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The complex parameter space of a two-mode oscillator model

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HIGHLIGHTS

- A complex parameter-space is revealed for the dynamics of a simple two-mode oscillator ensemble.
- Different types of collective responses are identified.
- The classical two-mode stochastic oscillator model is generalized.
- The possibility to obtain non-trivial synchronization in natural systems is discussed.

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ABSTRACT

The parameter-space of a simple model that exhibits nontrivial spontaneous synchronization is thoroughly investigated. The model considers two-mode stochastic oscillators, coupled through emitted pulses by a simple optimization rule. Different types of collective responses are identified as a function of two relevant model parameters that are related to the optimization threshold and the periods of the two oscillation modes. It is shown that the investigated system exhibits partial synchronization under unexpectedly general conditions.

1. Introduction

Spontaneous synchronization appears in a large variety of systems in nature [1–3]. Well-known examples include biological systems such as fireflies flashing in unison [4] or crickets chirping together [5], rhythmic applause [6,7], pacemaker cells in the heart [8], the menstrual cycles of women living together [9], oscillating chemical reactions, mechanically coupled metronomes, pendulum clocks hung on the same wall, and many other systems.

Several mathematical models have been proposed to explain and describe the spontaneous synchronization phenomena in large interacting ensembles. Most of these models can be grouped into one of two broad categories that are distinguished by the nature of coupling between the oscillators: those that are based on phase coupling and those that are based on a pulse-like coupling.

The prototypical model for phase coupled oscillators is the Kuramoto model [10]. However, there are many systems in nature where one cannot define an associated periodic phase variable, thus the Kuramoto model is not a suitable description for them. In the case of systems where the interaction between oscillators is pulse-driven (such as fireflies, firing neurons, rhythmic clapping, etc.), *integrate and fire* type synchronization models are used [11–14].

A novel model that leads to synchronization in a non-trivial manner was introduced by Nikitin et al. [15]. Originally the model was inspired by the study of rhythmic applause [6,7], but it is relevant for all those complex systems where the units are oscillators with fluctuating periods, and can operate in different oscillation modes. In the simplest version of the model, the oscillation modes differ in their frequencies. Such systems are frequent in nature: a few well known examples are the ensembles of thalamocortical relay neurons [16], the unicellular alga Gonyaulax polyedra [17], or the American snowy cricket [18]. In this family of models, similarly to the integrate and fire models, the oscillators are coupled through emitted pulses. Interaction between the oscillators does not however favor synchronization

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in a direct way. Instead, the dynamics of the system is aimed to keep the average output in the system close to a desired f^* threshold level. This optimal threshold level is approached by switching between the oscillation modes. Whenever the average output pulse level in the system is lower than the desired one, the oscillators are working in a higher frequency mode to increase the average output intensity per unit time. Correspondingly, when the output level is higher than f^*, the oscillators are working in a lower frequency mode. Synchronization appears unexpectedly, as a side effect of this optimization. Numerical studies have shown that synchronization appears only for a certain parameter range of the model [19,20]. These studies were investigating the influence of the chosen f^* threshold level in the optimization dynamics. Some preliminary studies were done to investigate the influence of the randomness as well. In the case of bimodal oscillators, the effect of changing the ratio of the frequencies of the two modes was however not investigated at all in previous works, and also the phase space of the model was not mapped with sufficiently high accuracy before. In the present work we focus on exploring the behavior of the model as a function of the f^* threshold level and the ratio of mode periods, and explore the phase space with a much improved accuracy. We have found that this apparently simple model of bimodal oscillators has a parameter space with a rather complex and surprisingly non-trivial structure.

2. The two-mode stochastic oscillator model

2.1. Description of the model

The basic version of the model considers an ensemble of N identical bimodal, globally coupled, stochastic oscillators [15]. At any time, an oscillator can either be active, emitting a signal of strength $1/N$, or inactive, emitting no signal. Therefore the total output level of the system can vary between 0 and 1. These oscillators can be intuitively thought of as flashing units. For simplicity, from now on we shall refer to active ones as *lit* and inactive ones as being in an *unlit* or *dark* state. In accordance with this intuitive picture, the sum of the units' output levels can be thought of as the total light intensity in the system.

The units are stochastic bimodal oscillators. They can operate in two oscillation modes, one with a shorter and one with a longer period. These will be referred to as mode 1 and mode 2, respectively. The periods of the modes are random, and their mean values are denoted by τ_1 and τ_2. Let us define in the following the dynamics of the units in a more rigorous manner.

