval (t_1, t_2) containing t_0 for the existence of the local solution U.

Theorem 7.2 *Let $A : R^{n+1} \times [0,T] \to R^n$ be such that for each $(x,u) \in R^{n+1}$ the function $A(x,u,\cdot)$ is continuous and we have*

$$\|A(x,u,t) - A(y,v,t)\| \leq L(c)(\|x-y\| + |u-v|)$$

for all $x, y \in R^n$ with $\|x\| \leq c, \|y\| \leq c, |u| \leq c, |v| \leq c$ and each $t \in [0,T]$, where $L(c) > 0$ is the locally Lipschitz constant. Let $U_0 \in R^n$ and $t_0 \in (0,T]$ fixed. Assume that $(v_N)_{N\in N}$ is a sequence from $C[0,T]$ which converges uniformly to $v \in C[0,T]$, i.e.

$$\lim_{N\to\infty} \sup_{t\in[0,T]} |v_N(t) - v(t)| = 0.$$

We consider the equations

$$U_N(t) = U_0 + \int_{t_0}^{t} A(U_N(s), v_N(s), s)ds, \quad N \in N \tag{17}$$

and

$$U(t) = U_0 + \int_{t_0}^{t} A(U(s), v(s), s)ds. \tag{18}$$

The equations (17) *and* (18) *have local solutions, i.e. there exists a maximal interval $(t_1,t_2) \subset [0,T]$ (which does not depend on N) containing t_0 and functions $U_N, U : R^n \times (t_1,t_2) \to R^n$ such that* (17) *and* (18) *are satisfied for each $t \in (t_1,t_2)$. Moreover,*

$$\lim_{N\to\infty} \sup_{t\in(t_1,t_2)} \|U_N(t) - U(t)\| = 0.$$

For any $\tau > 0$ let $M = \max\limits_{t\in[0,T]} \|A(0,0,t)\|$. Since $(v_N)_{N\in N}$ converges uniformly to v in $C[0,T]$, it follows that there exists $m > 0$ such that

$$\sup_{t\in[0,T]} |v_N(t)| + \sup_{t\in[0,T]} |v(t)| \leq m \qquad \text{for each } N \in N.$$

We consider

$$\delta = \min\left\{1, \frac{m}{(\|U_0\| + 2m)L(\|U_0\| + m) + M}, t_0, T - t_0\right\}.$$

We define the mapping $\mathcal{F}_N : C([t_0-\delta, t_0+\delta], R^n) \to C([t_0-\delta, t_0+\delta], R^n)$

$$(\mathcal{F}_N Y)(t) := U_0 + \int_{t_0}^{t} A(Y(s), v_N(s), s)ds, \quad t \in [t_0-\delta, t_0+\delta].$$

We prove that $\mathcal{F}_N$ maps the ball $\mathcal{B}(0,R)$ of radius $R = \|U_0\| + m$ centered at 0 of the space $C([t_0-\delta, t_0+\delta], R^n)$ into itself. For $Y \in \mathcal{B}(0,R)$ and for each $t \in [t_0-\delta, t_0+\delta]$ we have the following estimates

$$\begin{aligned}
\|\mathcal{F}_N(Y)(t)\| &\leq \|U_0\| + \left| \int_{t_0}^{t} \|A(Y(s), v_N(s), s) - A(0,0,s)\| + \|A(0,0,s)\| \Big) ds \right| \\
&\leq \|U_0\| + (L(R)(R+m) + M)|t-t_0| \leq \|U_0\| + m = R.
\end{aligned}$$

Therefore, $\mathcal{F}_N Y \in \mathcal{B}(0,R)$. It is easy to verify that for each $Y, Z \in \mathcal{B}(0,R)$ and each $t \in [t_0-\delta, t_0+\delta]$ we have

$$\|\mathcal{F}_N(Y)(t) - \mathcal{F}_N(Z)(t)\| \leq L(R)|t-t_0| \sup_{t\in[t_0-\delta,t_0+\delta]} \|Y(t) - Z(t)\|.$$

