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ORDER-DISORDER TRANSITIONS IN A MINIMAL MODEL OF SELF-SUSTAINED COUPLED OSCILLATORS

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Abstract. Spontaneous synchronization of interacting pendulum clocks offers a fascinating and pedagogical demonstration for order-disorder type phase transitions. A minimal model consisting of self-sustained harmonic oscillators placed on a common platform is used to model such systems. Varying the frequency of the oscillators, the mass of the platform and the friction coefficient of the platform we illustrate transitions from partially synchronized to unsynchronized states. Finite size effects are studied and the results are discussed in view of recent experiments performed with metronomes. We find a highly non-trivial trend for the synchrony level as a function of the number of oscillators placed on the platform: there is a an optimal number of oscillators for which the maximum synchrony level is reached.

Key words: Spontaneous synchronization, coupled oscillators, nonlinear systems.

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1. INTRODUCTION

Ever since being observed by Huygens [1] in 1665, spontaneous synchronization of coupled pendulums has been widely studied as an example of complex behavior arising in a simple mechanical system. Pendulums, metronomes, simple oscillators and other variations of the Huygens clocks systems have been studied and discussed. On the spectrum of simplicity of the studied models the most basic one is the Kuramoto model [2], which constitutes a system of ideal coupled rotators and demonstrates spontaneous synchronization and order-disorder type phase transitions. On the other hand, some of the most realistic and detailed theoretical models of this type of system was studied by Kapitaniak *et al.* [3, 4], taking into account multiple parameters and developing a realistic model that agreed well with the experiments performed on pendulum clocks. The goal of the present study is to find the middle ground and take a look at a system that is a rather simple model of coupled self-sustained oscillators, yet still possesses the key characteristics of the more realistic

models, such as the presence of damping, driving, and a variance in their natural frequencies. The goal is to determine whether some trends found in previous experimental studies are general features of such type of system or are the result of the specifics of the system. We are particularly interested in reproducing the experimentally observed order-disorder type phase transitions with a fixed ensemble of self-sustained oscillators.

One of the experimentally most studied ensemble of self-sustained oscillators is a system of metronomes coupled *via* a movable platform. As it is illustrated on Figure 1 several setups are imaginable: one can put the metronomes on a platform that can roll on two cylinders (bear cans) as illustrated in Figure a, on a platform that is suspended in a bridge-like manner (Figure b.) or on a rotating disc (Figure c.). Metronomes have the advantage that their synchronicity is easy to follow acoustically

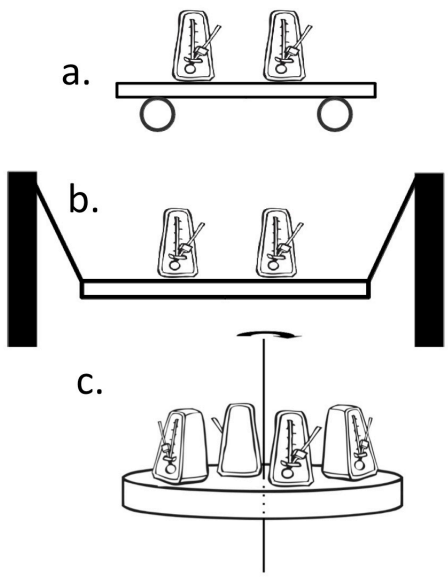

Fig. 1 – Simple experimental setups for demonstrating the emergent synchronization in coupled metronome systems. Metronomes are placed on a common movable platform. (a) the platform can freely roll on two cylinders; (b) the platform is suspended on a bridge-like manner; (c) the platform can freely rotate around a vertical axes.