At the beginning of each period, the oscillators are dark. After a while, they light up and stay lit until the end of the period. A full oscillation period has a stochastic duration. For the sake of more precise mathematical description, let us consider three phases during a full oscillation period, labeled A, B and C, respectively. During phase A and B the units are dark, while during phase C they are lit. The duration of phase A, τ_A, is a random variable drawn from the interval $[0, 2\tau^*]$ with a uniform distribution. Let us denote the mean value of τ_A by $\langle\tau_A\rangle = \tau^*$. Phase A is merely a means of describing the stochasticity of the duration of an oscillation period. In this paper we shall assume that $\tau^* \ll \tau_1$. The duration of phase B, τ_B, can have two values, τ_{B1} and τ_{B2}, corresponding to the two oscillation modes. The duration of the lit phase, τ_C, is fixed. The average lengths of the periods of the modes is the sum of the mean durations of these three phases: $\tau_1 = \langle\tau_A\rangle + \langle\tau_{B1}\rangle + \langle\tau_C\rangle = \tau^* + \tau_{B1} + \tau_C$ and similarly $\tau_2 = \tau^* + \tau_{B2} + \tau_C$. Since the units stay lit for a greater fraction of the short period mode than the long one, the average light intensity will be larger when the units are oscillating in the short period mode.

The coupling between the oscillators is realized through an interaction that strives to optimize the total light intensity in the system, denoted f. At the beginning of each period, a unit decides which mode to follow based on whether the total light intensity, f, is greater or smaller than a threshold level f^*:

- If $f \le f^*$, the shorter period mode will be chosen. Since an oscillating unit stays lit for a greater fraction of a full period when it is operating in the short mode, this will help in increasing the average light intensity in the system.
- If $f > f^*$, the longer period mode will be chosen, reducing the average total light intensity in the system.

By this dynamic, each oscillating unit individually aims to achieve a total output intensity as close to f^* as possible, based on their instantaneous measurements of the output level. As a side effect of this optimization procedure, synchronization can emerge: the total output intensity of the system becomes a periodic function and the units will flash in unison [15,19–21].

The simple model presented in the previous paragraphs differs from the original one described in [15,19] only in the distribution of the duration of the stochastic phase, τ_A. In the original model, τ_A was exponentially distributed, and the behavior of the system was studied as a function of the variables $\tau^* = \langle\tau_A\rangle$ and f^*. The present paper focuses on the case when $\tau^* \ll \tau_1$, therefore the precise statistical distribution of τ_A does not influence the results significantly. The reason for choosing a uniform distribution for this study is that using a distribution defined on a bounded interval simplifies numerical modeling of the system. Contrarily with the previous works, in this paper the system is studied as a function of the parameter f^* and the ratio of the average periods of the two oscillation modes, τ_2/τ_1.

There are several variations possible on the basic version of the model. Some of these variations have been previously shown to also lead to synchronization. In [21] a version of the model with the same duration of the dark phase and a variable duration lit phase was studied, while in [20] it was shown that synchronization emerges also when using multimodal oscillators or when the f^* parameter is locally fluctuating. The lit phase can occur at the beginning or at the end of the oscillation period, leading to different behaviors. Finally, in this paper we will show that synchronization will occur even when a reversed optimization is used, that aims to achieve an output as different from the threshold f^* as possible.

Three versions of the bimodal oscillator model will be considered here: *model* 1. the basic model described above with a fixed-duration lit phase, and a variable duration dark phase; *model* 2. a model with variable duration lit phase and a fixed-duration dark phase; and *model* 3. fixed-duration lit phase and variable duration dark phase with a reversed choice of the long or short modes depending on the f^* value. This last case will be referred to as "anti-optimization" because the oscillators strive to achieve an output as different from f^* as possible. Partial synchronization will emerge in all three cases.

2.2. The order parameter

We need a quantitative measure to characterize the synchronization level of the system. The order parameter used in previous studies [15,19,20] measures the periodicity level of the output signal in a tedious manner, estimating the *periodicity level*, p of the global signal. When considering numerical modeling to simulate the system, the output level is computed at discrete points in time. Unfortunately the periodicity measure p used in previous studies turned out not to be practical when the output signal is highly periodic and is known at discrete points only. The finite time resolution limits the precision of finding of the best period, which in turn might have a significant effect on the computed value of the periodicity level p. The behavior of p as a function of the model parameters will no longer be characteristic of the dynamics, but will reflect the discretization of time variables. This becomes apparent only when modeling a larger number of oscillators than has been done previously and sampling the parameter space with a higher resolution.

Therefore, here we choose a different approach and define a simpler order parameter to characterize the synchronization level in the system. It has been observed in previous numerical studies that when synchronization emerges, the amplitude of the total light intensity function, $f(t)$, will be high in a few time-moments and as a direct consequence of this $f(t)$ fluctuates strongly in time. On the other hand, when no synchronization emerges, the output pulses are uniformly distributed, leading to a constant output level as a function of time, therefore the fluctuation of $f(t)$ is small. The standard deviation of the global light intensity

$$\sigma = \lim_{x\to\infty} \sqrt{\frac{1}{x}\int_0^x (f(x) - \langle f(x)\rangle)^2\, dx} \tag{1}$$

where

$$\langle f(x)\rangle = \lim_{x\to\infty} \frac{1}{x}\int_0^x f(x)\, dx, \tag{2}$$

captures this fluctuation, and suggests a simple way to characterize the synchronization level of the units. We assume in the following that the value of σ as the order parameter for synchronization. This proved to be a robust measure that is not sensitive to outliers in the signal and characterizes intuitively well the "flashing" behavior of the system. A nonzero σ value corresponds to a partially synchronized flashing dynamics.

