Nontrivial spontaneous synchronization

R. Sumi,[1,2] Z. Néda,[1] A. Tunyagi,[1] Sz. Boda,[1] and Cs. Szász[3]
[1]*Faculty of Physics, Babeş-Bolyai University, RO-400084 Cluj, Romania*
[2]*E-Austria Research Institute, RO-300223 Timisoara, Romania*
[3]*Department of Electrical Engineering, Technical University of Cluj-Napoca, RO-400020 Cluj-Napoca, Romania*
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The collective behavior of an ensemble of multimode stochastic oscillators is investigated. The oscillators are pulse coupled; they are able to emit pulses and to detect the pulses emitted by the others. As a function of the output intensity in the system they can operate in different modes having different pulsing periods. The system is designed to optimize the output intensity around a fixed f^* output threshold. In order to do so a simple dynamics is considered. Whenever the total output intensity in the system is lower than f^*, a mode with a higher interpulse period is chosen. If the light intensity in the system is higher than f^*, a mode with a lower interpulse period is selected. As a side effect of this simple optimization rule, for a given f^* interval a nontrivial synchronization of the oscillators is observed. The synchronization level is studied by computer simulations, investigating the influence of model parameters (number of modes, stochasticity of the oscillators, the f^* threshold value, and interaction topology). An experimental realization of this system is also considered; an ensemble of electronic oscillators communicating with light pulses was constructed and studied. The experimental system behaves in many ways similar to the theoretically considered multimode stochastic oscillator ensemble.

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I. INTRODUCTION: TRIVIAL VERSUS NONTRIVIAL SYNCHRONIZATION

The spontaneous synchronization of coupled oscillators is a well-known collective behavior. It appears without an external periodic driving, and it is the consequence of the interactions between the units. A wide variety of natural and social systems offers fascinating examples: pendulum clocks, flashing fireflies, oscillating chemical reactions, menstrual cycles of women living together, lightning activity in distant storm cells, rhythmic applause, chirping crickets, pacemaker cells in the heart, neurons, etc. For an extended review on synchronization phenomena one can consult several review works [1–3]. Physics developed successful models for understanding this widespread phenomenon. The used models can be categorized by the relevant properties of the oscillators or the nature of the coupling between them. In the present work a different model category is studied. The oscillators are multimode units and the coupling between them is realized through a simple optimization dynamics.

Spontaneous synchronization of identical oscillators coupled by phase-difference minimizing forces is a trivial phenomenon. Since in such cases the interaction always minimizes the phase difference between the units, the synchronized state is a natural fix point of the dynamics. In such cases the interesting aspect is the transient dynamics leading to synchronization.

The problem becomes more interesting, and also more realistic for applications to natural systems, if oscillators with different natural frequencies are considered. In such cases it is not obvious at all that synchronization can be achieved in the thermodynamic limit (large number of oscillators), since the units are different and each of them would like to impose its natural frequency. In general, for globally coupled systems one obtains a second-order phase-transition type behavior of the order parameter as a function of the coupling strength. The order parameter in such cases is a nondimensional quantity, characterizing the synchronization level in the system. For interaction strengths less than a critical value, the system does not synchronize (disordered phase). For interaction values larger than the critical interaction, partial synchronization emerges (ordered phase). The value of the critical coupling depends on how different the natural frequencies of the oscillators are, i.e., it depends on the standard deviation of the natural frequencies. For locally coupled oscillators the problem is even more complex and the existence of the synchronized phase depends largely on the coupling topology. There are two main categories of models which are widely used for describing the synchronization of nonidentical oscillators. One is for phase-coupled oscillators (Kuramoto-type models) and the other one is for pulse-coupled oscillators (integrate-and-fire-like models). The *Kuramoto model* [4] considers a large number ($N, N \to \infty$) of coupled rotators. Their natural frequencies, ω_i, are distributed according to a $g(\omega)$ probability density. Coupling is considered uniform between the units, and the dynamics is governed by the following set of coupled first-order differential equations:

$$\dot{\Theta}_i = \omega_i + K\sum_{j=1}^{N} \sin(\Theta_j - \Theta_i), \quad i = 1, \ldots, N, \tag{1}$$

where K is the coupling constant and Θ_i is the phase of unit i. The complex order parameter of the system is defined as

$$re^{i\phi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\Theta_j}. \tag{2}$$

ϕ represents the collective phase of the synchronized state and $r \in [0,1]$ is the scalar order parameter characterizing the

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synchronization level. The advantage of the chosen harmonic form of the coupling is that the model can be solved analytically [4]. The model exhibits a second-order phase transition as a function of the K coupling parameter. For $K < K_c$, the system will not synchronize, and the stable solution is $r=0$. For coupling above a K_c critical value, partial synchronization of the rotators will appear ($r>0$). For many natural systems, however, considering the coupling phaselike is not realistic, since the interaction between the units is pulselike (fireflies, clapping, firing of neurons, etc.). The *integrate-and-fire* type models [5,6] are appropriate for such kind of oscillators. The basic assumption is that again each oscillator has a natural frequency $\omega(i)$, a phase $\Phi(i)$, and a state variable (or energy) $E(i)$. The phase and state variables are connected through an $F(x)$ monotonically increasing functional relation. Once the phase of an oscillator reaches 2π, the oscillator emits a pulse and its phase is reset to 0. The effect of this pulse on the other oscillators is that their state variable is instantaneously increased (or decreased) by a K value. As a result of the $F(x)$ relationship the phases of the oscillators are also increased. It is easy to realize that under some specific conditions the pulse of one oscillator can trigger an avalanche of pulses (firing), leading finally to a phase synchronization of the units. A proper order parameter characterizing the synchronization level in the system can be defined as the relative size of the largest avalanche (number of units firing together divided by the total number of units). Similarly to the Kuramoto model, for globally coupled units and for a wide variety of $F(x)$ functions, there exists a critical K_c coupling threshold. For $K < K_c$ the system does not synchronize ($r=0$) and for $K > K_c$ partial synchronization emerges ($r>0$). For locally coupled units the topology of the interactions is important and as a result of this, synchronization might or might not appear. A widely used variant of this model was elaborated for neurons [7,8]. The model assumes that if the membrane potential reaches a threshold V_{th}, the neuron will fire, this spike corresponds to the action potential. After firing, the membrane potential is reset to a $V_0 < V_{th}$ value. Once fired, the neuron needs some time to recover before firing again. Here the slow collection and quick release of the voltage is called integrate-and-fire behavior. FitzHugh [8] and Nagumo *et al.* [9] derived the FitzHugh-Nagumo equations [7] for such oscillators, which are also able to describe spontaneous synchronization.