and they are relatively cheap devices. Studies on synchronization in such systems were first done by Pantaleone [5] who reported the emergence of only in-phase synchronization. This was somehow in contrast with earlier studies of pendulum clocks [4], that found both in-phase and anti-phase synchronization. Theoretical studies of coupled metronomes by Ulrichs *et al.* [6] found Kuramoto type phase transitions and

reported also finite-size effects. More recent experimental and theoretical studies of metronomes placed on a rotating platform by Boda *et al.* [7, 8], confirmed the presence of phase transitions in this system. They found an order-disorder type phase transition as a function of the nominal frequencies fixed on the metronomes, which becomes more prominent with the increase of the number of coupled metronomes. Additionally, according to their studies, there are order-disorder transitions observable (i) when rotating the oscillating plane of the metronome's pendulum relative to the radial direction on the disc, (ii) as a function of the metronomes distance relative to the vertical rotation axes, (iii) as a function of the friction coefficient of the platform and (iv) as a function of the moment of inertia of the platform relative to the vertical rotation axes. Interestingly in some cases, like (i)-(ii) the order-disorder transition gets sharper with increasing metronome number, and in other cases like (iii)-(iv) the transition gets smoother and less prominent with the increase of the number of units. This study will attempt to numerically reproduce some of these phase transitions in a simple model system. Specifically we will focus on the order-disorder transition that takes place as function of the natural frequencies of the oscillators and the one that takes place as a function of the the friction coefficient and mass of the platform. Finite size effects will also be considered.

2. THEORETICAL MODEL

We consider a minimal model that captures the essence of the emerging synchronization phenomena observed for metronomes. Some of the order-disorder transitions observed in experiments and the more realistic model are successfully reproduced by this simplified approach.

We consider a platform of mass M with two identical masses m attached to it through ideal springs with spring constant k. This system represents two identical and ideal harmonic oscillators coupled by the common platform (Figure 2). We denote by x_1 and x_2 the spring deformation values and x_0 is the absolute coordinate (relative to the chosen inertial Reference frame) of the platform.

As a first study, let us assume that friction and driving are absent. The Lagrange function for this system is

$$L = \frac{1}{2}M\dot{x}_0^2 + \frac{1}{2}m(\dot{x}_1+\dot{x}_0)^2 + \frac{1}{2}m(\dot{x}_2+\dot{x}_0)^2 - \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2, \tag{1}$$

where the first term is the kinetic energy of the platform, the second and third terms stands for the kinetic energy of the oscillators relative to the chosen inertial reference frame, and the last two terms are the potential energies of the oscillators. The Euler-

Lagrange equations of motion can be written in the form

$$\begin{aligned} M\ddot{x}_0 &= k(x_1+x_2) \\ m\ddot{x}_1 &= -kx_1 - m\ddot{x}_0 \\ m\ddot{x}_2 &= -kx_2 - m\ddot{x}_0 \end{aligned} \tag{2}$$

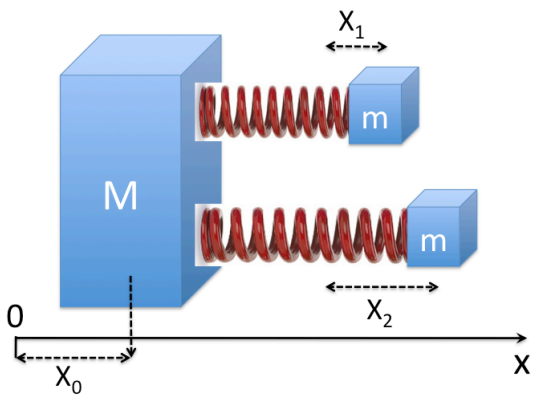

Fig. 2 – The coupled oscillator system.

Eliminating the $\ddot{x}_3$ terms, the system allows for an exact analytical solution. Assuming the initial positions of the oscillators $x_1(0) = 1$, $x_2(0) = a$, and that they are in rest relative to the platform ($\dot{x}_1(0) = 0$, $\dot{x}_2(0) = 0$) the exact solutions for $x_1(t)$ and $x_2(t)$ are

$$\begin{aligned} x_1(t) &= \tfrac{1}{2}\left((1-a)\cos\tfrac{\sqrt{k}t}{\sqrt{m}} + (1+a)\cos\tfrac{k(2m+M)t}{\sqrt{kmM(2m+M)}}\right) \\ x_2(t) &= \tfrac{1}{2}\left((a-1)\cos\tfrac{\sqrt{k}t}{\sqrt{m}} + (1+a)\cos\tfrac{k(2m+M)t}{\sqrt{kmM(2m+M)}}\right) \end{aligned} \tag{3}$$