3. Understanding synchronization in the two-mode oscillator model

Why spontaneous synchronization emerges in the two-mode stochastic oscillator model can be understood by following a simple theoretical argument, at least in some special cases.

For the sake of simplicity let us assume first that $\tau^* = 0$, i.e. there is no stochasticity in the system. Let us also assume that $\tau_2 > \tau_1$ and $\tau_2 < 2\tau_1$.

Following the dynamical rules of model 1, we can state the following evident facts:

1. The total output pulse amount ($P_{\text{out}}(\tau_2)$) during a time period τ_2 is

$$P_{\text{out}}(\tau_2) = \int_0^{\tau_2} f(t)\, dt = N \times (\tau_C/N) = \tau_C, \tag{3}$$

since during this τ_2 time interval each oscillator can and will fire only one time.

2. Each oscillator that finishes its pulse at a time moment t' so that $f(t') > f^*$ will continuously emit pulses with a stable τ_2 period. In other words, any time period during which $f(t)$ is continuously larger than f^* can only increase in duration in subsequent oscillations and definitely cannot decrease.

Taking into account the above facts, let us first investigate whether an $f(t) = K$ constant output intensity is stable in the $N \to \infty$ limit. If $K > f^*$ this is possible only in the case when $P_{\text{out}}(\tau_2) > \tau_2 f^*$, leading to the condition $f^* < \tau_C/\tau_2$. On the other hand for $K < f^*$, everybody would fire with period τ_1, and the necessary condition is $\tau_1 f^* > P_{\text{out}}(\tau_1) = \tau_C$, leading to $f^* > \tau_C/\tau_1$. This means that even in the thermodynamic limit ($N \to \infty$) for $\tau_C/\tau_2 < f^* < \tau_C/\tau_1$ one cannot obtain a global signal with $\sigma = 0$ order parameter, thus in this finite interval some level of synchronization should be observable. Moreover, following a similar argument, one can easily show, that in this f^* interval, the $f(t)$ global output will have to intersect the $f = f^*$ line. Those domains where $f(t) > f^*$ will have to grow (since they cannot decrease), and those domains where $f(t) < f^*$ will constantly change, either nucleating new points where $f(t) > f^*$, or decreasing. In the final stable configuration, a pulse-like signal has to be formed, since for this f^* interval it is not possible to achieve $f(t) > f^*$ for any $t \in [t_0, t_0 + \tau_2]$ interval. This pulse-formation is schematically illustrated in the appropriate f^* region in Fig. 1.

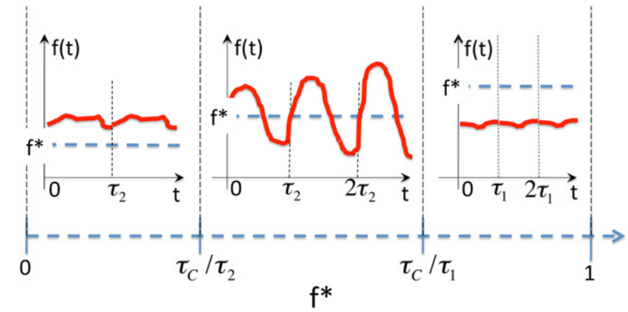

Fig. 1. Dynamics of the two-mode oscillator system in some special cases and for different f^* parameter values in the noise free limit ($\tau^* = 0$). For $f^* < \tau_C/\tau_2$ any signal with $f(t) > f^*$ repeats itself with a τ_2 period. For $f^* > \tau_C/\tau_1$ a signal with $f(t) < f^*$ is periodic with the period τ_1. In the intermediate f^* case, $\tau_C/\tau_2 < f^* < \tau_C/\tau_1$, in most of the cases a synchronization of pulses will emerge.

More generally, for the $\tau^* = 0$ case the following statements are also true:

- A signal with a pulse-like shape ($f(t) > f^*$ or $f(t) = 0$ for $t \in [t_0, t_0 + \tau_2]$) becomes periodical with a period τ_2. The maximum number of such pulses, ν, in one period should be less than (or equal to) $\lfloor 1/f^* \rfloor$, where $\lfloor x \rfloor$ denotes the integer part of x. This, later condition results from the fact that the minimum length of a pulse is τ_C and the total output amount is $P_{\text{out}}(\tau_2) = \tau_C$. Evidently, the $\nu\tau_C f^* < \tau_C$ inequality is satisfied, yielding $\nu < 1/f^*$.
- For $f^* < \tau_C/\tau_2$, any signal satisfying $f(t) > f^*$ for an interval $t \in [t_0, t_0 + \tau_2]$, repeats itself periodically with a period τ_2. In Fig. 1 we sketch this behavior for the $0 < f^* < \tau_C/\tau_2$ region.
- For $f^* > \tau_C/\tau_1$, any signal satisfying $f(t) < f^*$ on an interval $t \in [t_0, t_0 + \tau_1]$ repeats itself periodically with a period τ_1. This fact is suggested in Fig. 1 for the $\tau_C/\tau_1 < f^* < 1$ region.