Although spontaneous synchronization of nonidentical oscillators coupled by phase difference minimizing interactions is not obvious, the synchronization that will appear for large coupling values is again trivial. The interaction is obviously favoring synchronization, and one would naturally expect this collective behavior to appear in the large coupling limit. In layman's terms one could say that the desire for synchronization is directly built into the model.

A highly nontrivial model for synchronization was recently introduced by Nikitin *et al.* [10,11] to offer a realistic description of the fascinating dynamics of rhythmic applause and synchronization in several other biological systems with multiple oscillating modes. In this model there are no phase-minimizing interactions that would naturally lead to synchronization. Synchronization appears as an unexpected side effect of a simple optimization process. The oscillators are realistic in many ways; their natural frequencies are different and stochastically fluctuating in time, the oscillators can operate in several modes with different natural frequencies. In the present paper we investigate in detail this highly nontrivial synchronization. We analyze the influence of the number of modes and the interaction topology on the synchronization level. A simple experimental realization was built and the obtained collective behavior is compared to the results predicted by the model.

II. TWO-MODE STOCHASTIC OSCILLATOR MODEL

The cycle of these oscillators is composed of three parts; let us denote them as A, B, and C [10,11]. State A is the stochastic part. The time interval spent in this part is a stochastic variable, and its length is denoted by τ_A. It can be different for any two oscillators, and it can also be different for the same oscillator in successive cycles. This property makes the oscillators realistic, since in nature there are no perfectly identical individuals, and the period of any oscillation fluctuates in time. We thus consider τ_A as a stochastic variable with probability density

$$P(\tau_A) = \frac{1}{\tau^*} e^{-(\tau_A/\tau^*)} \tag{3}$$

($\tau^* = \langle \tau_A \rangle$).

More precisely, state A should be imagined and modeled with an escape dynamics of a stochastic field driven particle from a potential well of depth U. If the stochastic force field is totally uncorrelated, with $\langle \xi \rangle = 0$ and $\langle \xi(t)\xi(t+\tau)\rangle_t = D\delta(\tau)$, we get the distribution of escape times given in Eq. (3) with $\tau^* \propto e^{U/D}$. In analogy with the well-known FitzHugh-Nagumo system, state A corresponds to the stochastic recovery time of a neuron. This causes all experimentally observed fluctuations in rhythmic human activities.

State B represents a waiting time or the part where the desired rhythm is imposed. The oscillators spend most of their time in this state. In the two-mode version τ_B, the time interval spent in state B can take two possible values: τ_{B_I} or $\tau_{B_{II}}$, and this leads to the two coexisting modes. We have chosen $\tau_{B_{II}} = 2\tau_{B_I}$ here. The qualitative results of the collective dynamics remain however unaffected by the chosen ratio of τ_{B_I}/τ_{BII}, assuming of course that this ratio is not one.

In state C the oscillator emits a pulse which is detected by all the other units. The output (pulse) of oscillator i is considered thus to be $f_i = 1/N$ (N is the number of oscillators in the system) if the oscillator is in state C, and $f_i = 0$ if the oscillator is in state A or B.

The rules for the collective time evolution of the system are simple and realistic ones. They are designed for keeping an f^* average output intensity: (1) each oscillator starts with randomly selected phases and modes; (2) there is a fixed (desired) output intensity, f^*, for the system; (3) at each time step the total output is monitored $f = \sum_{i=1}^{N} f_i$; (4) After finishing state A each oscillator will choose to follow either mode I or mode II. If at that moment $f < f^*$, the oscillator will operate in mode I increasing the total average output by shortening the period. If $f > f^*$ the oscillator will follow

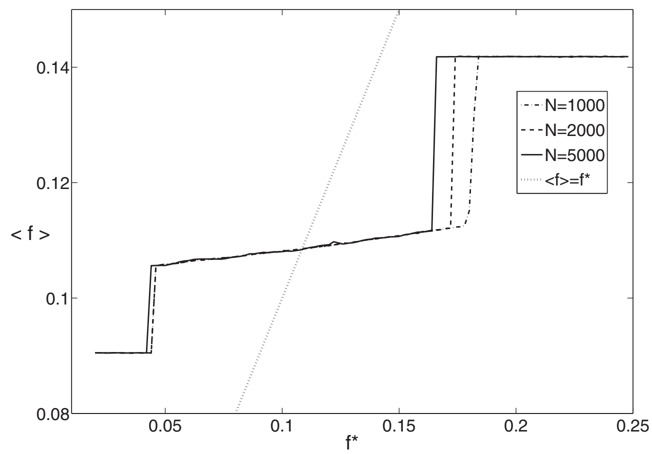

FIG. 1. The average global output of the system as a function of f^*. The dotted line illustrates the desired trend. Two-mode oscillators with $\tau^*=0.2$ are considered and simulations with different number, N, of units are done.

mode *II* with a longer period, decreasing the total average output. Since the intensity of the oscillator's pulse is fixed, this is the only possible way to influence the value of the average output.

The above dynamics tries to keep the average output close to f^*. In Fig. 1 we illustrate this by plotting the simulated time average of the systems global output, $\langle f\rangle$, as a function of f^*. From Fig. 1 it results that the dynamics is not perfectly efficient, since the desired $\langle f\rangle=f^*$ dependence is approximated quite roughly.