The exact solution shows, that synchronization is possible in such systems only if the system starts in synchrony. For all other initial conditions the system will not synchronize. In case the oscillators are different, synchronization is never observed. Damping and driving are thus essential in order to get a stable phase-locking synchronization. One can also argue using the second law of thermodynamics that order cannot emerge in such system. If one considers the oscillators and the platform a closed system, emerging order would mean a spontaneous decrease in the total entropy of the system, which would contradict the second law of thermodynamics. Dissipation through friction and driving makes the entropy balance more complicated and would finally increase the total entropy, so the second law of thermodynamics is not violated.

Let us assume from now on the practically interesting case when there are many different oscillators, and the differences between the oscillators are introduced through the spring constants. The equations of motions for such a system can be derived in a similar manner for arbitrary number of oscillators placed on the platform. The Lagrangian of the system writes as

$$L = \frac{1}{2}M\dot{x}_0^2 + \sum_{i=1}^{N}\frac{1}{2}m(\dot{x}_i + \dot{x}_0)^2 - \sum_{i=1}^{N}\frac{1}{2}k_i x_i^2, \tag{4}$$

The Euler Lagrange equations of motion yield

$$\begin{aligned} M\ddot{x}_0 &= \sum_{i=1}^{N} k_i x_i \\ m\ddot{x}_i &= -k_i x_i - m\ddot{x}_0 \end{aligned} \tag{5}$$

Adding damping and driving to the system makes spontaneous synchronization possible. The equations of motion can be now generalized by adding friction and driving terms:

$$\begin{aligned} M\ddot{x}_0 &= \sum_{i=1}^{N} k_i x_i + C_1 \sum_{i=1}^{N} \dot{x}_1 - C_0\dot{x}_0 - \sum_{i=1}^{N} F_i \\ m\ddot{x}_i &= -k_i x_i - m\ddot{x}_0 - C_1\dot{x}_i + F_i \end{aligned} \tag{6}$$

C_0 and C_1 are friction coefficients for the motion of the platform and of the oscillators, respectively, and F_i denote driving forces applied to the oscillators. A pulse-like driving acting at the moment when $x_i = 0$ ($i \in [1, N]$) in the direction of the oscillators motion is used. This is similar with the driving experienced in pendulum clocks or metronomes. We assume thus

$$F_i = P\dot{x}_i\delta(x_i). \tag{7}$$

P is a coefficient, which characterizes the strength of the given pulses and $\delta(x)$ is the Dirac function. The term $\dot{x}_i$ is needed in order to ensure a constant momentum input, independently of the metronomes' amplitude. It also ensures that the excitation is given in the right direction (direction of motion). It is easy to see that the total momentum transferred, M_{trans}, to the metronomes in half a period ($T/2$) is always P.

$$M_{trans} = \int_{t}^{t+T/2} P\delta(x_i)\dot{x}_i dt = \int_{-x_{max}}^{x_{max}} P\delta(x_i)dx_i = P \tag{8}$$

The equations of motion with damping and driving cannot be solved analytically, we consider thus a numerical integration by using the Mathematica software. Since the Dirac delta function is not suitable for use with numerical methods, we approximated it with a steeply decaying Gaussian form $F_i = P\dot{x}_i \exp(-10x_i^2)$.

The numerical integration for the equations of motions (6) exhibits a high dependence on its parameters rather than the initial position of one of the oscillators. For simulation time $t \gg T$ (where T is the natural period of the oscillators alone) the system shows either no synchronization or a stable in-phase or anti-phase synchronization, depending on parameters M, k, C_0, C_1.

Here we will be interested only in a narrow parameter region where, only in-phase synchronization is observable, in agreement with the experiments performed on the metronomes. Results for anti-phase synchronization in case of two oscillators was discussed elsewhere [9]. Since we focus only on in-phase synchronization, the synchronization level in the system is monitored by using the classical Kuramoto order parameter, r defined with

$$re^{i\Phi} = \sum_{i=1}^{N} e^{i\theta_i}, \tag{9}$$

where θ_i is the phase of oscillator i ($\theta_i = \arcsin(x_i/A_i)$ and A_i is the amplitude of oscillator i). The r value computed by us was actually a suitable time average of the r values computed from eq. (9).