Using the contraction principle exactly as in the proof of Theorem 7.1, it follows that $\mathcal{F}_N$ has a unique fixed point in $\mathcal{B}(0,R)$, which is defined on $[t_0-\delta, t_0+\delta]$. This fixed point is the local solution U_N of (17). We observe that this interval of existence of the local solution U_N does not depend on N, and $U_N \in \mathcal{B}(0,R)$ for each $N \in N$. Exactly in the same way we can prove that on the same interval $[t_0-\delta, t_0+\delta]$ there exists a solution $U \in \mathcal{B}(0,R)$ satisfying (18). Let $(t_1,t_2) \subset (0,T]$ be the maximal interval (which does not depend on N) containing t_0 such that (17) and (18) are satisfied for each $t \in (t_1,t_2)$ and there exists $R > 0$ (independent of N) such that $U_N, U \in \mathcal{B}(0,R)$. Then for large N we have

$$\begin{aligned}
\|U_N(t) - U(t)\| &\leq \left| \int_{t_0}^{t} \|A(U_N(s), v_N(s), s) - A(U(s), v(s), s)\| ds \right| \\
&\leq \left| \int_{t_0}^{t} L(R)(\|U_N(s) - U(s)\| + \|v_N(s) - v(s)\|) ds \right|.
\end{aligned}$$

By the Gronwall lemma we get

$$\sup_{t\in(t_1,t_2)} \|U_N(t) - U(t)\| \leq \sup_{t\in(t_1,t_2)} \|v_N(t) - v(t)\| e^{L(R)(t_2-t_1)}$$

Therefore,

$$\lim_{N\to\infty} \sup_{t\in(t_1,t_2)} \|U_N(t) - U(t)\| = 0.$$

References

[1] J. Bertoin, *Sur une intégrale pour les processus á α variation bornée.* Annals of Probability **17** (1989), 1521–1535.

[2] L. Coutin and L. Decreusefond, *Stochastic Differential Equations Driven by a Fractional Brownian Motion.* Tech. Report 97C004, ENST Paris (1997).

[3] L. Coutin and Z. Qian, *Stochastic Differential Equations for Fractional Brownian Motions*, C. R. Acad. SCI. Paris, t. **331**, Srie I (2000), 75–80.

[4] L. Decreusefond and A.S. Üstünel, *Stochastic Analysis of the Fractional Brownian Motion.* Potential Analysis **10** (1999), 177–214.

[5] K. Dzhaparidze and H. van Zanten, *A Series Expansion of Fractional Brownian Motion.* Probab. Theory Relat. Fields **130** (2004), 39–55.

[6] M. Errami and F. Russo, *n-Covariation, Generalized Dirichlet Processes and Calculus with respect to Finite Cubic Variation Process.* Stochastic Process. Appl. **104** (2003), 259–299.

[7] M.L. Kleptsyna, P.E. Kloeden and V.V. Anh, *Existence and Uniqueness Theorems for FBM Stochastic Differential Equations.* Information Transmission Problems. **34** (1998), 41–51.

[8] F. Klingenhöfer and M. Zähle, *Ordinary Differential Equations with Fractal Noise.* Proc. Amer. Math. Soc. **127** (1999), 1021–1028.

[9] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Springer Verlag, New York (1983).

[10] F. Russo and P. Vallois, *Forward, Backward and Symmetric Stochastic Integration.* Probab. Theory Related Fields **97** (1993), 403–421.

[11] F. Russo and P. Vallois, *The Generalized Covariation Process and Itô Formula.* Stoch. Processes and Appl. **59** (1995), 81–104.

[12] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications.* Gordon and Breach, 1993.

[13] G. N. Watson, *A Treatise on the Theory of Bessel Functions.* Cambridge University Press, 1944.

[14] M. Zähle, *Integration with respect to Fractal Functions and Stochastic Calculus I.* Probab.Theory Relat. Fields **111** (1998), 333–374.

[15] M. Zähle, *Ordinary Differential Equations with Fractal Noise.* Proc. Am. Math. Soc. **127** (1999), 333–374.

[16] M. Zähle, *Integration with respect to Fractal Functions and Stochastic Calculus II.* Math. Nachr. **225** (2001), 145–183.