As an unexpected side effect, for a broad f^* and τ^* interval nontrivial spontaneous synchronization appears. Synchronization in this case means that a large number of oscillators will emit pulses at the same time. As a consequence of this the total output signal f exhibits a periodic behavior. We illustrate this phenomenon in Fig. 2. For the f^* and τ^* parameter values chosen in Fig. 2(a) the oscillators are working synchronously since the total output intensity, $f(t)$, has a periodic pulsation. In contrast to this, for the f^* and τ^* parameters chosen in Fig. 2(b) there is no synchronization.

In order to characterize the synchronization level numerically, we use a simple measure that is appropriate for describing the periodicity level of the output [10,11]. Let us denote the output signal as a function of time by $f(t)$. The

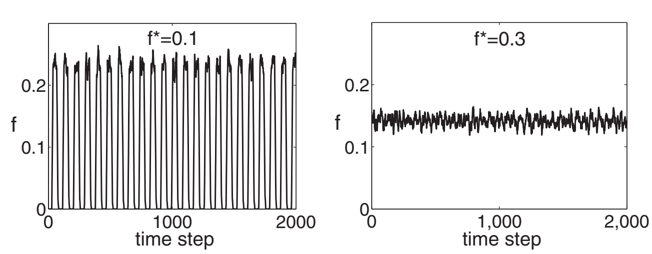

FIG. 2. The total output as a function of time for partially synchronized (left) and unsynchronized (right) oscillators. The two cases differ in the value of the f^* parameter: for the partially synchronized oscillators $f^*=0.10$ and for the unsynchronized ones $f^*=0.30$. In both cases $N=2000$ oscillators were considered and $\tau^*=0.2$ was chosen.

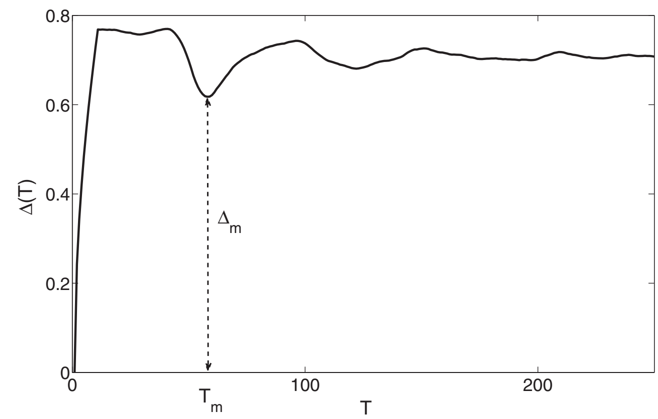

FIG. 3. General shape of the $\Delta(T)$ function.

first step is to define an error function, $\Delta(T)$, which characterizes quantitatively how strongly the $f(t)$ signal differs from any periodic signal with period T,

$$\Delta(T) = \frac{1}{2M}\lim_{x\to\infty}\frac{1}{x}\int_0^x |f(t) - f(t+T)|dt, \tag{4}$$

where

$$M = \lim_{x\to\infty}\frac{1}{x}\int_0^x |f(t) - \langle f(t)\rangle|dt, \tag{5}$$

$$\langle f(t)\rangle = \lim_{x\to\infty}\frac{1}{x}\int_0^x f(t)dt. \tag{6}$$

The general shape of the $\Delta(T)$ curve is sketched in Fig. 3. For any $f(t)$ signal, the $\Delta(T)$ function initially increases, after that decreases, and reaches a minimum at T_m (we denote the minimum as Δ_m). T_m is the best approximation for the $f(t)$ signal's period, and the p periodicity level of the signal is characterized by

$$p = \frac{1}{\Delta_m}. \tag{7}$$

As discussed earlier, the p periodicity level is appropriate for characterizing the synchronization level of the oscillators. It has the advantage that it can be computed solely from the $f(t)$ global output. One can calculate the value of p for the whole system and also for one single oscillator (p_1) working in the long period mode (where the effect of randomness on the period is weaker). The ratio $p/p1$ will then characterize the enhancement in the periodicity due to the considered coupling. This ratio will be used hereafter to characterize the synchronization level of the oscillator ensemble.

In the previous numerical studies performed on this system [10] it was shown that in the f^*-τ^* phase space there is a compact islandlike region where partial synchronization is achieved (Fig. 4). It was also shown that in the synchronized regime the total output of the two-mode oscillator system is more periodic than the output of one single unit in the long period mode (i.e., $p\geq p1$). Moreover, the previous studies revealed that the periodicity level of the system (the value of

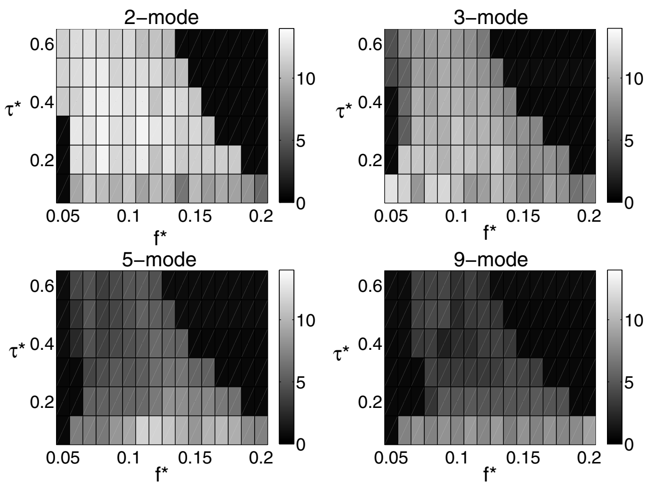

FIG. 4. Synchronization level in the system as a function of the f^* and τ^* parameters. Grayscale code, where lighter colors correspond to higher synchronization levels. The collective dynamics of $N=2000$ oscillators are simulated.

$p/p1$) monotonically increases with the number of units in the system (see for example Fig. 6). This enhancement in the periodicity level offers challenging perspectives for practical applications.