3. REALISTIC PARAMETERS

In order for this system to be realistic, and the results be comparable with the experimental one we need to find an equivalent for each parameter from the real system of metronomes on a rotating platform. The values for the experimental setup are discussed in [7, 8]. According to [7, 8] the realistic equations of motion for the system of N metronomes displayed on disk of radius R and J moment of inertia relative to the vertical rotational axes is written as

$$(J + NmR^2)\ddot{\phi} + mr\sum_i h_i[\ddot{\theta}_i \cos\theta_i - \dot{\theta}_i^2 \sin\theta_i] + c_\phi \dot{\phi} + \sum_i \mathbb{M}_i = 0 \tag{10}$$

$$[mh_i^2 + J_i]\ddot{\theta}_i + mr\ddot{\phi}h_i \cos\theta_i + mgh_i \sin\theta_i + c_\theta \dot{\theta}_i = \mathbb{M}_i. \tag{11}$$

(The inertial momentum of the metronomes boxes relative to the vertical rotation axes is also included in J.) In the equations of motion from above m and h_i denotes the mass and length of the physical pendulum for the i-th metronome, respectively. It was assumed that all pendulum has the same mass, and their difference is quantified by the values of h_i, solely. θ_i denotes the displacement angle of the i-th pendulum relative to the vertical direction, and ϕ is the angular displacement of the platform. c_ϕ and c_θ are coefficients characterizing the friction in the rotation of the platform and pendulums, respectively. $\mathbb{M}_i$ are instantaneous excitation terms defined as

$$\mathbb{M}_i = W\delta(\theta_i)\dot{\theta}_i, \tag{12}$$

with W a fixed constant.

Now, in our simplified model the mass m will be fixed at 0.025 kg, which is the experimentally measured mass of one metronomes pendulum. The mass of the metronomes boxes (largest part of their mass) is included in the M mass of the platform. Spring constants are selected so that the frequency of the oscillations corresponds to the real metronome frequency range, as since we fixed m it will only be defined by the value of the spring constants. The platform's M mass will be varied between 0.5 kg and 5 kg, values similar to the ones used in the experiments.

Using energy dissipation arguments the friction coefficient C_1 can be obtained from the friction coefficients measured for the swinging of the metronomes pendulum ($c_\theta = 0.00005$ kg·m^2/s, see [8])

$$C_1 \approx \frac{c_\theta \pi}{2A}, \tag{13}$$

where A is the desired amplitude of the oscillations in the present model.

Friction coefficient C_0 is selected in a way that preserves the ratio of C_0/C_1 at the same value as c_θ/c_ϕ ($c_\phi = 0.00001$ kg·m^2/s is the experimentally determined friction coefficient for the rotation of the platform, see [7]). As we fixed most parameters, the driving force in this system will define the end amplitude, A, so we select the driving coefficient in a way that will result in a plausible oscillation amplitude. Thus, the parameters of the system are $\langle k_i \rangle \in [0.5, 5]$ kg/s^2, corresponding to a range of nominal frequencies of $85 - 270$ BPM (BPM stands here for Beats Per Minute). The standard deviation of the spring constants is taken as $\Delta k \approx 0.1$ kg/s^2 as found from examining the standard deviation of the experimentally used metronomes natural frequencies. We fix the amplitude of the oscillations at 0.1 m which leads to the parameters $C_1 = 0.0008$ kg·m/s , $C_0 = 0.0001$ kg·m/s and $P = 0.0008$ kg·m/s. The idea behind the present study is to use oscillators with fixed parameters where only their frequencies can be tuned (similarly with the experiments), and vary the physical properties of the platform. In such sense in the followings the value of C_0 will also be varied.