The above facts indicate that in the noise free limit, independently of the initial starting conditions, there is an intermediate $\tau_C/\tau_2 < f^* < \tau_C/\tau_1$ region, where a non-zero order parameter is expected, this meaning a partial synchronization of the pulses of the two-mode oscillators. This synchronization phenomenon can be related to the synchronization observed previously in chaotic oscillators or rotors with irrationally related frequencies [22].

It is not trivial to understand the influence of a nonzero $\tau^* \neq 0$ noise level on the system's dynamics, since it will induce two competing effects. First, all the above mentioned stable and periodic dynamics are perturbed, and as a result of this one expects in the partially synchronized regime a less periodic output signal. On the other hand however, noise can help the system to reach the stable attractors, and in the thermodynamic limit ($N \to \infty$) will help the system to forget sooner the peculiarities of the initial starting conditions. By this, the dynamics becomes independent of the chosen starting configuration, leading to a dynamical system that can be controlled solely through the f^*, τ^* and τ_2/τ_1 parameters. One would expect thus that a small amount of randomness (meaning $\tau^* \ll \tau_1$) in the dynamics will not preclude the possibility to obtain partially synchronized states.

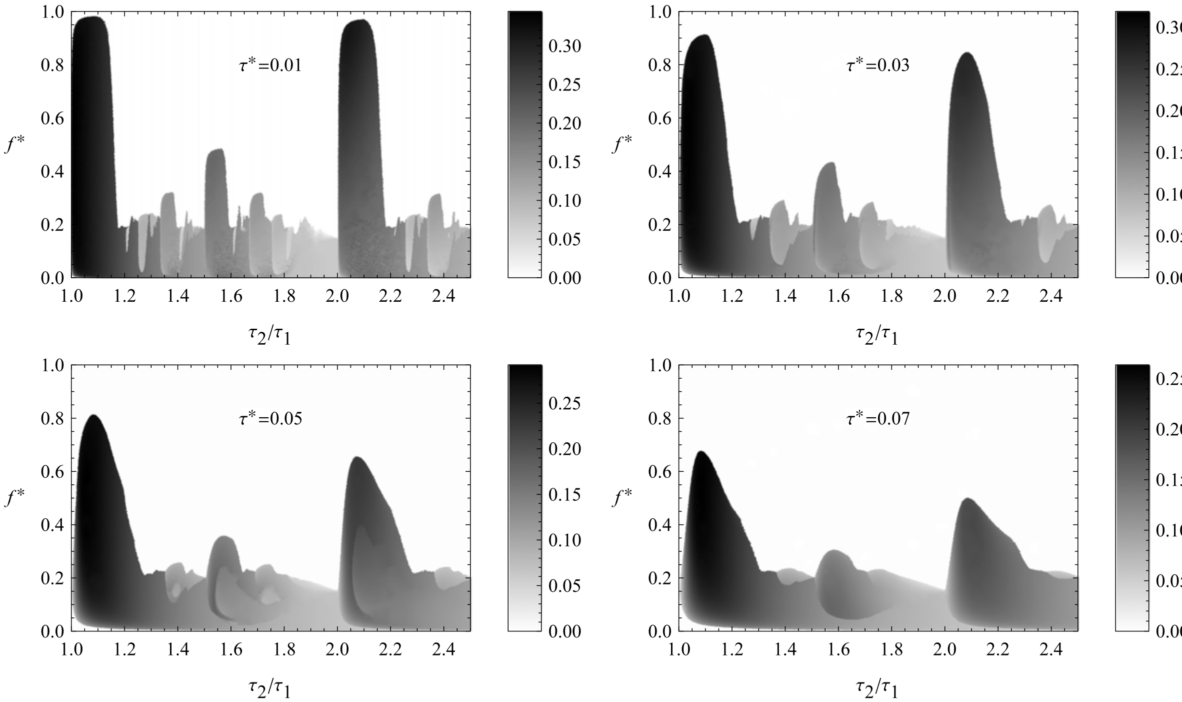

Fig. 2. The parameter space of the original two-mode oscillator model is shown for four different values of the τ^* parameter. The shades of gray indicate the value of σ: white regions correspond to no synchronization, for more details see the attached colorbar. The other simulation parameters were $N = 10\,000$ and $\tau_C = 0.15$.