The multimode stochastic oscillator model is realistic for many social and biological entities [12–16]. As immediate examples, one can mention the thalamocortycal relay neurons, the unicellular alga Gonyaulax polyhedra, or the American snowy cricket. In a previous study [11] it was also shown that the two-mode stochastic oscillator model can be successfully applied for describing the fascinating collective social behavior in the rhythmic applause.

III. *n*-MODE STOCHASTIC OSCILLATOR MODEL

A natural question that one should answer is how the results are modified when there are more than two possible oscillation modes for the units. Will the level of the obtained synchronization increase or decrease in such cases? How will the f^*-τ^* parameter space region, in which synchronization is present, modify when the number of modes is increasing?

In order to answer these questions, three-, five-, and nine-mode stochastic oscillators will be considered. States A and C will remain the same as in the two-mode case. We assume that in the three-mode case there are three possible τ_B values: $\tau_{B_I}=0.4$, $\tau_{B_{II}}=0.6$, and $\tau_{B_{III}}=0.8$. In the five-mode case we have chosen $\tau_{B_I}=0.4$, $\tau_{B_{II}}=0.5$, $\tau_{B_{III}}=0.6$, $\tau_{B_{IV}}=0.7$, and $\tau_{B_V}=0.8$, and in the nine-mode case: $\tau_{B_I}=0.4$, $\tau_{B_{II}}=0.45$, $\tau_{B_{III}}=0.5$, $\tau_{B_{IV}}=0.55$, $\tau_{B_V}=0.6$, $\tau_{B_{VI}}=0.65$, $\tau_{B_{VII}}=0.7$, $\tau_{B_{VIII}}=0.75$, and $\tau_{B_{IX}}=0.8$.

The dynamical rules for the time evolution of the system is similar to the ones considered for two-mode oscillators. Let us denote the number of possible modes by M. After finishing phase A, each oscillator has the possibility to remain either in its last operation mode or to choose a new mode. Let us assume that in the previous cycle the oscillator operated in mode k ($k\in\{1,2,\ldots,M\}$), and at the end of phase A, the total output intensity in the system is f. If $f>f^*$ the oscillator will choose the mode $k+1$ with a higher period in order to decrease the total output intensity (approach better the desired f^* value). If there is no mode with a higher period (i.e., $k=M$), the oscillator will remain in mode M. In the case $f<f^*$ the oscillator will choose to work in mode $k-1$ with a lower period, increasing by this the average output intensity (for $k=1$ the oscillator will continue to operate in mode 1). For the case of the "n-mode" system one can define other dynamical rules as well. A natural rule would be to allow the units to jump on the available modes in a less gradual manner. As a function of the difference between f and f^* the oscillator would choose a mode with a more suitable period. Another possibility would be to allow the oscillators to choose randomly one mode in the direction imposed by f and f^*. If $f>f^*$ a mode with a higher period would be randomly selected, and for $f<f^*$ a mode with a lower period would be randomly chosen. These are only few examples on how the dynamical rules for the n-mode system could be defined, and definitely all these systems are worth a detailed study. In the present study we limit ourselves to the first described steplike dynamics.

Computer simulations were used to investigate the collective behavior of these systems. The length of the C phase of the cycle was fixed as $\tau_C=0.1$ units and the time step in the simulation was taken to be $\Delta t=0.01$. The values of f^* and τ^* were changed in uniform steps, and the previously introduced $p/p1$ synchronization level was calculated for each mesh point in the f^*-τ^* parameter space. Initially 32 000 transient time steps are simulated to heat up the system. After this additional 32 000 time steps were simulated for each parameter value, and the value of $p/p1$ was computed from this data.

In agreement with the previously obtained results (the two-mode case [10]), in the f^*-τ^* parameter space there is an islandlike region where synchronization is present (Fig. 4). In Fig. 4 we illustrate with a grayscale code the value of the $p/p1$ synchronization level. Lighter colors correspond to stronger synchronization levels.

The figure suggests that as the number of modes increases, the size of the island in the f^*-τ^* parameter space where partial synchronization is present will decrease. The grayscale code also suggests that the more oscillating modes we have, the weaker the synchronization will be.

Let us investigate now the distribution of the oscillating modes chosen by the oscillators for different values of the f^* parameter ($\tau^*=0.2$, $N=2000$). Results are plotted in Fig. 5. The height of the histograms presented in Fig. 5 is proportional to the number of oscillators found to be working in a given oscillating mode. For each case four f^* values are considered in ascending order. For $f^*=0.03$, the oscillators usually work in the long period mode and there is no synchronization; for $f^*=0.08$ and $f^*=0.15$, the oscillators will switch between the allowed modes and this is the interval where partial synchronization arises; for $f^*=0.22$ all oscillators choose the short period mode which results in an unsynchronized behavior. As expected, synchronization appears when the coupling is efficient, and the oscillators continuously shift between the allowed modes.

We focus now on the $p/p1$ synchronization level as a function of the number of oscillators in the system, N (Fig.

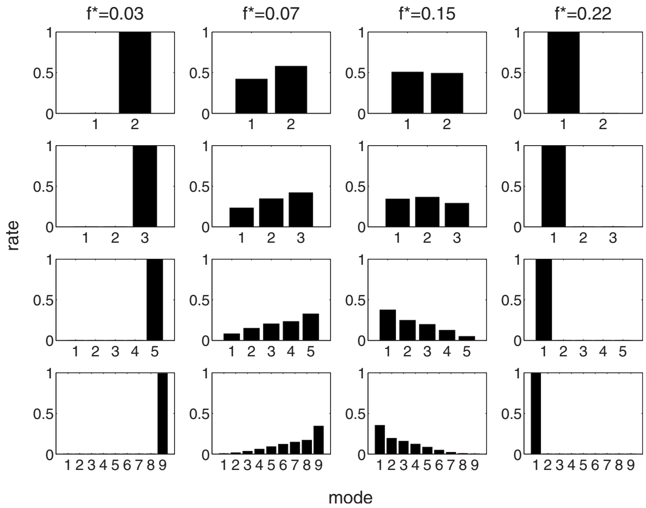

FIG. 5. Distribution of the followed oscillating modes for different values of f^* and different numbers of possible modes. Simulations are for systems with $N=2000$ and $\tau^*=0.2$. Partial synchronization appears when all modes are active and the oscillators continuously switch between them.