4. NUMERICAL RESULTS

Using the presented model we studied the variation of the r order parameter as a function of the oscillators spring constant (k), the mass (M) and friction coefficient (C_0) of the platform. The aim of these studies was to reproduce the order-disorder phase transitions revealed in the experiments performed with metronomes [7, 8].

The results for dependence of the order parameter as a function of the mass of the platform and spring constant are shown in Figure 3. In order to obtain these results we have mapped the $M - k$ parameter space and for each mesh point we have

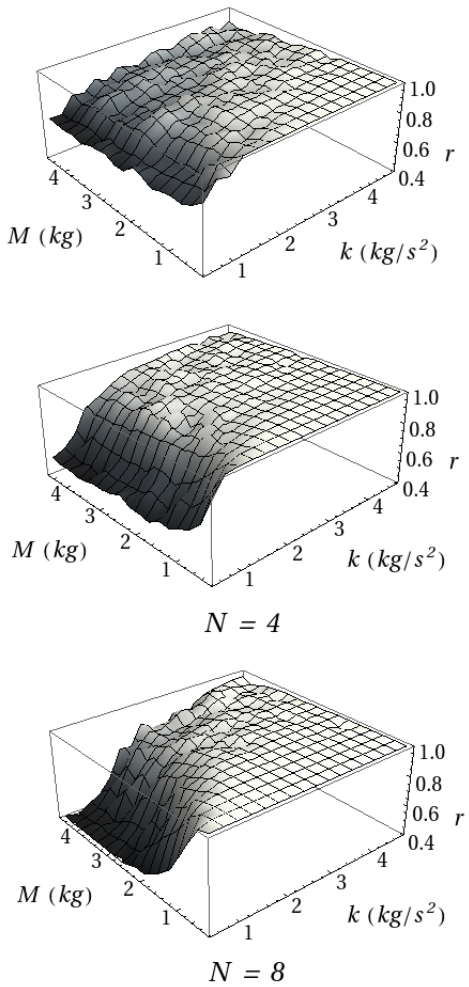

Fig. 3 – The Kuramoto order parameter as a function of the mass of the platform and spring-constants value. Results for different number of oscillators, N, as it is indicated in the figures. ($C_0 = 0.0001$ kg·m/s).

averaged the results of 32 numerical experiments. In order to reveal finite size effects the oscillator numbers, N, on the platform was also varied. The results from Figure 3 indicates the increase in synchrony level with increasing the spring constant (and consecutively the natural frequency of the oscillators) and by decreasing the mass of the platform. These results are in good agreement with the experimental results presented in [7, 8]. By increasing the frequency of the oscillators we have more coupling pulses per unit time, so evidently the synchronization level should increase. Similarly, by increasing the mass (inertia) of the platform the coupling between the

oscillators is decreasing, which is manifested in a decrease in the synchronization level. The order-disorder transition in the *M*-*k* space becomes more pronounced and steep with the increase in the number of oscillators, a phenomenon also observed in the experiments. This finite size effect suggests that in the limit of $N \to \infty$ one would expect a genuine order-disorder type phase transition in this system.

The synchronization level depends also on the value of the C_0 friction coefficient. As one would naturally expect, increasing the friction coefficients of the platform leads to smaller amplitude oscillations, thus a decreased coupling leading to decreased synchronization level. The results from Figure 4 illustrates this. With low number of oscillators the order-disorder transition is rather smooth, but becomes steeper with increasing N, stabilizing for $N > 12$. There is an unexpected peak however in the order parameter as a function of N. As it is visible on Figure 5 the location of the peak is at $N = 6$, independently on the value of C_0. This trend has not been yet confirmed experimentally, as in experiments with real metronomes the mass of the platform increases with larger number of metronomes due to the weight of their boxes while in our calculations we assumed that M stays the same. The non-monotonic finite-size effects are more pronounced for smaller mass of the platform, as it is illustrated in Figure 5.