4. Details of the computer simulation method

Due to the inherent difficulties discussed in the previous section, the stochastic multimode oscillator system was investigated by computer simulations. The value of the σ order parameter was investigated as a function of the threshold parameter, f^*, and the ratio of the average periods of the two oscillation modes, τ_2/τ_1, when τ^* is much smaller than τ_1 and τ_2. In order for the system to reach equilibrium, and in order that the stationary state be independent of the initial state, it is necessary that $\tau^* > 0$. For all simulations we have fixed the values of τ_1 and τ^* to be $\tau_1 = 1.0$ and $\tau^* = 0.03$. It is important to note that this value of τ^* does not approximate perfectly the $\tau^* \to 0$ limit. Reducing its value further will result in still noticeable changes in the structure of the phase space as it is illustrated in Fig. 2. However, since the time needed to reach equilibrium was found to grow proportionally with $1/(\tau^*)^2$, the available computation resources imposed a limit on reducing the value of τ^*.

The order parameter, σ, was calculated from a time average on an interval given by a large *integer* number of periods (when the signal was periodic), for increased accuracy.

4.1. Efficient simulation methods

Previous studies [15,19–21] used a direct method of simulating the ensemble of bimodal oscillators. The easiest way to simulate the ensemble on a computer is by updating the state of each oscillator in discrete time steps.

Obtaining the results presented in this paper required a huge computational effort and it was made possible by choosing an efficient way to model the system and map the parameter space. The most direct way to simulate the oscillators is the following: let the model be discretized in time and let Δt be the time step. Then, let the state of each oscillator be stored separately, making the oscillator the basic unit of the model, and in each time step update the state of each oscillator. This method requires computer time proportional to $Nt_{\max}/\Delta t$ to simulate the dynamics of N oscillators for a time-interval $t_{\max}$, i.e. the simulation will slow down proportionally to the time resolution Δt.

A significant speedup can be achieved if the basic unit of the simulation is chosen to be the events that happen to oscillators instead of oscillator states. Events can be an oscillator turning on (lighting up), an oscillator turning off (darkening), or the start of a new oscillation period. Events are processed sequentially, in chronological order. Processing an "on" event causes the total output intensity to increase by $1/N$, while an "off" event causes it to decrease by $1/N$. A "period start" event causes a new event to be created in the future. Since there is an upper bound on the time length of a period, events need to be stored only up to a fixed time ahead in the future. A fixed-length array, containing only the number of each type of event for successive periods of time of length Δt can be used for this. This method requires a time proportional to $\alpha Nt_{\max} + \beta t_{\max}/\Delta t$, where the constants α and β depend on implementation details. Since in practical implementations $\beta \ll \alpha$, increasing the time resolution does not significantly increase the simulation time, making accurate and fast simulation possible.

4.2. Mapping the parameter space

Another essential optimization technique used in the present study was sampling the parameter space adaptively. The most common way of mapping the parameter space of a system is by simulating the dynamics of the system and calculating the value of the order parameter for each point in a rectangular lattice of points in the parameter space. Increasing the resolution of the lattice twofold causes a 2^d-fold increase (with d the dimensionality of the parameter space) in the number of sample points and consequently a similar increase in the required computation time.

A better approach is using adaptive sampling, i.e. increasing the number of sample points only in the regions where the behavior of the order parameter is "interesting". An adaptive sampling

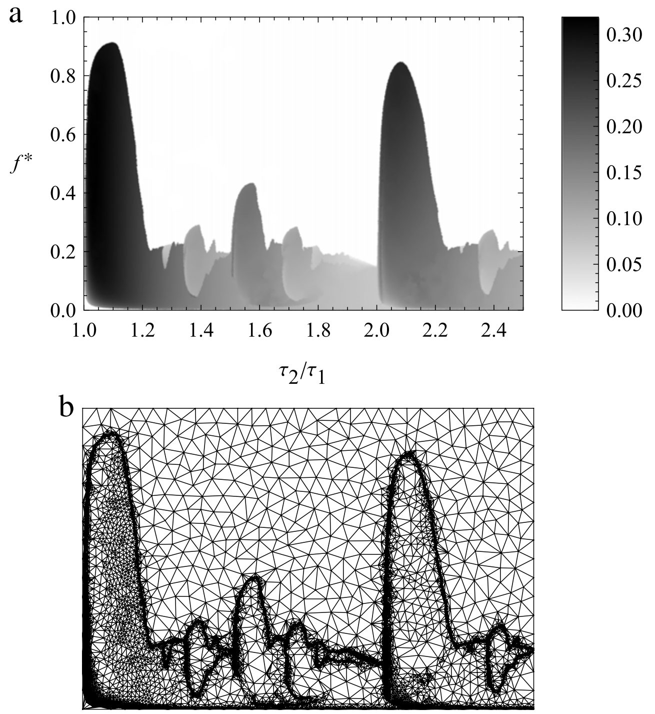

Fig. 3. The investigated parameter space of the model 1 system. The other simulation parameters were $N = 10\,000$ and $\tau_C = 0.15$. (a) The order parameter σ as a function of the threshold level f^* and the ratio of the average periods of oscillation modes τ_2/τ_1. (b) Illustration of the adaptively subdivided mesh in the same parameter space as figure (a).