6). The f^* and τ^* parameters are fixed in such a way that partial synchronization is present. Independently of the number of considered modes, the synchronization level is monotonically increasing as a function of the number of oscillators in the system. Again, it can be observed that by increasing the number of possible modes the synchronization level decreases.

For a fixed number of oscillators and a fixed τ^* value, synchronization appears and disappears abruptly as a function of f^*, resembling a phase-transition phenomenon. Figure 7 illustrates this sharp variation, which becomes even sharper when more oscillators are present in the system. In agreement with the results sketched in Fig. 4, the synchronization level and the f^* interval, where the oscillators are synchronized, both decrease when the number of oscillating modes is increased.

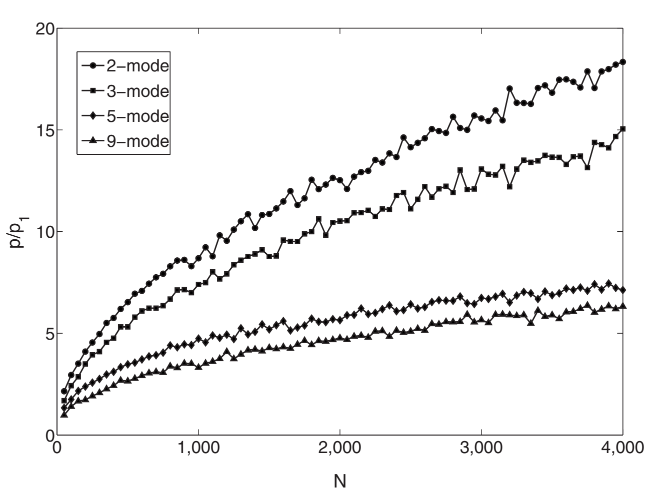

FIG. 6. The p/p_1 synchronization level as a function of the number of oscillators (N) in the system. Simulations are for systems with $f^*=0.1$ and $\tau^*=0.2$.

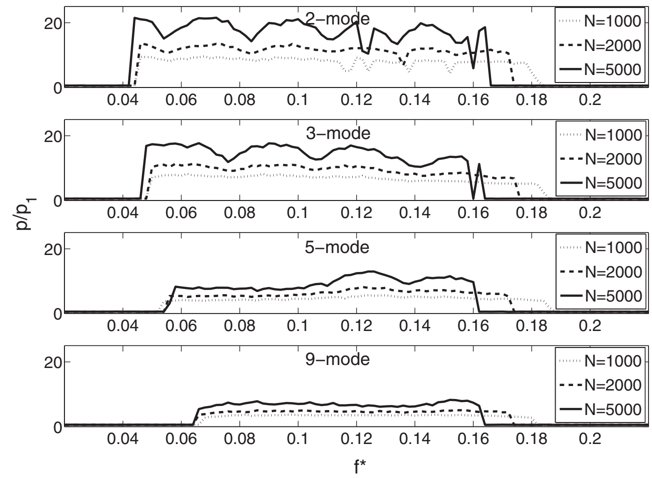

FIG. 7. The p/p_1 synchronization level of the oscillators as a function of f^*. Results for different numbers of modes. On each graph the different line styles correspond to different oscillator numbers as indicated in the legend. $\tau^*=0.2$ in all cases.

It is also worth studying what the best approximation for the T_m period of the global signal will be. Independently of the τ^* and f^* parameter values in the partially synchronized regime, its value was always around 0.9 units. This corresponds to the period of the mode with the largest τ_B value. There are however some f^* and τ^* values for which the $\Delta(T)$ function has a deep minimum also around 0.3 or 0.45 time units. The global minimum remains, however, always at 0.9 time units.

IV. LOCAL COUPLING

In many statistical physics models (such as the Kuramoto model), the dimensionality of the system and the topology of the interaction network are crucial. Let us investigate here what will happen with our multimode stochastic oscillator system if local coupling is considered instead of a global one. This means that each oscillator i detects only the pulses emitted by some of its neighbors, and we will compare the fixed f^* value with the local $f_{\{i\}}$ output given by these. The dynamics of the system is similar to the previous cases, the only difference is that optimization is done locally. Simulations were considered for two-mode oscillators with $\tau^*=0.1$ fixed. The main difference in the dynamics relative to the globally coupled case is that the strength of the pulse emitted by each oscillator is $1/S$ (and not $1/N$ as in the globally coupled case), where S denotes the number of interacting neighbors. This modification is necessary in order to compare the results for different interaction radii and different number of units, and also for getting the same average interaction strength in the thermodynamic limit.

Partial synchronization of the oscillator ensemble appears already in a one-dimensional (1D) chain. For a one-dimensional system one does not expect an order-disorder type phase transition in equilibrium systems. The system investigated here is however a typical nonequilibrium one, where the order arises as a result of the specific dynamical rules and not as a result of the competition between thermal

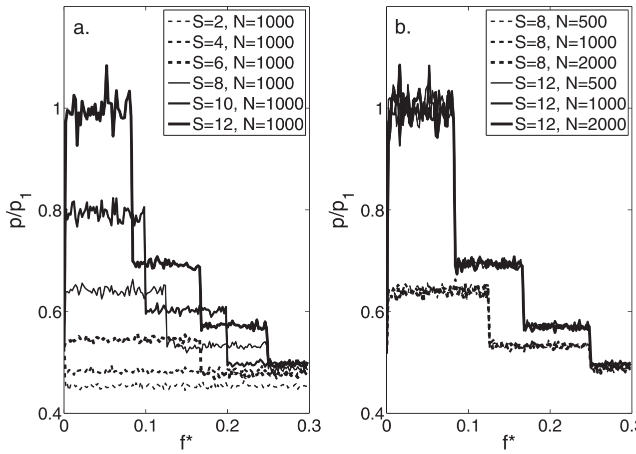

FIG. 8. Synchronization level as a function of f^*. Results for a 1D chain. (a) Different curves are results for different number of interacting neighbors (S), as indicated in the legend. (b) Curves are for two different values of S and different number of oscillators as indicated in the legend. $\tau^*=0.1$ for all cases.