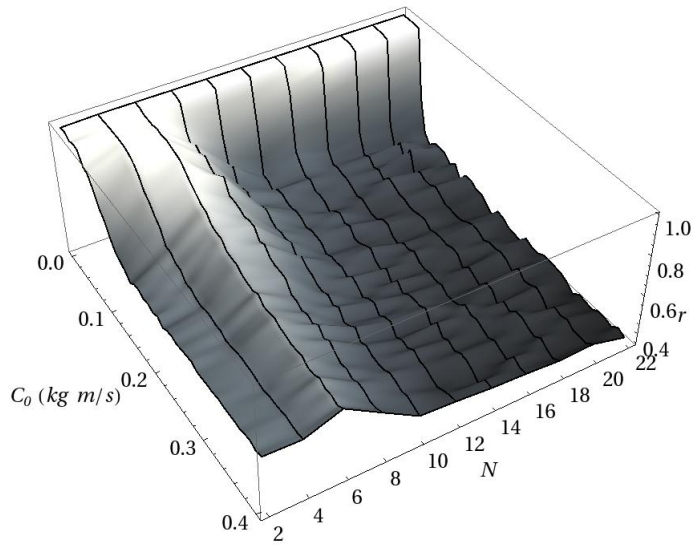

Fig. 4 – The Kuramoto order parameter as a function of the friction coefficient (C_0) of the platform and the number of oscillators on the common platform. ($k = 5$ kg/s^2 and $M = 1$ kg.)

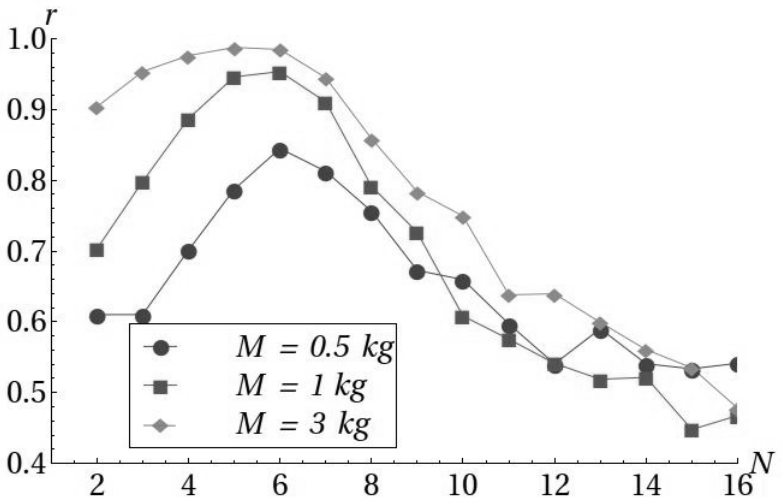

Fig. 5 – The non-monotonic trend of the Kuramoto order parameter as a function of the number of oscillators for three different values of the platforms' mass. ($C_0 = 0.1$ kg/s^2, $k = 5$ kg/s^2).

5. CONCLUSIONS

Emergent synchronization is a fascinating collective behavior, widely present in nature and in physics experiments. Synchronization of coupled pendulum clocks is probably the oldest problem in this field. Nowadays such phenomena is pedagogically illustrated by simple experiments using metronomes instead of pendulum clocks [5, 7, 8]. Here, we discussed a minimal model which is able to reproduce successfully the spontaneous synchronization and order-disorder type phase-transitions observed in metronome systems [7, 8]. The parameters of this simple model were fixed to match those of a realistic system of coupled metronomes. The simplicity of this minimal model allowed an easier numerical analysis and identification of the main parameters that will govern the obtained synchronization level.

It was found that the spontaneous synchronization level of the considered model setup can be controlled by the frequency of the oscillators, mass of the platform and the friction coefficient for the movement of the platform. Since we have fixed the mass of the oscillators, their frequency is controlled by the value of the spring constant, k. Several order-disorder type phase-transition were identified. Studying finite-size effects we obtained intriguing non-monotonic trends. We found an optimal oscillator number where the synchronization level is maximal. The obtained phase-transition type variations are in good agreement with the experimental results recently obtained by our group on coupled metronome systems [7, 8]. Future experiments could confirm also the predicted non-monotonic trend of the order-parameter as a function of the number of oscillators on the platform. We believe that the mini-

mal model discussed here, could be useful in a better understanding of the collective behavior observed in self-sustained coupled oscillators.

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