method is defined by two choices: 1. a quantitative definition of "interesting" regions, i.e. the criterion for adding more sampling points 2. choosing the exact location of new sample points. Appropriate choices for these should depend on the behavior of the order parameter as a function of system parameters in the given model. In our system, the parameter space is two-dimensional and the value of the order parameter is constrained to be in the interval [0, 1]. The order parameter varies smoothly inside regions that are separated by abrupt and discontinuous transitions, as seen in Fig. 3(a) for the case of model 1. Due to the Monte Carlo simulation technique used, the results might be noisy. Based on these considerations, a simple adaptive sampling scheme was chosen. We start with an arbitrary set of roughly equally spaced sample points and compute the function value (order parameter) in them. Then in each refinement step, compute the Delaunay triangulation of the point set, and insert a new sample point in the midpoint of each triangle edge if the edge is longer than a threshold and the function values in the two ends differ by more than another threshold.

This method will trace the shape of the discontinuities very well. Since most sample points get inserted close to the discontinuities, the increase in the number of points is close to linear than quadratic in the resolution. Adaptive sampling makes it possible to map the parameter space of the system with high precision with relatively few sample points. The adaptively subdivided mesh for the case of model 1 is illustrated in Fig. 3(b). The disadvantage of using such a method is that only those features that have a size comparable to the resolution of the initial mesh will be discovered with certainty. The features that *are* discovered are mapped with high precision, and the method can produce a detailed looking output, as in Fig. 3(a). One has to keep in mind however that the precision of the mapping differs from region to region.

5. Results

As described already in Section 2, three versions of the model were studied:

model 1 is the basic model described in [15,19]. The duration of the lit phase, $\tau_C = 0.15$, is fixed while the duration of the dark phase, τ_B, can have a greater and a smaller value (τ_{B2} and τ_{B1}). The oscillation modes are chosen so as to *optimize* the output intensity f towards f^*, i.e. minimize the difference between f and f^*. The mapped phase space is shown in Fig. 4(a).

model 2 has a fixed duration dark phase. This means that τ_B is fixed while τ_C can have a greater and a smaller value, τ_{C2} and τ_{C1}. The oscillation modes are chosen to minimize the difference between f and f^* (*optimization*). The phase space is shown in Fig. 4(b).

model 3 is similar to model 1, except that the oscillation modes are chosen so as to make f as different from f^* as possible: if $f \leq f^*$ then τ_2 is chosen, while if $f > f^*$ then τ_1 is chosen (*anti-optimization*). The phase space of this model is shown in Fig. 4(c).

The time resolution chosen for all simulations presented in the above figures was $\Delta t = 1/1000$. Simulations were performed up to time $t_{\max} = 10\,000$ to ensure that the system reaches a steady state and the order parameter does not change any more. Then, the period of the signal was estimated using a simple auto-correlation method and the order parameter σ was calculated based on an integer number of periods covering approximately the last 500 time units in the data. The simulations were run for an ensemble containing a number of oscillators ranging from $N = 10$ to $N = 100\,000$. We found that the order parameter curves do not change significantly when increasing N above 3000. This is nicely visible if one studies finite-size effects for horizontal and vertical sections in Fig. 4(a). Characteristic results are shown in Fig. 5(a) and (b), respectively.

The obtained results indicates that the f^*–τ_2/τ_1 parameter space of the system has a complex structure in the case of all three models, and consists of several partially synchronized regions. The regions are separated by discontinuities in the value of the order parameter σ. In each of these regions, the output intensity function of the system, $f(t)$, is periodic but has a different shape. Some of the shapes of $f(t)$ that occur for different parameter values in model 1 and model 2 are shown in Figs. 6 and 7. These widely different global signals suggest that the dynamics of this simple system is extremely rich and many different phases are possible. The abrupt appearance of synchronization on the region boundaries and the sudden changes in the shape of the output function resemble phase transitions.

6. Summary

In this work we have studied three variations of the stochastic bimodal oscillator model, and mapped their behavior as a function of the imposed threshold level, f^*, and the ratio of the average periods of the oscillation modes, τ_2/τ_1. It was found that the ratio of the oscillation modes, which was not considered as a parameter of this model before, has a significant influence on the behavior of the system. The f^*–τ_2/τ_1 parameter space has a complex structure with several regions, separated by sharp discontinuities of the chosen order parameter. The shape of the total output intensity function differs between these regions.