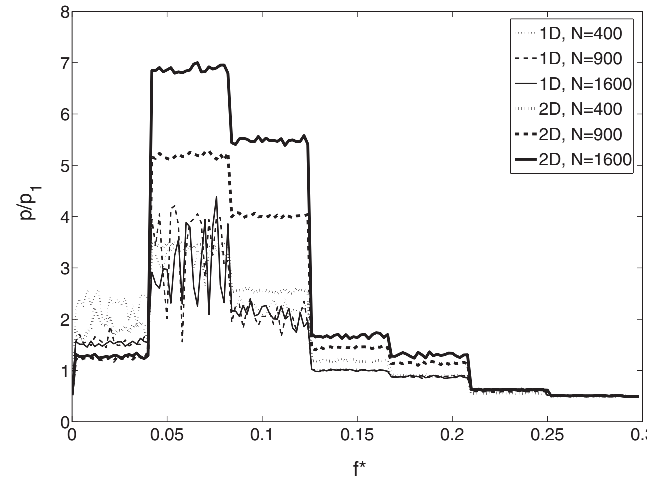

FIG. 10. Comparing the synchronization levels for the 1D and 2D interaction topologies. In both cases interactions with $S=24$ neighbors and various system sizes, N (as indicated in the legend), are considered. $\tau=0.2$ for all curves.

fluctuations and an ordering interaction. Results for different interaction radii, S, and different chain lengths, N, are presented in Fig. 8.

The results indicate that partial synchronization appears in a given f^* interval for all $S>4$ cases. Figure 8 also suggests a discontinuous steplike variation in p/p_1 as a function of f^*. The number of clearly distinguishable steps is roughly $S/2$. The synchronization level is clearly increasing with S, but it saturates quickly as a function of the number of units, N.

For oscillators placed on square lattice, the results are similar (Fig. 9). Again, a steplike variation for p/p_1 is obtained as a function of f^*. There is however an important difference relative to the 1D case; in the partially synchronized regime p/p_1 increases monotonically with the size of the system (N), resembling in this sense the globally coupled case. By comparing the results for the 1D and two-dimensional (2D) topologies (Fig. 10), this difference is evident. In the simulations leading to the results plotted in Fig. 10 each unit is interacting with $S=24$ nearest neighbors, and different system sizes, N, are considered. One will observe that both for the 1D and 2D topologies synchronization appears in the same f^* interval, the characteristic steps are at the same f^* values, but finite-size effects are qualitatively different.

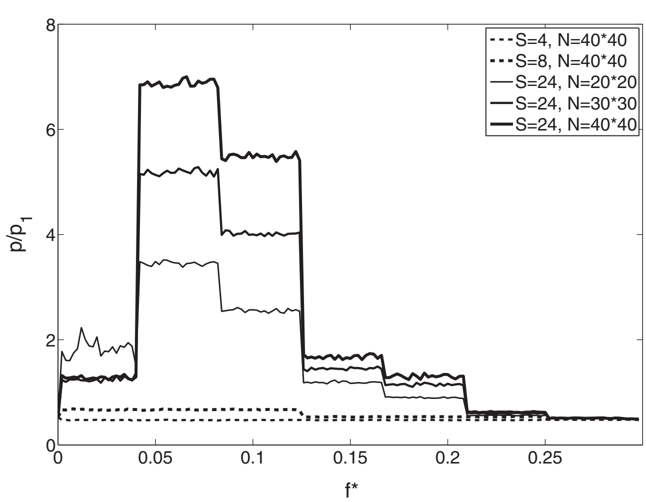

FIG. 9. Synchronization level as a function of f^* for a 2D square lattice topology. Different curves represent results for various number of interacting neighbors, S, and system sizes N. $\tau^*=0.1$ for all curves.

We conclude thus that similarly to the globally coupled systems, in the locally coupled case partial synchronization is also possible if the number of interacting neighbors is big enough. Interestingly, and in contrast to the globally coupled system, the $p/p1$ order parameter varies as a function of f^* in a steplike manner. As expected, the results plotted in Figs. 8 and 9 indicate that as the dimensionality and the number of interacting neighbors is increasing, the results are getting qualitatively similar to the ones obtained for globally coupled systems.

V. LOCALLY VARIABLE THRESHOLDS

A practically relevant objection against the discussed multimode stochastic oscillator model concerns the considered uniform nature of f^*. For a real system, the f^* threshold parameter could be slightly different for the units and could also fluctuate in time. A question that naturally arises is whether synchronization is still possible under such circumstances.

To answer this question, the two-mode globally coupled system was investigated by computer simulations. First a quenched uniform distribution was considered, assigning random $f_i^* \in [f_0^*-\delta, f_0^*+\delta]$ local thresholds to each oscillator, i. The dynamics of the oscillators was the same as for the globally coupled system in Sec II. Results for the $p/p1$ order parameter as a function of f_0^* and for various amounts of the disorder are plotted in Fig. 11(a). The results presented in this figure are for $\tau^*=0.2$ and the level of disorder is indicated by the $d=\delta/f_0^*$ ratio.

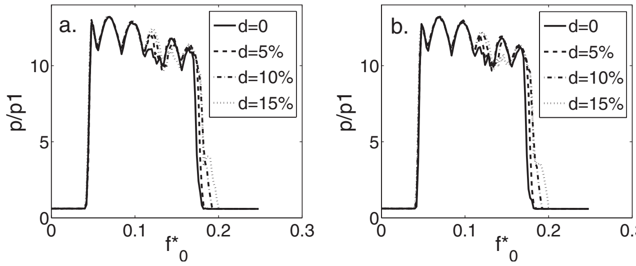

FIG. 11. The p/p_1 synchronization level as a function of f_0^*. Figure 11(a) is for quenched disorder, and Fig. 11(b) is for f^* values fluctuating in time. Different curves are for different disorder level, characterized by the $d=\delta/f_0^*$ ratio. For all curves $\tau^*=0.2$ and the number of oscillators are $N=2000$.