An interesting finding of the present work is that synchronization in such models appears under an unexpectedly wide range of conditions. All previous studies have considered a dynamics which intends to minimize the difference between the total output

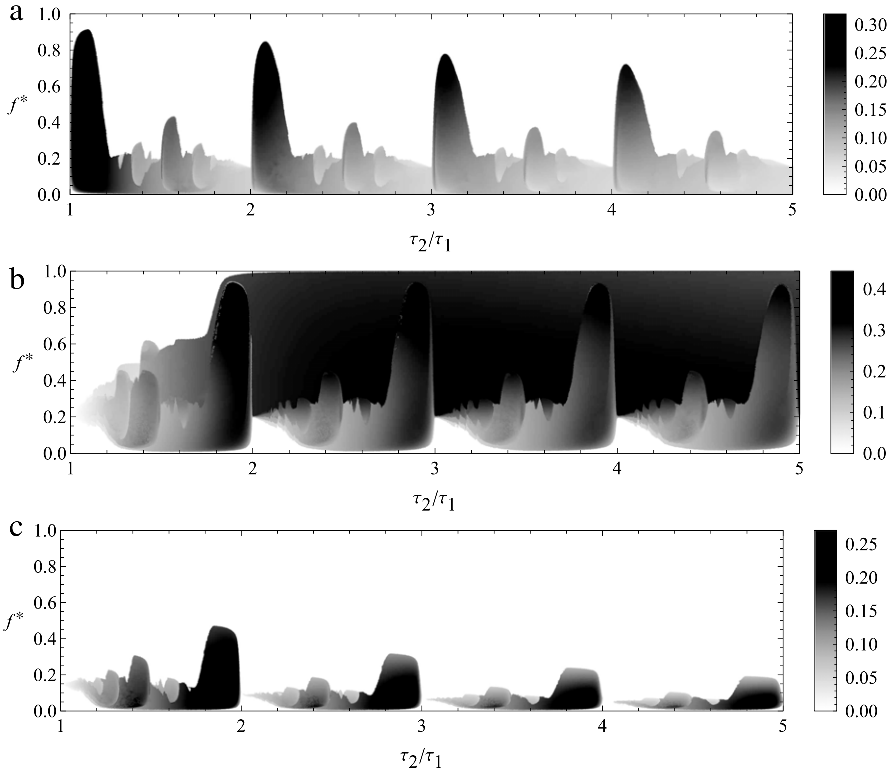

Fig. 4. The parameter space of model 1, 2 and 3. The order parameter σ is shown as a function of the threshold level f^* and the ratio of the average periods of oscillation modes τ_2/τ_1. The shades of gray indicate the value of the σ order parameter: white regions correspond to $\sigma = 0$, i.e. no synchronization, see also the attached legend. (a) The parameter space of model 1. The simulation parameters were $N = 10\,000$ oscillators and $\tau_C = 0.15$. (b) The parameter space of model 2. $N = 10\,000$, $\tau_B = 0.8$. (c) The parameter space of model 3. $N = 10\,000$, $\tau_C = 0.15$.

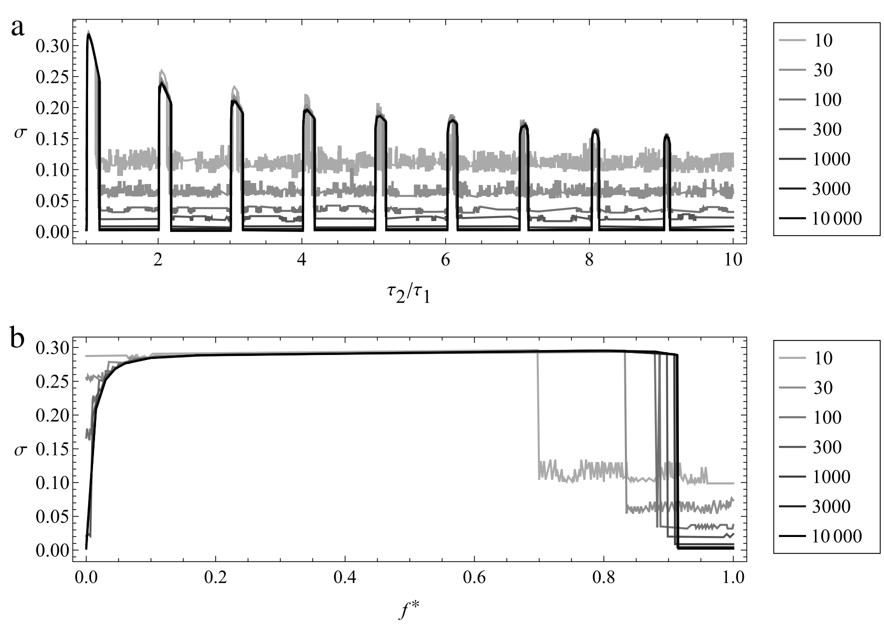

Fig. 5. Finite-size effects for the system. The order parameter σ in model 1 as a function of: (a) τ_2/τ_1 for $f^* = 0.5$; and (b) f^* for $\tau_2/\tau_1 = 1.1$, for different numbers of oscillators. The curves do not change visibly when the number of oscillators is increased above $N = 3000$. The $\tau_C = 0.15$ value was used.

intensity of the system and a threshold level f^*. Here, we have found that synchronization will emerge even when an "anti-optimizing" dynamics is used that will maximize the difference between the output level and f^*. One can also study models where the stochasticity is introduced in the lit phase, C, or models where both the lit and unlit phases are characterized with a stochastic time-length. We have found that despite its simplicity, the behavior of such models changes in an unexpectedly complex manner as a function of the studied parameters, and as a result of this the parameter space of such models is also rather complex.

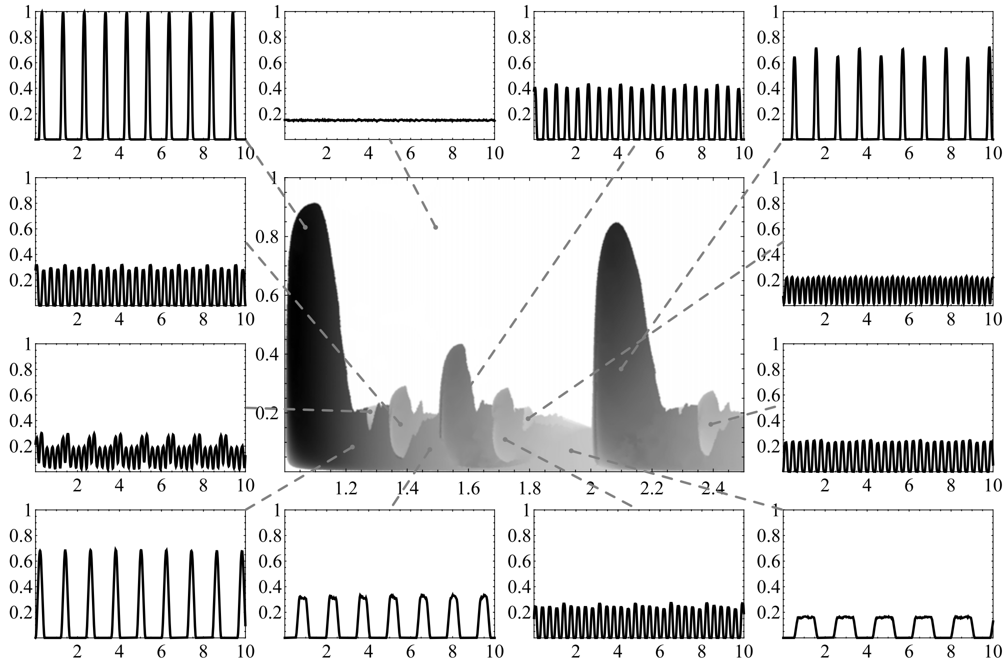

Fig. 6. The shape of the output signal for model 1 is shown for various points in the f^*–τ_2/τ_1 parameter space. ($N = 10\,000$ oscillators and $\tau_C = 0.15$). The value of the σ order parameter is illustrated with the same grayscale code as in Fig. 4(a).

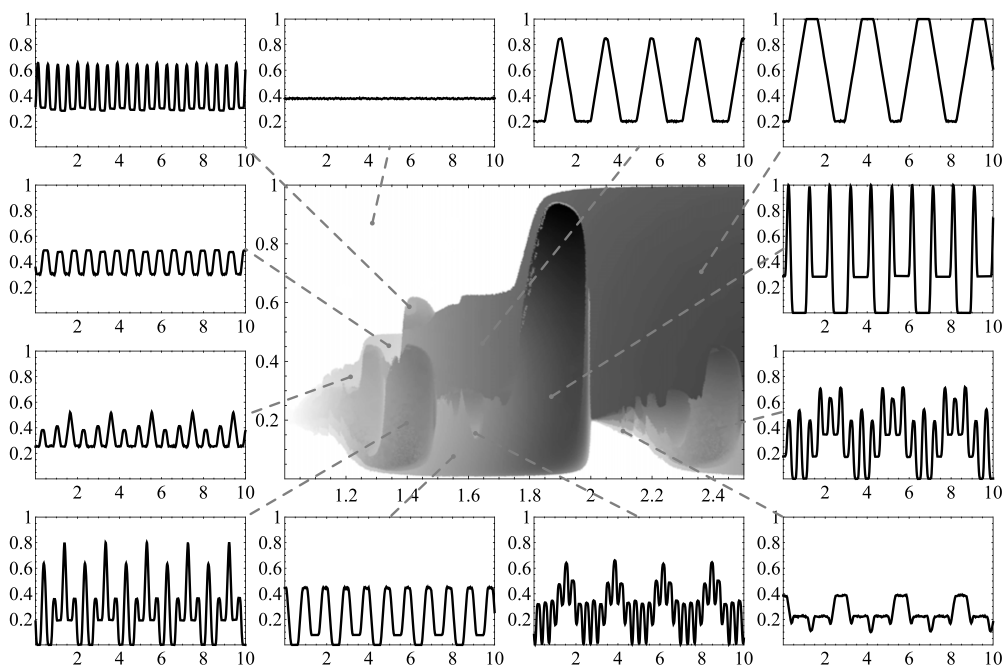

Fig. 7. The shape of the output signal for model 2 is illustrated for various points in the f^*–τ_2/τ_1 parameter space. ($N = 10\,000$ oscillators, $\tau_B = 0.8$). The value of the σ order parameter is illustrated with the same grayscale code as in Fig. 4(b).

Acknowledgments

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