Simulation results suggest that a small amount of quenched disorder will not alter the observed collective behavior significantly. Our simulation results also revealed that for $d>15\%$ fluctuations, the collective behavior becomes much more complicated, and the simple order-disorder type transitions disappear.

Results for the case when the f_i^* values have timelike fluctuations are similar. One can choose again the f^* values with a uniform distribution in the $f_i^* \in [f_0^*-\delta, f_0^*+\delta]$ interval and investigate the synchronization level as a function of f_0^* by simulations. Results are plotted in Fig. 11(b). Similarly to the case of quenched disorder, up to a $d\leq 15\%$ disorder level, the synchronization properties are practically unchanged. For a higher disorder level, a more complex collective behavior is observable, and the clear order-disorder transitions will disappear.

VI. EXPERIMENTAL REALIZATION OF THE TWO-MODE SYSTEM

The highly nontrivial collective behavior of the previously described oscillator system challenged an experimental realization. Electronic oscillators capable of emitting and detecting light pulses were built (Fig. 12). Similarly to the theoretically studied system, these oscillators can have several oscillating modes, and their dynamics is similar to the considered oscillators in Secs. II and III.

The oscillators were developed around an 8 bit reduced instruction set computer (RISC)-core microcontroller from Atmel. Figure 12(a) shows the main elements: the microcontroller, the photoresistor, and the light-emitting diode (LED) is clearly visible. The circuit diagram of the oscillator is given in Fig. 13. To ensure a natural behavior, the internal *RC* oscillator of the microcontroller was chosen as time reference. The light intensity in the system is measured by a photoresistor in conjunction with three normal resistors of 10 $K\Omega$, 100 $K\Omega$, and 1 $M\Omega$, which enables several sensitivity ranges. It is also possible to setup a low pass filter (LPF) on the photoresistor signal by choosing one of the two 10 or 100 nF capacitors on the INT0 and INT1 pins. The reference signal U (which corresponds to the value of f^* in the model) applied to the oscillators and the voltage on the photoresistor [which depends on the selected $f(t)$ light intensity] can either be measured by a 10 bit resolution analogic-digital converter (ADC) or compared by a built-in hardware comparator. The output of the oscillator appears on the LED. A hardware pulse width modulation (PWM) can be used to alter this light intensity. The main advantage of this system is the software-oriented flexibility. The parameters are fixed and the hardware is setup by the program in the microcontroller. The stochasticity of the period is naturally satisfied due to the analog nature of the internal *RC* oscillator.

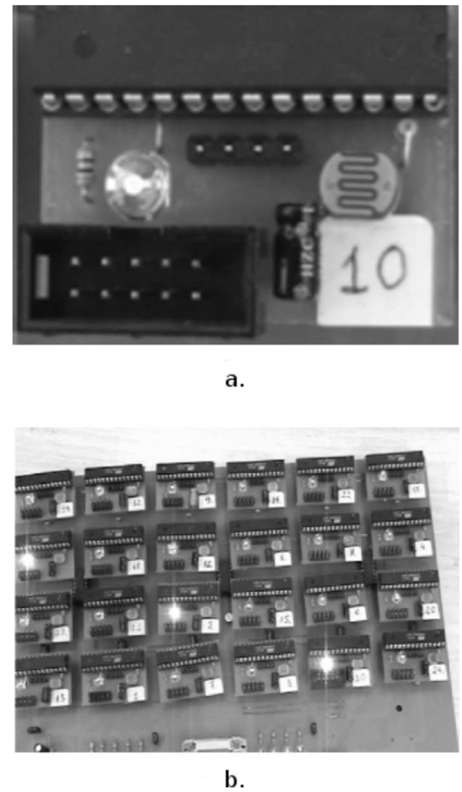

FIG. 12. (a) Picture of the considered electronic oscillators and (b) an ensemble composed by 24 oscillators placed on a circuit board.

The oscillators were programmed for reproducing the collective dynamics of the two-mode system experimentally. The parameters of the modes were chosen in agreement with the values considered in the simulations. The oscillators are placed on a circuit board [Fig. 12(b)], which is closed inside

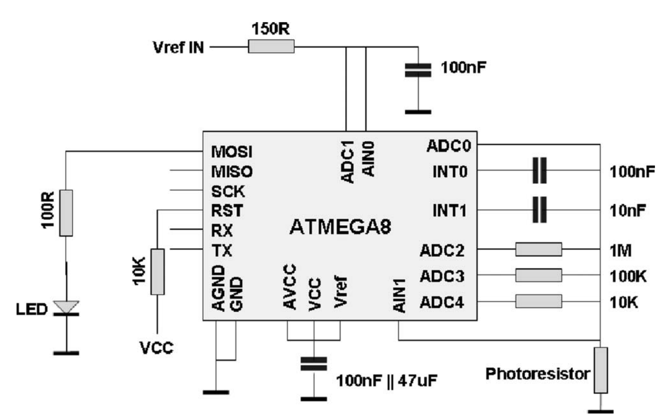

FIG. 13. Circuit diagram of an oscillator.

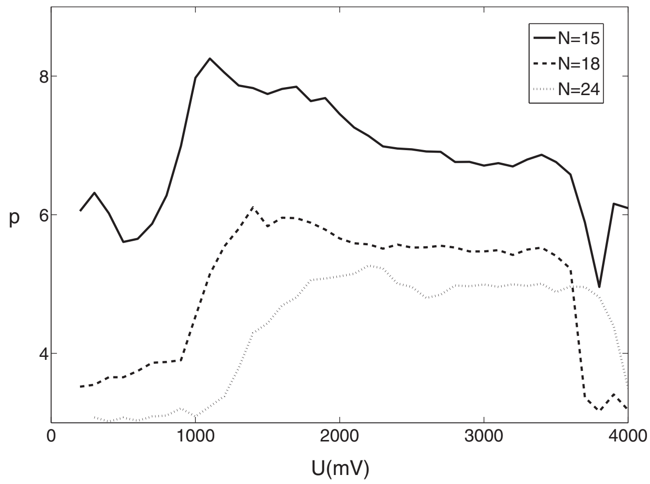

FIG. 14. Synchronization level, p, of the electronic oscillators as a function of the U reference voltage for different oscillator numbers.

a box to isolate the system from external light. In principle, each oscillator feels the light pulse emitted by the others. This is realized by placing a dispersive mirror on the top inside wall of the box. In this setup, the coupling between the oscillators is quasiglobal since the oscillators will detect the pulse emitted by the others with different intensity depending on their relative position on the circuit board. Through a simple interface, the circuit board is connected to a personal computer (PC). This interface allows the user to follow in time the state of each oscillator and to fix the relevant physical parameters and change them automatically in time. At the beginning of the experiment two reference signal values, the lowest (U_{min}) and the highest value (U_{max}), the ΔU step and the t_m total measuring time for each reference value are set. The driver program running on the PC will then do the experiments automatically for all specified U reference values and will save the state of the system with a 10 μs time resolution.

From the saved data one can easily compute the value of the p parameter [Eq. (7)], which characterizes the periodicity level of the global signal. This p value will be taken as the experimental order parameter for the synchronization level in the system. Experiments with $N=15$, 18, and 24 oscillators were performed. One can realize, already visually, that for a given U reference voltage interval this experimental setup reproduces the desired collective synchronization of the light pulses. Movies are shown in [17].

In Fig. 14 the value of p is plotted as a function of the U reference voltage. The shape of these curves is in agreement with the theoretical results plotted in Fig. 7; synchronization emerges only for a given interval of the U reference voltage value. On the other hand there is also an important disagreement. In all our simulation results, synchronization is enhanced by increasing the number of units. This observation is valid for both global and local coupling, and it is also valid for the case when the f^* values are fluctuating in time or space. Contrary to this, in the experimental realization it is found that by increasing the number of oscillators, the p periodicity of the global signal (or the level of synchronization in the system) decreases. This means that the collective dynamics in this electronic setup is different from the one observed in the model. A possible source for this difference is that in simulations all the elements are with linear characteristics, there are no reaction times or delays in the oscillator units. The output intensity of the units is steplike, the characteristic curve of the detecting device is linear, the signal travels with infinite velocity, and it is instantaneously detected. However, this is not true for the electronic system, where there are nonlinear characteristics for the LED and the photoresistor and time delays due to the finite reaction time of the active elements.

VII. CONCLUSIONS

Pulse emitting stochastic oscillators with several oscillating modes and with a simple optimization dynamics were investigated. In each cycle the oscillators are able to modify their oscillating mode (or oscillation period). This is done in order to optimize the intensity of the system's global output around a desired f^* threshold. As an unexpected secondary effect, nontrivial collective behavior appears in the system. This collective behavior is characterized by a periodic variation in the global output intensity and synchronous pulsing of the units. We called this behavior synchronization. It is hard to decide, however, if this self-organization is synchronization in a strict mathematical sense. To do this, one would first need to rigorously generalize the concept of synchronization to multimode oscillator units, which has not been done up to the present. Being aware of this, we use the term *nontrivial spontaneous synchronization* to characterize the obtained self-organization. The nontrivial adjective has thus a double meaning here. First it suggests that the observed collective behavior might not be a synchronization in a mathematically rigorous sense, and second it indicates that there is no obvious phase-minimizing force between the units.

The present study revealed that the observed nontrivial synchronization appears only for a given f^* threshold interval. The appearance and disappearance of the order (characterizing the synchronization level) as a function of f^* resemble an order-disorder type phase transition.

In this paper, this simple model was investigated considering several important aspects. First, a multimode generalization was studied. It has been shown that synchronization is not enhanced by increasing the number of modes. The synchronization is strongest in the two-mode version. In addition to this, the f^* parameter interval where synchronization is present becomes smaller as the number of possible modes is increased. For a system that contains continuous modes (infinite number of modes), synchronization is seemingly not possible at all. Synchronization is linked to the simple observation that the desired output intensity cannot be achieved in a stable manner with a discrete number of modes. Independently of the number of modes, the synchronization level was always enhanced by increasing the number of oscillators in the system.

As a second aspect, local versus global coupling was investigated. We learned that synchronization can appear even for a one-dimensional chain, assuming that the units interact with a sufficiently large number of neighbors. In the synchronized regime an interesting steplike variation in the synchronization level as a function of f^* is observed. It was found that whenever the dimensionality of the system is greater than one, the synchronization level is always enhanced by increasing the number of units.

Finally, the influence of locally varying or fluctuating f^* thresholds was investigated. It was found that for a not too high level of quenched disorder or timelike fluctuations in the f^* values, partial synchronization is still possible. The same type of order-disorder transitions are obtained as in the cases with a uniform value of the f^* threshold parameter.

A simple electronic realization of the two-mode oscillator system was also considered. Skeptics could visually convince themselves that this nontrivial synchronization is real. A detailed experimental investigation of the system shows that just like in the theoretical model, synchronization is present only for a given range of f^* values. However, contrary to the results obtained on the theoretical model, in the experimental system the synchronization becomes worst when the number of oscillators is increased. This contradictory result shows that the simple electronic realization is still different from the original two-mode oscillator model.

Besides the nontrivial nature of this model, the investigated system can be useful for describing several real systems. An immediate example is the fascinating social collective behavior in the rhythmic applause. As discussed in [11], all the experimentally observed aspects of rhythmic applause can be described by using the simple two-mode stochastic oscillator model. The interesting observation that the periodicity of the collective behavior is better than the periodicity of a simple unit can lead to several important practical applications as well. We consider thus that the present model can be of interest to a broad scientific community working in various research fields and can open a useful scientific discussion on nontrivial synchronizations in complex systems.

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